Supplementary Material

This supplementary material provides a comprehensive introduction and detailed derivations related to the paper. Since our algorithm is built upon matrix Lie group theory and Lie group-based invariant filter, the associated derivations are not as straightforward as those in vector spaces. Therefore, the primary goal of this material is to provide a clear demonstration of the key theoretical foundations and derivation steps, making it easier for readers to follow and understand the underlying mathematical principles.

1 Preliminaries and Notations

In this section, we provide a brief overview of the notations and the fundamentals of matrix Lie group theory that form the basis for deriving our algorithm.

1.1 Notations

Let $\mathbf{I}_r(\mathbf{0}_r)$ denote the $r \times r$ identity (zero) matrix; $\mathbf{0}_{m\times n}$ denote the $m \times n$ zeros matrix; $\mathbf{Tr}(\cdot)$ denote the trace of a matrix. When applied to a set, $|\cdot|$ denotes the cardinality. We use $\mathbf{q} \in \mathbb{R}^r$ to represent a vector of dimension r with all real entries. Given a 3×1 vector $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^\top$, its skew-symmetric matrix is defined as

$$\lfloor \mathbf{q} \times \rfloor = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{1}$$

1.2 Matrix Lie Group and Lie Algebra

A matrix Lie group $\mathcal{G} \in \mathbb{R}^{N\times N}$ is a subset of square invertible matrices with the following three properties holding:

$$\begin{aligned} \mathbf{I}_N &\in \mathcal{G}; \\ \forall X \in \mathcal{G}, X^{-1} &\in \mathcal{G}; \\ \forall X_1, X_2 \in \mathcal{G}, X_1 X_2 &\in \mathcal{G} \end{aligned} \tag{2}$$

The corresponding Lie algebra of a matrix Lie group $\mathcal{G}$ denoted as $\mathfrak{g}$, is a vector space with the same dimension as $\mathcal{G}$. For any element $\boldsymbol{\xi}$ in $\mathfrak{g}$ denoted as $\boldsymbol{\xi} \in \mathbb{R}^{dim(\mathfrak{g})}$, it can be transformed to its

Lie group using the exponential map $\exp_{\mathcal{G}}(\cdot) : \mathbb{R}^{dim\mathfrak{g}} \to \mathcal{G}$ as

$$\exp_{\mathcal{G}}(\boldsymbol{\xi}) = \exp(\boldsymbol{\xi}^{\wedge}) = \sum_{k=1}^{\infty} \frac{\boldsymbol{\xi}^{\wedge}}{k!} \tag{3}$$

where the "hat" operator $(\cdot)^{\wedge}$ denotes the linear mapping $\mathbb{R}^{dim\mathfrak{g}} \to \mathfrak{g}$, that transform the element in the Lie algebra $\mathfrak{g}$ to the corresponding matrix form. $\exp(\cdot)$ denotes the matrix exponential. The inverse function of the exponential map, that is, the logarithm map $\log_{\mathcal{G}} : \mathcal{G} \to \mathbb{R}^{dim(\mathfrak{g})}$, can be defined as

$$\log_{\mathcal{G}}(\exp_{\mathcal{G}}(\boldsymbol{\xi})) = \boldsymbol{\xi} \tag{4}$$

We also encourage readers to refer to [1] for a more comprehensive introduction to matrix Lie groups and Lie algebras, including topics such as group-affine dynamics and the log-linear property of the invariant error.

2 System Propagation

Recall the definition of the state $\mathbf{X}_{I_k}$

$$\begin{aligned} \mathbf{X}_{I_k} &= (\mathbf{T}_k, \mathbf{B}_k), \quad \mathbf{B}_k = \begin{bmatrix} \mathbf{b}_{a_k}^{\top} & \mathbf{b}_{\omega_k}^{\top} \end{bmatrix}^{\top} \\ \mathbf{T}_k &= \left[\begin{array}{c|ccc} {}^{G}_{I_k}\mathbf{R} & {}^{G}\mathbf{v}_{I_k} & {}^{G}\mathbf{p}_{I_k} & {}^{G}\mathbf{p}_{u} \\ \hline \mathbf{0}_3 & & \mathbf{I}_3 & \end{array} \right] \end{aligned} \tag{5}$$

and its kinematics model

$$\begin{aligned} {}^{G}_{I_k}\dot{\mathbf{R}} &= {}^{G}_{I_k}\mathbf{R}\lfloor {}^{I_k}\boldsymbol{\omega}\times \rfloor, \quad {}^{G}\dot{\mathbf{v}}_{I_k} = {}^{G}_{I_k}\mathbf{R}({}^{I_k}\mathbf{a}) + {}^{G}\mathbf{g} \\ {}^{G}\dot{\mathbf{p}}_{I_k} &= {}^{G}\mathbf{v}_{I_k}, \quad {}^{G}\dot{\mathbf{p}}_{f} = \mathbf{0}, \quad \dot{\mathbf{b}}_{\omega_k} = \mathbf{w}_{\omega}, \quad \dot{\mathbf{b}}_{a_k} = \mathbf{w}_{a} \end{aligned} \tag{6}$$

2.1 State transition and covariance propagation

Given the state $\mathbf{X}_{I_k}$ and its corresponding estimate $\hat{\mathbf{X}}_{I_k}$, we first formulate the estimation error as

$$\widetilde{\mathbf{X}}_{I_k} = (\boldsymbol{\eta}_k, \widetilde{\mathbf{B}}_k) \tag{7}$$

where $\boldsymbol{\eta}_k$ represents the corresponding right invariant error as

