**An Investigation of the Impact of Strategizing in Problem-Solving**

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**Abstract:** This clinical interview investigated the impact of strategizing during a problem-solving task. High-school mathematics teachers (n = 3) were given the task of solving an adaptation of a Sudoku problem, observed at their attempt, and then asked to respond to several questions specifically aimed to compare and contrast the strategies employed by the subject. Findings indicated that implementing a strategy is positively correlated to success in solving the given task, and negatively correlated to the time it takes to succeed at the task.

**Introduction**

In the past decade, mathematics teachers around the country have become more aware of the central role that problem solving must play in the lives of their students to be better prepared to operate in life outside of the classroom. This message has been supported by the National Assessment of Educational Progress (NAEP) data and the Third International Mathematics and Science Study (TIMSS) study (Sakshaug, 2002). Although NAEP data shows that students are developing mathematics skills over time, TIMSS data shows that students are not strong in problem solving and reasoning. In order to be lifelong users of mathematics, it is vital that students have more experience in these areas. We must focus our studies on what characteristics make good problem solvers successful, and consequently how to teach our students to acquire those characteristics.

**Literature Review**

Problem solving, according to Principles and Standards for School Mathematics, “means engaging in a task for which the solution method is not known in advance (NCTM, 2000).” Research started to evolve on the topic of problem solving with Polya’s description of problem solving heuristics in the 1950s. He wrote about four steps: understanding the problem, devising a plan, carrying out the plan, and looking back (Polya, 1957). In this paper, I will focus on ‘devising a plan,’ and more specifically, how having a strategy well-planned out separates successful problem solvers from those that are unsuccessful.

Although researchers have been studying problem solving now for more than 50 years, there is much to still be studied. In the 70s and 80s, researchers focused on individual problem-solving competence and performance. More specifically, they sought to know *what* distinguished good from poor problem solvers (alternatively, successful from unsuccessful or expert from novice). Schoenfeld concentrated on the areas of cognitive science and artificial intelligence, and his research includes the following five distinguishing factors.

1. Good problem solvers know more than poor problem solvers and what they know, they know differently – their knowledge is well connected and composed of rich schemata.
2. Good problem solvers tend to focus their attention on structural features of problems; poor problem solvers on surface features.
3. Good problem solvers are more aware than poor problem solvers of their strengths and weaknesses as problem solvers.
4. Good problem solvers are better than poor problem solvers at monitoring and regulating their problem-solving efforts.
5. Good problem solvers tend to be more concerned than poor problem solvers about obtaining “elegant” solutions to problems.

(Lester, 1994)

 Bransford’s chapter in How People Learn reiterates these distinguishing factors as ‘principles of expertise.’ He warns that the principles are interrelated and cannot be separated. For example, experts notice features and meaningful patterns of information that are not noticed by novices; experts are also able to retrieve important aspects of their knowledge with little attentional effort (Bransford, 2000). But, just because we understand what characteristics separate experts from novices, it is not possible to teach expert models to novices and assume they will learn how to effectively solve problems as experts do. Rather, they need to gradually develop both the content and the ability to monitor their own approach to problem solving, namely metacognition, before being able to grasp specific expert strategies.

 Teaching problem solving begins by focusing on strengthening the connection between common sense and mathematics. Teachers’ goals for children should include making mathematics support, extend, and refine common sense, NOT replace common sense or even supplement it with a set of clever but ‘magical’ tricks and methods (Goldenberg, 2003). Thinking itself must be at the core of learning. It is impossible to teach someone to think. It *is* possible to teach someone a trick or a method, but until that student learns how to make decisions about when and why it is appropriate to use particular tricks and methods, that knowledge will be useless.

 As the connection between common sense and mathematics matures, techniques to problem solving also naturally mature. Examples of problem-solving techniques are: working out particular cases, recalling similar problems, looking for patterns, or representing the problem in another way. Good problem solvers challenge their solutions, even the correct ones (Levasseur, 2003). Although techniques to reach a solution are not explicit in good problems, good problem solvers internalize the techniques they try as they maneuver through a problem and continuously adapt their habits with experience.