$$\begin{aligned}
\boldsymbol{\eta}_k &= \hat{\mathbf{T}}_k \mathbf{T}_k^{-1} = \left[\begin{array}{c|ccc} {}^G_{I_k}\widetilde{\mathbf{R}} & \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_2 & \boldsymbol{\Gamma}_3 \\ \hline \mathbf{0}_3 & & \mathbf{I}_3 & \end{array}\right] \\
{}^G_{I_k}\widetilde{\mathbf{R}} &= {}^G_{I_k}\hat{\mathbf{R}}({}^G_{I_k}\mathbf{R})^{-1} \\
\boldsymbol{\Gamma}_1 &= {}^G\hat{\mathbf{v}}_{I_k} - {}^G_{I_k}\widetilde{\mathbf{R}}{}^G\mathbf{v}_{I_k} \\
\boldsymbol{\Gamma}_2 &= {}^G\hat{\mathbf{p}}_{I_k} - {}^G_{I_k}\widetilde{\mathbf{R}}{}^G\mathbf{p}_{I_k} \\
\boldsymbol{\Gamma}_3 &= {}^G\hat{\mathbf{p}}_u - {}^G_{I_k}\widetilde{\mathbf{R}}{}^G\mathbf{p}_u
\end{aligned} \tag{8}$$

and

$$\widetilde{\mathbf{B}}_k = \left[(\hat{\mathbf{b}}_{a_k} - \mathbf{b}_{a_k})^\top, (\hat{\mathbf{b}}_{\omega_k} - \mathbf{b}_{\omega_k})^\top\right]^\top \tag{9}$$

The combined error term $\widetilde{\mathbf{X}}_{I_k}$ is commonly referred to as the augmented right-invariant error, as it includes both a right-invariant error term and an appended error vector. By applying the log-linear property of the invariant error [2], errors $\boldsymbol{\eta}_k$ can be approximated using a first-order approximation as follows:

$$\boldsymbol{\eta}_k = \exp_{\mathcal{G}}(\boldsymbol{\xi}_{I_k}) \approx \mathbf{I}_6 + \boldsymbol{\xi}_{I_k}^\wedge \in \mathbb{R}^{6\times 6} \tag{10}$$

where $(\cdot)^\wedge : \mathbb{R}^{\dim\mathfrak{g}} \to \mathfrak{g}$ be the linear map that transforms the error vector $\boldsymbol{\xi}_{I_k}$ and $\boldsymbol{\xi}_{c_k}$ defined in the Lie algebra to its corresponding matrix representation [2] as

$$\begin{aligned}
\boldsymbol{\xi}_{I_k} &\triangleq \begin{bmatrix}(\boldsymbol{\xi}_{\theta_k})^\top & (\boldsymbol{\xi}_{v_k})^\top & (\boldsymbol{\xi}_{p_k})^\top & (\boldsymbol{\xi}_{u_k})^\top\end{bmatrix}^\top \in \mathbb{R}^{12} \\
\boldsymbol{\xi}_{\theta_k} &= \widetilde{\boldsymbol{\theta}}_k = \log({}^G_{I_k}\widetilde{\mathbf{R}}) \in \mathbb{R}^3 \\
\boldsymbol{\xi}_{v_k} &= {}^G\hat{\mathbf{v}}_{I_k} - (\mathbf{I}_3 + \lfloor\widetilde{\boldsymbol{\theta}}_k\times\rfloor){}^G\mathbf{v}_{I_k} \in \mathbb{R}^3 \\
\boldsymbol{\xi}_{p_k} &= {}^G\hat{\mathbf{p}}_{I_k} - (\mathbf{I}_3 + \lfloor\widetilde{\boldsymbol{\theta}}_k\times\rfloor){}^G\mathbf{p}_{I_k} \in \mathbb{R}^3 \\
\boldsymbol{\xi}_{u_k} &= {}^G\hat{\mathbf{p}}_u - (\mathbf{I}_3 + \lfloor\widetilde{\boldsymbol{\theta}}_k\times\rfloor){}^G\mathbf{p}_u \in \mathbb{R}^3,
\end{aligned} \tag{11}$$

Give the error definitions in (9), (11), we define the error state vector of the state $\mathbf{X}_{I_k}$ as

$$\widetilde{\mathbf{x}}_{I_k} \triangleq \begin{bmatrix}\boldsymbol{\xi}_{I_k}^\top & \widetilde{\mathbf{B}}_k^\top\end{bmatrix}^\top \in \mathbb{R}^{18} \tag{12}$$

To compute the linearized error dynamics, denoted as

$$\frac{d}{dt}\widetilde{\mathbf{x}}_{I_k} = \mathbf{F}_k\widetilde{\mathbf{x}}_{I_k} + \mathbf{G}_k\mathbf{n}_k, \tag{13}$$

we first make the first-order approximation (**cf.** (10)) to each individual term of the invariant error $\boldsymbol{\eta}_k$, which yields

$$\begin{aligned}
\frac{d}{dt}\left({}^{G}_{I_k}\widetilde{\mathbf{R}}\right) &= \frac{d}{dt}\left({}^{G}_{I_k}\hat{\mathbf{R}}\right)({}^{G}_{I_k}\mathbf{R})^\top + ({}^{G}_{I_k}\hat{\mathbf{R}})\frac{d}{dt}\left({}^{G}_{I_k}\mathbf{R}\right)^\top \\
&= {}^{G}_{I_k}\hat{\mathbf{R}}\lfloor {}^{I_k}\boldsymbol{\omega}_m - \hat{\mathbf{b}}_{\omega_k}\times\rfloor({}^{G}_{I_k}\mathbf{R})^\top + ({}^{G}_{I_k}\hat{\mathbf{R}})\left({}^{G}_{I_k}\mathbf{R}\lfloor {}^{I_k}\boldsymbol{\omega}_m - \mathbf{b}_{\omega_k} - \mathbf{n}_{\omega_k}\times\rfloor\right)^\top \\
&= {}^{G}_{I_k}\hat{\mathbf{R}}\lfloor {}^{I_k}\boldsymbol{\omega}_m - \hat{\mathbf{b}}_{\omega_k}\times\rfloor({}^{G}_{I_k}\hat{\mathbf{R}})^\top {}^{G}_{I_k}\widetilde{\mathbf{R}} - ({}^{G}_{I_k}\hat{\mathbf{R}})\lfloor {}^{I_k}\boldsymbol{\omega}_m - \mathbf{b}_{\omega_k} - \mathbf{n}_{\omega_k}\times\rfloor({}^{G}_{I_k}\hat{\mathbf{R}})^\top {}^{G}_{I_k}\widetilde{\mathbf{R}} \\
&\approx ({}^{G}_{I_k}\hat{\mathbf{R}})\lfloor -\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}\times\rfloor({}^{G}_{I_k}\hat{\mathbf{R}})^\top(\mathbf{I}_3 + \lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor) \\
&\approx \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}\times)\rfloor(\mathbf{I}_3 + \lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor) \\
&\approx \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}\times)\rfloor \\
\Rightarrow \frac{d}{dt}(\lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor) &\approx \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})\times\rfloor \\
\Rightarrow \frac{d}{dt}(\boldsymbol{\xi}_{\theta_k}) &\approx {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{\Gamma}_1) &= \frac{d}{dt}\left({}^{G}\hat{\mathbf{v}}_{I_k} - {}^{G}_{I_k}\widetilde{\mathbf{R}}{}^{G}\mathbf{v}_{I_k}\right) \\
&= {}^{G}_{I_k}\hat{\mathbf{R}}({}^{I_k}\mathbf{a}_m - \hat{\mathbf{b}}_{a_k}) + {}^{G}\mathbf{g} - \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})\times\rfloor {}^{G}_{I_k}\widetilde{\mathbf{R}}{}^{G}\mathbf{v}_{I_k} \\
&\quad - {}^{G}_{I_k}\hat{\mathbf{R}}\left({}^{I_k}\mathbf{a}_m - \mathbf{b}_{a_k} - \mathbf{n}_{a_k}\right) - {}^{G}_{I_k}\widetilde{\mathbf{R}}{}^{G}\mathbf{g} \\
&\approx {}^{G}_{I_k}\hat{\mathbf{R}}(\widetilde{\mathbf{b}}_{a_k} + \mathbf{n}_{a_i,k}) - \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})\times\rfloor {}^{G}\mathbf{v}_{I_k} + (\mathbf{I}_3 - {}^{G}_{I_k}\widetilde{\mathbf{R}}){}^{G}\mathbf{g} \\
&\approx {}^{G}_{I_k}\hat{\mathbf{R}}(\widetilde{\mathbf{b}}_{a_k} + \mathbf{n}_{a_k}) + \lfloor {}^{G}\mathbf{v}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}) + (\mathbf{I}_3 - {}^{G}_{I_k}\widetilde{\mathbf{R}}){}^{G}\mathbf{g} \\
\Rightarrow \frac{d}{dt}(\boldsymbol{\Gamma}_1) &\approx -\lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor {}^{G}\mathbf{g} + {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{a_k} + \mathbf{n}_{a_k}) + \lfloor {}^{G}\mathbf{v}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}) \\
&\approx \lfloor {}^{G}\mathbf{g}\times\rfloor\boldsymbol{\xi}_{\theta_k} + {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{a_k} + \mathbf{n}_{a_k}) + \lfloor {}^{G}\hat{\mathbf{v}}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}) \\
\Rightarrow \frac{d}{dt}(\boldsymbol{\xi}_{v_k}) &\approx \lfloor {}^{G}\mathbf{g}\times\rfloor\boldsymbol{\xi}_{\theta_k} + {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{a_k} + \mathbf{n}_{a_k}) + \lfloor {}^{G}\hat{\mathbf{v}}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}),
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{\Gamma}_2) &= \frac{d}{dt}\left({}^{G}\hat{\mathbf{p}}_{I_k} - {}^{G}_{I_k}\widetilde{\mathbf{R}}{}^{G}\mathbf{p}_{I_k}\right) \\
&= {}^{G}\hat{\mathbf{v}}_{I_k} - \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_i,k})\times\rfloor {}^{G}_{I_k}\widetilde{\mathbf{R}}{}^{G}\mathbf{p}_{I_k} - {}^{G}_{I_k}\widetilde{\mathbf{R}}{}^{G}\mathbf{v}_{I_k} \\
&\approx {}^{G}\hat{\mathbf{v}}_{I_k} - \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_i,k} + \mathbf{n}_{\omega_k})\times\rfloor(\mathbf{I}_3 + \lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor){}^{G}\mathbf{p}_{I_k} - (\mathbf{I}_3 + \lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor){}^{G}\mathbf{v}_{I_k} \\
&\approx {}^{G}\hat{\mathbf{v}}_{I_k} - \lfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})\times\rfloor {}^{G}\mathbf{p}_{I_k} - (\mathbf{I}_3 + \lfloor\boldsymbol{\xi}_{\theta_k}\times\rfloor){}^{G}\mathbf{v}_{I_k} \\
&\approx {}^{G}\hat{\mathbf{v}}_{I_k} - {}^{G}\mathbf{v}_{I_k} + \lfloor {}^{G}\mathbf{p}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}) + \lfloor {}^{G}\mathbf{v}_{I_k}\times\rfloor\boldsymbol{\xi}_{\theta_k} \\
&\approx \boldsymbol{\xi}_{v_k} + \lfloor {}^{G}\hat{\mathbf{p}}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}) \\
\Rightarrow \frac{d}{dt}(\boldsymbol{\xi}_{p_k}) &\approx \boldsymbol{\xi}_{v_k} + \lfloor {}^{G}\hat{\mathbf{p}}_{I_k}\times\rfloor {}^{G}_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\frac{d}{dt}(\mathbf{\Gamma}_3) &= \frac{d}{dt}\left({}^G\hat{\mathbf{p}}_u - {}^G_{I_k}\widetilde{\mathbf{R}}{}^G\mathbf{p}_u\right) \\
&\approx -\lfloor {}^G_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})\times\rfloor {}^G_{I_k}\widetilde{\mathbf{R}}{}^G\mathbf{p}_u \\
&\approx -\lfloor {}^G_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})\times\rfloor (\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k}\times\rfloor)({}^G\hat{\mathbf{p}}_u) \\
&\approx \lfloor {}^G\hat{\mathbf{p}}_u\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k}) \\
\Rightarrow \frac{d}{dt}(\boldsymbol{\xi}_u) &\approx \lfloor {}^G\hat{\mathbf{p}}_u\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}(-\widetilde{\mathbf{b}}_{\omega_k} + \mathbf{n}_{\omega_k})
\end{aligned} \tag{17}$$