 Organizing your own thinking is the most essential skill to develop in order to become a more skilled problem solver. Many students believe that you either know the answer to a given problem, or you don’t, and if you don’t know, there isn’t anything you can do about it. But, recall by the definition of ‘problem,’ it is *not* a problem if you initially know the solution or of an immediate path that leads to the solution. It is crucial that problem-solvers become comfortable with the fact that problem-solving is a time-consuming activity that depends on thinking and not a memorized set of techniques.

There are three phases of thinking in the process of problem-solving: entry, attack, and review (Collier, 2000). There are not strict lines that distinguish between the three steps. The entry phase is the phase in which one attempts to understand the problem. Students seem to think this phase is relatively unimportant and only consists of reading the problem and understanding the definitions therein. Rather, this is the most important phase. The first part of this phase is the *play* phase in which you are just getting used to the problem setting and not yet consciously trying to develop a plan to solve the problem. The later part of the entry phase does involve more purposefully directed activities to develop an action plan, but this later part can not take place before playing around and understanding exactly what the problem is asking.

The second phase, the attack, is when the solution is obtained or it is realized that one must revert to the entry phase to better understand the problem before attacking. In this phase, the subject works strategically, systematically, deliberately, and purposefully to find a path to the solution of the problem. Sometimes this results in getting ‘stuck,’ but getting ‘stuck’ serves the purpose to indicate that there must be an alternate path that must be found and followed.

Lastly, the review phase is the phase of reflection - where the most real learning takes place. It is more than checking an answer to ensure its correctness; it is the final stage when the student proves that their solution works by constructing a more general argument. This may lead to a more ‘elegant’ solution or the appreciation of a flexible strategy for similar problems.

**Analysis**

 Although the participants were hand-picked to be as uniform as possible, there were still differences in their problem-solving strategies and success. Theresa, Mary, and Patricia were all high school or college mathematics teachers for at least three years and all had all majors in mathematics. Patricia actually taught for almost 20 years, and Theresa and Mary had Masters degrees in mathematics education.

 Theresa completed the Sudoku adaptation in less than a half hour, Mary in just under 2 hours, and Patricia quit the puzzle after two hours without completing it. All three had been familiar with Sudoku puzzles, but none regularly worked on them. It was most interesting to observe and ask the subjects about their strategies. None were identical, and it was clear some led to the solution more effectively than others.

Theresa, the quickest to solve the puzzle, took the longest to get started. She stared at the grids for a long moment before deciding to make a list on scratch paper of the different ways three and four digits added to the given sums in the puzzles. She explained to me that she was looking for places where, when considering both grids, the outlined sectors overlapped at only one digit. Then she would look at her lists to see if she could verify which digit was uniquely shared. She continued on faster than I could follow her every move until the puzzle was complete about 20 minutes later. She then took a few seconds to make sure that the sums indeed checked out and that both puzzles matched, handed her solution to me and smiled. I asked her if she was 100% confident this was a unique solution, and she said that it was impossible to be anything else. She went on to explain to me that she ‘proved’ each digit she entered by using her lists, the sums outlined in the puzzles, and reflecting on the possibilities given the overlaps.

Mary also successfully completed the puzzle, but in an especially longer period of time. Mary started by just plugging in numbers into one puzzle to see if they would check out in the other. She erased and erased and erased, making small changes as she concentrated on one puzzle and then the other. After she realized that this strategy was not leading her to the solution in a timely manner, she also had the idea of making lists of the ways to attain the outlined sums. She flipped her paper over and listed out the combinations, but then realized that it was distracting to keep flipping her paper from front to back – so she asked for another copy of the puzzle. While I watched her work through the puzzle, she seemed to be making similar considerations as Theresa did. She voiced that she wanted to be able to ‘prove’ each number that she decided to write down, but was not able to in each step. She told me that she was able to eliminate some of the numbers, but often had to guess between the remaining options and just go on to see if her choice worked out. When it didn’t work out, she had to decide how many steps back she made the error. She stressed the fact that she became much better at keeping notes of her guesses the longer she worked, so that she could monitor her own strategy. She also stressed that fact that she wanted to quit but that she prides herself in not being a quitter, so persevered. After she handed me her solution and I asked her if she was sure it was the ONLY solution, she said she was willing to bet that it was but was frustrated because she was not able to justify each and every move that she made.