Then, a linearized error state system can be constructed from the above equations as

$$\frac{d}{dt}\widetilde{\mathbf{x}}_{I_k} = \mathbf{F}_k\widetilde{\mathbf{x}}_{I_k} + \mathbf{G}_k\mathbf{n}_k, \tag{18}$$

where the jacobian $\mathbf{F}_k$ can be computed as

$$\mathbf{F}_k = \begin{bmatrix} \mathbf{F}_A & F_{B_k} \\ \mathbf{0}_{6\times12} & \mathbf{0}_6 \end{bmatrix}, \quad \mathbf{F}_A = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{g}\times\rfloor & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}, \quad \mathbf{F}_{B_k} = \begin{bmatrix} -{}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{v}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & -{}^G_{I_k}\hat{\mathbf{R}} \\ -\lfloor {}^G\hat{\mathbf{p}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{p}}_u\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \end{bmatrix} \tag{19}$$

and the jacobian $\mathbf{G}_k$ can be computed as

$$\mathbf{G}_k = \begin{bmatrix} \mathbf{Ad}_{\hat{\mathbf{X}}_k} & \mathbf{0}_{12\times6} \\ \mathbf{0}_{6\times12} & \mathbf{0}_6 \end{bmatrix}, \quad \mathbf{Ad}_{\hat{\mathbf{X}}_k} = \begin{bmatrix} {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\hat{\mathbf{v}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\hat{\mathbf{p}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 & {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \\ \lfloor {}^G\hat{\mathbf{p}}_u\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_{I_k}\hat{\mathbf{R}} \end{bmatrix} \tag{20}$$

By considering the time interval δt between the time t_k and t_{k-1} is a small constant value, the corresponding discrete-time state transition matrix $\mathbf{\Phi}_{k|k-1}$ from time t_{k-1} to t_k can be computed using the matrix exponential as

$$\mathbf{\Phi}_{k|k-1} = \exp\left(\begin{bmatrix} \mathbf{F}_A & \mathbf{F}_{B_k} \\ \mathbf{0}_{6\times12} & \mathbf{0}_6 \end{bmatrix}\delta t\right) = \begin{bmatrix} \mathbf{\Phi}_A & \mathbf{\Phi}_{B_k} \\ \mathbf{0}_{6\times12} & \mathbf{I}_6 \end{bmatrix} \tag{21}$$

where

$$\begin{aligned}
\mathbf{\Phi}_A &= \mathbf{I}_{12} + \mathbf{F}_A\delta t + \frac{1}{2!}\mathbf{F}_A^2\delta t^2 + \cdots + \frac{1}{n!}\mathbf{F}_A^n\delta t^n \\
\mathbf{\Phi}_{B_k} &= \mathbf{F}_{B_k}\delta t + \frac{1}{2!}\mathbf{F}_A\mathbf{F}_{B_k}\delta t^2 + \cdots + \frac{1}{n!}\mathbf{F}_A^{n-1}\mathbf{F}_{B_k}\delta t^n
\end{aligned} \tag{22}$$

Since $\mathbf{F}_A^3 = \mathbf{0}$, the above equation yields

$$\mathbf{\Phi}_A = \mathbf{I}_{12} + \mathbf{F}_A\delta t + \frac{1}{2!}\mathbf{F}_A^2\delta t^2 = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{g}\times\rfloor\delta t & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor {}^G\mathbf{g}\times\rfloor\delta t^2 & \mathbf{I}_3\delta t & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{\Phi}_{B_k} &= \mathbf{F}_{B_k}\delta t + \frac{1}{2!}\mathbf{F}_A\mathbf{F}_{B_k}\delta t^2 + \frac{1}{3!}\mathbf{F}_A^2\mathbf{F}_{B_k}\delta t^3 \\ &= \begin{bmatrix} -{}^G_{I_{i,k}}\hat{\mathbf{R}} & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{v}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & -{}^G_{I_k}\hat{\mathbf{R}} \\ -\lfloor {}^G\hat{\mathbf{p}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{p}}_u\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \end{bmatrix}\delta t + \frac{1}{2}\begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{v}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & -{}^G_{I_k}\hat{\mathbf{R}} \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}\delta t^2 \\ &\quad + \frac{1}{6}\begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}\delta t^3 = \begin{bmatrix} -{}^G_{I_k}\hat{\mathbf{R}}\delta t & \mathbf{0}_3 \\ \Psi_A & -{}^G_{I_k}\hat{\mathbf{R}}\delta t \\ \Psi_{B_k} & -\frac{1}{2}{}^G_{I_k}\hat{\mathbf{R}}\delta t^2 \\ -\lfloor {}^G\hat{\mathbf{p}}_u\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}\delta t & \mathbf{0}_3 \end{bmatrix} \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Psi_A &= -\lfloor {}^G\hat{\mathbf{v}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}\delta t - \frac{1}{2}\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}\delta t^2 \\ \Psi_{B_k} &= -\lfloor {}^G\hat{\mathbf{p}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}\delta t - \frac{1}{2}\lfloor {}^G\hat{\mathbf{v}}_{I_k}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}\delta t^2 - \frac{1}{6}\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_k}\hat{\mathbf{R}}\delta t^3 \end{aligned} \tag{24}$$

Then, we can compute the state transition matrix $\mathbf{\Phi}_{k|0}$ as

$$\begin{aligned} \mathbf{\Phi}_{k|0} &= \mathbf{\Phi}_{k|k-1}\mathbf{\Phi}_{k-1|k-2}\cdots\mathbf{\Phi}_{1|0} \\ &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{15} & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{g}\times\rfloor\delta t & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{25} & \mathbf{\Phi}_{26} \\ \frac{1}{2}\lfloor {}^G\mathbf{g}\times\rfloor\delta t^2 & \mathbf{I}_3\delta t & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{35} & \mathbf{\Phi}_{36} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{45} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathbf{\Phi}_{15} &= -\sum_{\tau=0}^{k} {}^G_{I_\tau}\hat{\mathbf{R}}\delta t \\ \mathbf{\Phi}_{25} &= -\sum_{\tau=0}^{k}\left(\lfloor {}^G\hat{\mathbf{v}}_{I_\tau}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t + \frac{1}{2}\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^2\right) \\ \mathbf{\Phi}_{35} &= -\sum_{\tau=0}^{k}\left(\lfloor {}^G\hat{\mathbf{p}}_{I_\tau}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t + \frac{1}{2}\lfloor {}^G\hat{\mathbf{v}}_{I_\tau}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^2 + \frac{1}{6}\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^3\right) \end{aligned} \tag{26}$$

and

$$\begin{aligned}
\mathbf{\Phi}_{45} &= -\sum_{\tau=0}^{k} \lfloor {}^{G}\hat{\mathbf{p}}_u \times \rfloor {}^{G}_{I_{i,\tau}}\hat{\mathbf{R}}\delta t \\
\mathbf{\Phi}_{26} &= -\sum_{\tau=0}^{k} {}^{G}_{I_\tau}\hat{\mathbf{R}}\delta t \\
\mathbf{\Phi}_{36} &= -\frac{1}{2}\sum_{\tau=0}^{k} {}^{G}_{I_\tau}\hat{\mathbf{R}}\delta t^2
\end{aligned} \tag{27}$$

In addition to the matrix exponential approach described above, the state transition matrix can also be derived analytically. We refer interested readers to [1] for further details.

3 Measurement Update

3.1 Visual Measurements

The visual measurement model is given by

$$\mathbf{z}_{c_k} = \mathbf{\Pi}({}^{C_k}\mathbf{p}_f) + \mathbf{n}_{c_k} \tag{28}$$

where

$$\begin{aligned}
{}^{C_k}\mathbf{p}_f \triangleq \begin{bmatrix} x & y & c \end{bmatrix}^\top &= {}^{G}_{I_k}\mathbf{R}^\top({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I_k}) \\
\mathbf{\Pi}\left({}^{I_k}\mathbf{p}_f\right) = \mathbf{\Pi}\left(\begin{bmatrix} x & y & z \end{bmatrix}^\top\right) &= \begin{bmatrix} x/z & y/z \end{bmatrix}^\top
\end{aligned} \tag{29}$$

To proceed with the measurement update, we first linearize the above equation as

$$\widetilde{\mathbf{z}}_{c_k} = \mathbf{H}_{x_k}\widetilde{\mathbf{x}}_{I_k} + \mathbf{H}_{f_k}{}^{G}\widetilde{\mathbf{p}}_f + \mathbf{n}_{c_k} \tag{30}$$

where the measurement jacobians $\mathbf{H}_{x_k}$ and $\mathbf{H}_{f_k}$ can be computed using the chain rule

$$\mathbf{H}_{x_k} = \frac{\partial \widetilde{\mathbf{z}}_{c_k}}{\partial {}^{C_k}\widetilde{\mathbf{p}}_f}\frac{\partial {}^{C_k}\widetilde{\mathbf{p}}_f}{\partial \widetilde{\mathbf{x}}_{I_k}} \quad , \mathbf{H}_{f_k} = \frac{\partial \widetilde{\mathbf{z}}_{c_k}}{\partial {}^{C_k}\widetilde{\mathbf{p}}_f}\frac{\partial {}^{C_k}\widetilde{\mathbf{p}}_f}{\partial {}^{G}\widetilde{\mathbf{p}}_f} \tag{31}$$

In particular, the jacobian $\frac{\partial \widetilde{\mathbf{z}}_{c_k}}{\partial {}^{C_k}\widetilde{\mathbf{p}}_f}$ can be computed as

$$\frac{\partial \widetilde{\mathbf{z}}_{c_k}}{\partial {}^{C_k}\widetilde{\mathbf{p}}_f} \triangleq \mathbf{H}_{pc} = \begin{bmatrix} 1/z & 0 & -x/z^2 \\ 0 & 1/z & -y/z^2 \end{bmatrix} \tag{32}$$

Recall that $\boldsymbol{\eta}_k = \exp_{\mathcal{G}}(\boldsymbol{\xi}_{I_k}) \approx \mathbf{I}_6 + \boldsymbol{\xi}_{I_k}^{\wedge} \Rightarrow \hat{\mathbf{T}}_k \approx (I_6 + \boldsymbol{\xi}_{I_k}^{\wedge})\mathbf{T}_k$. We can compute the jacobians $\frac{\partial^{C_k}\widetilde{\mathbf{p}}_f}{\partial \widetilde{\mathbf{x}}_{I_k}}$, $\frac{\partial^{C_k}\widetilde{\mathbf{p}}_f}{\partial^{G}\widetilde{\mathbf{p}}_f}$ as

$$
\begin{aligned}
{}^{C_k}\hat{\mathbf{p}}_f &= {}^{C}_{I}\mathbf{R}^{G}_{I_k}\hat{\mathbf{R}}^{\top}({}^{G}\hat{\mathbf{p}}_f - {}^{G}\hat{\mathbf{p}}_{I_k}) \\
&\approx {}^{C}_{I}\mathbf{R}^{G}_{I_k}\mathbf{R}^{\top}(\mathbf{I}_3 - \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor)\left((\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor)^{G}\mathbf{p}_f + {}^{G}\widetilde{\mathbf{p}}_f - (\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor)^{G}\mathbf{p}_{I_k} - \boldsymbol{\xi}_{p_k}\right) \\
&\approx {}^{C}_{I}\mathbf{R}^{G}_{I_k}\mathbf{R}^{\top}\left((\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor)^{G}\mathbf{p}_f + {}^{G}\widetilde{\mathbf{p}}_f - (\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor)^{G}\mathbf{p}_{I_k} - \boldsymbol{\xi}_{p_k}\right) - {}^{G}_{I_k}\mathbf{R}^{\top}\lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor\left({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I_k}\right) \\
&= {}^{C}_{I}\mathbf{R}^{G}_{I_k}\mathbf{R}^{\top}\left({}^{G}\mathbf{p}_f + {}^{G}\widetilde{\mathbf{p}}_f - {}^{G}\mathbf{p}_{I_k} - \boldsymbol{\xi}_{p_k}\right) \\
&\Rightarrow {}^{C_k}\,\widetilde{\mathbf{p}}_f \approx {}^{C}_{I}\mathbf{R}^{G}_{I_k}\mathbf{R}^{\top}({}^{G}\widetilde{\mathbf{p}}_f - \boldsymbol{\xi}_{p_k}) \\
&\Rightarrow \frac{\partial^{C_k}\widetilde{\mathbf{p}}_f}{\partial \widetilde{\mathbf{x}}_{I_k}} = {}^{C}_{I}\mathbf{R}^{G}_{I_k}\mathbf{R}^{\top}\begin{bmatrix}\mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 6}\end{bmatrix}, \quad \frac{\partial^{C_k}\widetilde{\mathbf{p}}_f}{\partial^{G}\widetilde{\mathbf{p}}_f} = {}^{C}_{I}\mathbf{R}^{G}_{I_k}\mathbf{R}^{\top}
\end{aligned}
\tag{33}
$$