Patricia was the only subject that did not successfully complete the puzzle challenge. She told me that she was not willing to continue on after she had been working for two hours. She explained her strategy to me as starting from the middle and working her way out to the edges of the puzzles. She started with the puzzle on the right, the one that had sums of four digits. She filled in the innermost square and then concentrated on the puzzle with three-digit sums. After this step, she remained stuck until she became frustrated and gave up. She expressed that the puzzle would be far too hard to have middle-school-aged students try – but asked me how I would solve the puzzle. I told her that I wasn’t quick to finish myself but that by observing others’ strategies it seemed that making the lists and looking for overlaps was effective. When I asked her if there could be more than one solution, she came back to say that since she couldn’t even find one, she couldn’t imagine that there would be more than one.

Let us recall Collier’s three phases of thinking in the process of problem-solving: entry, attack, and review.

**Interview Protocol**

Below is the puzzle that the three subjects worked on followed by its solution. The puzzle was presented to the subjects, they were given time to read the directions and ask clarifying questions before working on the puzzle (I observed some of the time), and then I met back with the subjects to ask them about their experiences – specifically about what kinds of strategies they implemented to attempt solving the puzzle.

The puzzle was part of a collection of ‘Back to School Puzzles’ handed out to teachers at an NCTM conference. Although they were developed for students, it was determined that middle school level students would have the mathematical ability to work on them.



To the left, find the SOLUTION.

**Participant Background and Responses to Interview Questions**

\*Names of the 3 subjects were changed in order to protect their identity.

SUBJECT 1

Background – before problem solving experience

**Name:** Theresa

**Age:** 28

**Gender:** Female

**Occupation:** Online Resources

**Background in Mathematics:** Math major in college, Masters degree in Math Education, taught high-school mathematics (Algebra and Geometry) for three years

**Feelings toward mathematics in general:** Loves it, always came naturally, as a student herself liked to learn and make connections on her own instead of following prescribed drills from teacher

Interview Responses – after problem solving experience **(COMPLETED PUZZLE IN 25 mins)**

**Do you enjoy math puzzles? Are you familiar with Sudoku?** I am familiar with Sudoku although I don’t own any books or anything. I like to try new and different math puzzles.

 **What “gauged” your desire to give up?** I didn’t give up ☺

 **Did you focus on one grid or both simultaneously?** Both. You have to.

**Can you describe your strategy?**  I first made a list of all of the 3-number and 4-number combinations using 1 through 6 to obtain the given sums in the grids. Then, I looked on the grid to see where the sums with the fewest combinations overlapped. I filled in numbers as I justified squares that must be given numbers. I continued until the puzzle was complete.

**Did you believe that there is a “unique” solution to this puzzle or could there be more?** There is a unique solution. There are no other possible ways to fill in the puzzles. Each time I filled in a square, it was the only number that worked.

**Did you do anything before you started filling in blanks? How did you know you were ready to ‘start’?** I listed out all of the sums. I studied the grids to see which sums had the least overlap, and started there.