Hence the jacobians $\mathbf{H}_{x_k}$, $\mathbf{H}_{f_k}$ can be calculated as

$$
\begin{aligned}
\mathbf{H}_{x_k} &= \mathbf{H}_{pc}{}^{C}_{I}\mathbf{R}^{I_k}_{G}\mathbf{R}\begin{bmatrix}\mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_{3\times 9}\end{bmatrix} \\
\mathbf{H}_{f_k} &= \mathbf{H}_{pc}{}^{C}_{I}\mathbf{R}^{I_k}_{G}\mathbf{R}.
\end{aligned}
\tag{34}
$$

Note that the measurement Jacobian above is computed using the true state $\mathbf{X}_{I_k}$ as the linearization point. To perform the update, we must shift the linearization point to the corresponding state estimate $\hat{\mathbf{X}}_{I_k}$, which yields results identical to those presented in the paper.

3.2 UWB Ranging Measurement

The UWB ranging measurement model is given by

$$
\begin{aligned}
\mathbf{z}_{u_k} &= \|\mathbf{d}_k\| + \mathbf{n}_{u_k} + \mathbf{b}_{u_k} \\
\mathbf{d}_k &= {}^{G}\mathbf{p}_{I_k} + {}^{I_k}_{G}\mathbf{R}^{\top}\,{}^{I}\mathbf{p}_T - {}^{G}\mathbf{p}_u
\end{aligned}
\tag{35}
$$

Similarly to the visual measurement update, we also linearize the UWB ranging measurement as

$$
\widetilde{\mathbf{z}}_{u_k} = \mathbf{H}_{u_k}\widetilde{\mathbf{x}}_{I_k} + \mathbf{n}_u \tag{36}
$$

Following the chain rule, the measurement jacobian $\mathbf{H}_{u_k}$ can be computed as

$$
\mathbf{H}_{u_k} = \frac{\partial \widetilde{\mathbf{z}}_{u_k}}{\partial \widetilde{\mathbf{d}}_k}\frac{\partial \widetilde{\mathbf{d}}_k}{\partial \widetilde{\mathbf{x}}_{I_k}} \tag{37}
$$

where the jacobian $\frac{\partial \widetilde{\mathbf{z}}_{u_k}}{\partial \widetilde{\mathbf{d}}_k}$ is calculated as

$$
\frac{\partial \widetilde{\mathbf{z}}_{u_k}}{\partial \widetilde{\mathbf{d}}_k} \triangleq \mathbf{H}_{pu} = \frac{\left({}^{G}\mathbf{p}_{I_k} - {}^{G}\mathbf{p}_u + {}^{G}_{I_k}\mathbf{R}\,{}^{I}\mathbf{p}_T\right)^{\top}}{\|{}^{G}\mathbf{p}_{I_k} - {}^{G}\mathbf{p}_u + {}^{G}_{I_k}\mathbf{R}\,{}^{I}\mathbf{p}_T\|} \tag{38}
$$

and given that $\boldsymbol{\eta}_k = \exp_{\mathcal{G}}(\boldsymbol{\xi}_{I_k}) \approx \mathbf{I}_6 + \boldsymbol{\xi}_{I_k}^{\wedge}$, the jacobian $\frac{\partial \widetilde{\mathbf{d}}_k}{\partial \widetilde{\mathbf{x}}_{I_k}}$ is given by

$$\begin{aligned}
\mathbf{d}_k &= {}^G\hat{\mathbf{p}}_{I_k} + {}^G_{I_k}\hat{\mathbf{R}}\, {}^I\mathbf{p}_T - {}^G\hat{\mathbf{p}}_u \approx (\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor) {}^G\mathbf{p}_{I_k} + \boldsymbol{\xi}_{I_k} \\
&- (\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor) {}^G\mathbf{p}_u - \boldsymbol{\xi}_{u_k} + (\mathbf{I}_3 + \lfloor \boldsymbol{\xi}_{\theta_k} \times \rfloor) {}^G_{I_k}\mathbf{R}\, {}^I\mathbf{p}_T \\
&\Rightarrow \widetilde{\mathbf{d}}_k \approx \boldsymbol{\xi}_{I_k} - \boldsymbol{\xi}_{u_k} + \lfloor {}^G\mathbf{p}_u - {}^G\mathbf{p}_{I_k} - {}^G_{I_k}\mathbf{R}\, {}^I\mathbf{p}_T \times \rfloor \boldsymbol{\xi}_{\theta_k} \\
&\Rightarrow \frac{\partial \widetilde{\mathbf{d}}_k}{\partial \widetilde{\mathbf{x}}_{I_k}} = \begin{bmatrix} \lfloor {}^G\mathbf{p}_u - {}^G\mathbf{p}_{I_k} - {}^G_{I_k}\mathbf{R}\, {}^I\mathbf{p}_T \times \rfloor & \mathbf{0}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_{3\times 6} \end{bmatrix}
\end{aligned} \tag{39}$$

Thus, the measurement jacobian $\mathbf{H}_{u_k}$ is given as

$$\begin{aligned}
\mathbf{H}_{u_k} &= \mathbf{H}_{pu} \begin{bmatrix} \Lambda_k & \mathbf{0}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_{3\times 6} \end{bmatrix} \\
\mathbf{H}_{pu} &= \frac{\left({}^G\mathbf{p}_{I_k} - {}^G\mathbf{p}_u + {}^G_{I_k}\mathbf{R}\, {}^I\mathbf{p}_T \right)^\top}{\| {}^G\mathbf{p}_{I_k} - {}^G\mathbf{p}_u + {}^G_{I_k}\mathbf{R}\, {}^I\mathbf{p}_T \|} \\
\Lambda_k &= \lfloor {}^G\mathbf{p}_u - {}^G\mathbf{p}_{I_k} - {}^G_{I_k}\mathbf{R}\, {}^I\mathbf{p}_T \times \rfloor .
\end{aligned} \tag{40}$$

4 Observability Analysis

According to [3], the observability of the MSCKF system can be analyzed through the EKF-SLAM framework, as the two are theoretically equivalent. Thus, without loss of generality, we study the observability of the EKF-SLAM system with a single feature point ${}^G\mathbf{p}_f$ and one UWB anchor ${}^G\mathbf{p}_u$. The state of the EKF-SLAM system has a form very similar to $\mathbf{X}_{I_k}$, with the only difference being that the original state $\mathbf{T}_k$ is now augmented with a feature state, given by

$$\mathbf{T}_k = \left[\begin{array}{c|cccc} {}^G_{I_k}\mathbf{R} & {}^G\mathbf{v}_{I_k} & {}^G\mathbf{p}_{I_k} & {}^G\mathbf{p}_u & {}^G\mathbf{p}_f \\ \hline \mathbf{0}_4 & & \mathbf{I}_4 & & \end{array} \right] \in SE_4(3), \tag{41}$$

Following the proof of [4], the local observability matrix for the time-varying error state of the whole system is defined as

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}_0 \\ \mathcal{O}_1 \\ \vdots \\ \mathcal{O}_k \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \widetilde{\boldsymbol{\Phi}}_{1|0} \\ \vdots \\ \mathbf{H}_k \widetilde{\boldsymbol{\Phi}}_{k|0} \end{bmatrix}, \tag{42}$$

where $\mathbf{H}_k = \begin{bmatrix} \mathbf{H}_{c_k}^\top & \mathbf{H}_{r_k}^\top \end{bmatrix}^\top$ is the joint measurement jacobian; $\mathbf{H}_{c_k}$ represents the jacobian of the visual measurement, and $\mathbf{H}_{r_k}$ represents the jacobian of the UWB ranging measurement

$$\begin{aligned}
\mathbf{H}_{c_k} &= \mathbf{H}_{f_k} \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_{3\times 6} \end{bmatrix} \\
\mathbf{H}_{r_k} &= \mathbf{H}_{pu} \begin{bmatrix} \Lambda_k & \mathbf{0}_3 & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 6} \end{bmatrix},
\end{aligned} \tag{43}$$

$\widetilde{\mathbf{\Phi}}_{k,0}$ is the state transition matrix of the EKF-SLAM system. Note that the computation of the state transition matrix $\widetilde{\mathbf{\Phi}}_{k,0}$ for the above EKF-SLAM system is very similar to that of the state transition matrix $\mathbf{\Phi}_{k,0}$ presented in the previous section, which is given by

$$\begin{aligned}\widetilde{\mathbf{\Phi}}_{k,0} &= \begin{bmatrix} \widetilde{\mathbf{\Phi}}_A & \widetilde{\mathbf{\Phi}}_{B_k} \\ \mathbf{0}_{6\times 15} & \mathbf{I}_6 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widetilde{\mathbf{\Phi}}_{16} & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{g}\times\rfloor \delta t & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widetilde{\mathbf{\Phi}}_{26} & \widetilde{\mathbf{\Phi}}_{27} \\ \frac{1}{2}\lfloor {}^G\mathbf{g}\times\rfloor \delta t^2 & \mathbf{I}_3\delta t & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widetilde{\mathbf{\Phi}}_{36} & \widetilde{\mathbf{\Phi}}_{37} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \widetilde{\mathbf{\Phi}}_{46} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \widetilde{\mathbf{\Phi}}_{56} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}\end{aligned} \tag{44}$$

where

$$\begin{aligned}\widetilde{\mathbf{\Phi}}_{16} &= -\sum_{\tau=0}^{k} {}^G_{I_\tau}\hat{\mathbf{R}}\delta t \\ \widetilde{\mathbf{\Phi}}_{26} &= -\sum_{\tau=0}^{k} \left(\lfloor {}^G\hat{\mathbf{v}}_{I_\tau}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t + \frac{1}{2}\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^2 \right) \\ \widetilde{\mathbf{\Phi}}_{36} &= -\sum_{\tau=0}^{k} \left(\lfloor {}^G\hat{\mathbf{p}}_{I_\tau}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t + \frac{1}{2}\lfloor {}^G\hat{\mathbf{v}}_{I_\tau}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^2 + \frac{1}{6}\lfloor {}^G\mathbf{g}\times\rfloor {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^3 \right) \\ \widetilde{\mathbf{\Phi}}_{46} &= -\sum_{\tau=0}^{k} \lfloor {}^G\hat{\mathbf{p}}_{f}\times\rfloor {}^G_{I_{i,\tau}}\hat{\mathbf{R}}\delta t\end{aligned} \tag{45}$$

and

$$\begin{aligned}\widetilde{\mathbf{\Phi}}_{56} &= -\sum_{\tau=0}^{k} \lfloor {}^G\hat{\mathbf{p}}_{u}\times\rfloor {}^G_{I_{i,\tau}}\hat{\mathbf{R}}\delta t \\ \widetilde{\mathbf{\Phi}}_{27} &= -\sum_{\tau=0}^{k} {}^G_{I_\tau}\hat{\mathbf{R}}\delta t \\ \widetilde{\mathbf{\Phi}}_{37} &= -\frac{1}{2}\sum_{\tau=0}^{k} {}^G_{I_\tau}\hat{\mathbf{R}}\delta t^2\end{aligned} \tag{46}$$