**What was most frustrating in your experience with this puzzle?** It wasn’t frustrating; it was fun!

SUBJECT 2

Background – before problem solving experience

**Name:** Mary

**Age:** 26

**Gender:** Female

**Occupation:** Online Resources

**Background in Mathematics:** Math major in college, Masters degree in Math Education, taught college mathematics as lecturer for three years (Algebra, Calculus, Math for Elementary Education majors, Statistics) and taught one year in high school (Algebra and Trigonometry)

**Feelings toward mathematics in general:** Sort of obsessed. It’s lovely.

Interview Responses – after problem solving experience **(COMPLETED PUZZLE IN 1 hour 55 mins)**

**Do you enjoy math puzzles? Are you familiar with Sudoku?** I am familiar with Sudoku, but I don’t really do them. I think the only one I actually ever worked out was on a flight across the country. You don’t have to know math to do them really; just patience and strategy.

**What “gauged” your desire to give up?** I am not the kind of person to give up. I think I could have completed this puzzle quicker than I did, but I wanted to justify every number that I penciled in; and I found myself wanting to guess at some points. This bothered me because I felt like there must only be one way to do the puzzle, but I couldn’t prove it to myself.

 **Did you focus on one grid or both simultaneously?** I focused on both.

**Can you describe your strategy?** At first I didn’t really have one. I started just plugging numbers into both puzzles to see if I could make it work…but I couldn’t. Then, I decided to make a list of the ways to get the given sums. But, I wrote them on the back of the same paper and couldn’t see them…so I had to rewrite them before they became helpful. After I had this, I referred to it to find out where I could narrow the squares down to one or two choices. If I could prove it was a number, I put it down. If not, I made a guess, wrote lightly, and tried to continue on. I guessed a few times before finally getting it to work.

**Did you believe that there is a “unique” solution to this puzzle or could there be more?** I am definitely not going to try to find another one! I am pretty sure that there is not another one that could possibly work, but I am also frustrated that I had to make a few guesses along the way. I wonder if there is some sort of “proof” that goes along with this problem – a way to prove each and every number as you go along. Maybe I just overlooked some of these.

**Did you do anything before you started filling in blanks? How did you know you were ready to ‘start’?**  At first I didn’t, but I soon realized that I had to make a list of the sums. And do my best to minimize guessing.

**What was most frustrating in your experience with this puzzle?**  Getting almost done with the puzzle and then realizing that something doesn’t quite work in one of the puzzles. Because then I had to go back and try to figure out which guess was wrong and do it all over again. This went on and on and on and on…but I didn’t want to quit.

SUBJECT 3

Background – before problem solving experience

**Name:** Patricia

**Age:** 41

**Gender:** Female

**Occupation:** High-school math teacher, taught algebra and some geometry for almost 20 years.

**Background in Mathematics:** College degree in mathematics and several graduate-level credits in math education

**Feelings toward mathematics in general:** Essential and detailed. I love seeing students mature and learn the ‘language’ of mathematics. It’s like learning the rules of conjugating in a Romance language; once you get used to the rules, you can say/solve anything!

Interview Responses – after problem solving experience (DID NOT COMPLETE PUZZLE; quit after 2 hrs.)

**Do you enjoy math puzzles? Are you familiar with Sudoku?** I am familiar, but I don’t do puzzles often. I have enough to do balancing my planning, grading, and family life!

**What “gauged” your desire to give up?** Frustration. I feel like I know math and am a logical person, but I just couldn’t seem to get it. I got tired to trying. There is no way middle school students could get this done!

 **Did you focus on one grid or both simultaneously?** One and then the other, so I guess both…

**Can you describe your strategy?** I started in the middle with the four squares that added to 13. I tried all of the combinations in all of the possible places, but none worked out. There must be a better way to do this besides brute force.

**Did you believe that there is a “unique” solution to this puzzle or could there be more?** I have no idea. I couldn’t find any! I can’t imagine that there is more than one tho. It’s impossible to find one that works.

**Did you do anything before you started filling in blanks? How did you know you were ready to ‘start’?** I looked over the puzzle and decided to start in the middle and work my way out. I guess maybe I wasn’t ready to start because I never finished…haha.

**What was most frustrating in your experience with this puzzle?** Not finishing. Now I want you to teach me what I should have done to get there. Can I keep this? I will ask my students to see if they could have done it.

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