Then, we can compute each block raw $\mathcal{O}_k$ of the observability matrix as

$$\mathcal{O}_k \triangleq \begin{bmatrix} \mathcal{O}_{c_k} \\ \mathcal{O}_{r_k} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{c_k} \widetilde{\mathbf{\Phi}}_{k,0} \\ \mathbf{H}_{r_k} \widetilde{\mathbf{\Phi}}_{k,0} \end{bmatrix}$$
$$\mathcal{O}_{c_k} = \mathbf{H}_{f_k} \begin{bmatrix} \mathbf{M}_{c,1} & -\mathbf{I}_3 \delta t & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{M}_{c,2} & \mathbf{M}_{c,3} \end{bmatrix}$$
$$\mathcal{O}_{r_k} = \mathbf{H}_{pu} \begin{bmatrix} \mathbf{M}_{r,1} & \mathbf{I}_3 \delta t & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{M}_{r,2} & \mathbf{M}_{r,3} \end{bmatrix}, \tag{47}$$

where

$$\mathbf{M}_{c,1} = -\frac{1}{2} \lfloor {}^G\mathbf{g} \times \rfloor \delta t^2, \quad \mathbf{M}_{c,2} = \widetilde{\mathbf{\Phi}}_{56} - \widetilde{\mathbf{\Phi}}_{36}$$
$$\mathbf{M}_{c,3} = -\widetilde{\mathbf{\Phi}}_{37}, \quad \mathbf{M}_{r,1} = \Lambda_k + \frac{1}{2} \lfloor {}^G\mathbf{g} \times \rfloor \delta t^2,$$
$$\mathbf{M}_{r,2} = \Lambda_k \widetilde{\mathbf{\Phi}}_{16} + \widetilde{\mathbf{\Phi}}_{36} - \widetilde{\mathbf{\Phi}}_{46}, \quad \mathbf{M}_{r,3} = \widetilde{\mathbf{\Phi}}_{37} \tag{48}$$

Theorem 1 (Observability properties of CVIRO): *The right nullspace of the observability matrix $\mathcal{O}_k$, denoted as $\mathbf{N}$, spans the following unobservable subspace*

$$\mathbf{N} = \begin{bmatrix} {}^G\mathbf{g}^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}^\top \tag{49}$$

which denotes the global orientation and yaw translation.

Proof 1 *To verify that $\mathbf{N}$ is a non-space of $\mathcal{O}_k$, we have to ensure that $\mathcal{O}_k \mathbf{N} = \mathbf{0}$. In particular, we have*

$$\mathcal{O}_k \mathbf{N} = \mathbf{0} = \begin{bmatrix} \mathcal{O}_{c_k} \mathbf{N} \\ \mathcal{O}_{r_k} \mathbf{N} \end{bmatrix} \tag{50}$$

where

$$\mathcal{O}_{c_k} \mathbf{N} = \mathbf{H}_{f_k} \begin{bmatrix} \mathbf{M}_{c,1} & -\mathbf{I}_3 \delta t & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{M}_{c,2} & \mathbf{M}_{c,3} \end{bmatrix} \mathbf{N}$$
$$= \mathbf{H}_{f_k} \begin{bmatrix} -\frac{1}{2} \lfloor {}^G\mathbf{g} \times \rfloor \delta t^2 \, {}^G\mathbf{g} & (\mathbf{I}_3 - \mathbf{I}_3) \end{bmatrix} = \mathbf{0}_{2\times 4} \tag{51}$$

$$\mathcal{O}_{r_k} \mathbf{N} = \mathbf{H}_{pu} \begin{bmatrix} \mathbf{M}_{r,1} & \mathbf{I}_3 \delta t & \mathbf{I}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{M}_{r,2} & \mathbf{M}_{r,3} \end{bmatrix} \mathbf{N}$$
$$= \begin{bmatrix} \mathbf{H}_{pu} \left(\Lambda_k + \frac{1}{2} \lfloor {}^G\mathbf{g} \times \rfloor \delta t^2 \right) {}^G\mathbf{g} & \mathbf{H}_{pu} (\mathbf{I}_3 - \mathbf{I}_3) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{H}_{pu} \Lambda_k {}^G\mathbf{g} & \mathbf{0}_{1\times 3} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{\left({}^G\mathbf{p}_u - {}^G\mathbf{p}_{I_k} - {}^G_{I_k}\mathbf{R} \, {}^I\mathbf{p}_T \right)^\top}{\| {}^G\mathbf{p}_u - {}^G\mathbf{p}_{I_k} - {}^G_{I_k}\mathbf{R} \, {}^I\mathbf{p}_T \|} \lfloor {}^G\mathbf{p}_u - {}^G\mathbf{p}_{I_k} - {}^G_{I_k}\mathbf{R} \, {}^I\mathbf{p}_T \times \rfloor {}^G\mathbf{g} & \mathbf{0}_{1\times 3} \end{bmatrix}$$
$$= \mathbf{0}_{1\times 4} \tag{52}$$

As a result, $\mathbf{N}$ *is a non-space of* $\mathcal{O}_k$*. Moreover,* $\mathcal{O}_k$ *doesn't have any other non-space. Thus, we complete the proof. We can see that unlike the standard EKF-based SLAM system, where the unobservable subspace depends on the state and leads to inconsistency issues, the nullspace* $\mathbf{N}$ *remains a constant matrix, inherently preserving observability consistency.*

References

[1] R. Hartley, M. Ghaffari, R. M. Eustice, and J. W. Grizzle, "Contact-aided invariant extended kalman filtering for robot state estimation," *The International Journal of Robotics Research*, vol. 39, no. 4, pp. 402–430, 2020. [Online]. Available: https://doi.org/10.1177/0278364919894385

[2] A. Barrau and S. Bonnabel, "The invariant extended kalman filter as a stable observer," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1797–1812, 2017.

[3] M. Li and A. I. Mourikis, "High-precision, consistent ekf-based visual-inertial odometry," *The International Journal of Robotics Research*, vol. 32, no. 6, pp. 690–711, 2013.

[4] G. P. Huang, A. I. Mourikis, and S. I. Roumeliotis, "A first-estimates jacobian ekf for improving slam consistency," in *Experimental Robotics*, O. Khatib, V. Kumar, and G. J. Pappas, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 373–382.