Experience with Modified Barrier Function Methods for Linear Programming

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Abstract We describe some numerical results obtained by logarithmic Modified Barrier Function (MBF) methods for Linear Programming (LP) problems.

MBF methods consist of sequential unconstrained minimization of MBF followed by Lagrange multipliers update, while the positive barrier parameter can be fixed or one can change it from step to step.

The numerical results obtained are consistent with the MBF theory.

All LP problems have been solved with high accuracy under a *fixed* barrier parameter. We steadily observed the "hot start" phenomenon, which means that from some point on the approximation for MBF minimizer remains in the Newton's area after each Lagrange multipliers update.

Therefore few and from some point only one Newton step is enough for the Lagrange multipliers update. Every Lagrange multipliers update shrinks, in case of nondegenerate primal-dual LP, the primal-dual gap by a factor $0 < \gamma < 1$, that depends on input data and the size of the problem and can be made as small as one wants by choosing a large enough but *fixed* barrier parameter.

The results obtained show that MBF methods are numerically stable and the total number of Newton steps requires to shrink the primal-dual gap and the constraints violation to 10^{-10} - 10^{-14} are practically independent on the size of the problem.

Keywords: Linear Programming, Primal and Dual Problems, Modified Barrier Functions, Newton Method.

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1. Introduction

Interior Point Methods (IPMs), which are based on Classical Barrier Functions (CBFs) have the best known complexity bound for LP problems (see Gonzaga 1988, Todd 1988, Goldfarb and Hiao, 1991, Renegar and Shub, 1992, Nesterov and Nemirovsky, 1994 and bibliography in it).The primal-dual predictor-corrector version of the IPM (see Mechrotra 1992) is very efficient numerically (see Lustig, Marsten and Shanno, 1992, 1994; Forrest and Tomlin, 1992).

However, along with the very important self-concordant properties (see Nesterov and Nemirovsky, 1994), which guarantee the polynomial complexity of path-following methods, the CBFs have well known drawbacks (see Fiacco and McCormick, 1968). To eliminate some of these drawback, the Modified Barrier Functions (MBFs) were introduced and their theory has been developed for both LP and NLP (see Polyak, 1992a).

Based on the MBF theory two methods - Primal and Dual MBF methods - for solving LP problems have been developed (see Polyak, 1992b). These methods consist in sequential unconstrained minimization of the Primal or Dual MBF functions followed by Lagrange multipliers update, while the barrier parameter can be fixed or one can change it from step to step.

The distinctive feature of the MBF methods is their convergence due to the Lagrange multipliers update rather than to the barrier parameter update as it takes place in the Interior Point Methods (see Powell 1995, Jensen and Polyak, 1994, Polyak and Teboulle, 1995). It contributes to both the rate of convergence and numerical stability.

The numerical realization of the MBF methods for LP leads to Primal and Dual Newton MBF.

One uses Newton method to find an approximation for the MBF minimizer followed by the Lagrange multipliers update.

This paper describes the numerical experience with the Primal and Dual Newton MBF methods for solving LP problems with a fixed barrier parameter. The numerical results obtained are very much in the spirit of the MBF theory. They show the fundamental difference between CBFs and MBFs and methods that are based on these functions.

First, MBFs are defined on an extended feasible set and along with the barrier parameter they have an additional tool -the vector of Lagrange multipliers to control the numerical process.

Therefore, unlike the Classical Barrier IPMs, the Modified Barrier methods converge for *any* fixed positive barrier parameter. Moreover the objective function tends to optimality and any constraint violations tend to zero at R-linear rate whether the primal and Dual LP degenerate or not (see Powell 1995).

Second, in MBF methods the constraints violation does not compromise the convergence. It shrinks to zero not due to infinite barrier parameter increase, as it occurs in the Shifted Barrier methods, but due to the Lagrange multipliers update. It contributes to the numerical stability and allows to obtain results with high level of accuracy under the fixed barrier parameter.

Third, in all problems, which have been solved, we observed the "hot start" phenomenon which has been theoretically predicted only for non-degenerate LP problems (see Polyak , 1992a , 1992b).

Fourth, there is a substantial difference between the Primal and Dual MBF methods in terms of numerical performance. The dual MBF performs much better and we will try to explain this observation.

Fifth, the numerical results obtained show that the dual MBF method is numerically stable and the total number of Newton steps required to reduce the primal-dual gap and constraints violation to the order of 10^{-10} - 10^{-14} is practically independent of the size of the problem.

In the following section we introduce the problems and make the basic assumptions. In Sections 3 and 4 we describe the Primal MBF method and its numerical realization - the Primal Newton MBF method. In sections 5 and 6 we describe the Dual MBF and its numerical realization - the Dual Newton MBF method.

In Section 7 we discuss the numerical aspects of the Primal and Dual Newton MBF methods. We present our numerical results in Section 8 and compare those results to other IPM results in Section 9. We conclude the paper with some remarks.

2. Problem Formulation

Let $A = (B, N)$ be an $m \times n$ matrix ($n > m$), where B is an $m \times m$ matrix and N is an $m \times (n\text{-}m)$ matrix. Also let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We consider the following dual pair of LP problems:

$$x^* = \operatorname{argmin} \{ (a,x) \mid Ax = b,\ x \geq 0^n \} \tag{2.1}$$

and

$$v^* = \operatorname{argmax} \{ (b,v) \mid vA \leq a \} \tag{2.2}$$

We also define the primal feasible set $P = \{ x : Ax = b,\ x \geq 0^n \}$ and the primal optimal set $P^* = \{ x \in P : (a,x) = (a,x^*) \}$, as well as the dual feasible set $D = \{ v : vA \leq a \}$ and the dual optimal set $D^* = \{ v \in D : (b,v) = (b,v^*) \}$. The vector $u = a - vA$ represents the residuals of the dual problem. So if $u \geq 0^n$ then $v \in D$.

The primal and dual LP problems are non-degenerate if rank (A) = rank $(B) = m$ and the following complementary slackness condition holds in the strict form, i.e.

$$x_j^* > 0 \ \Leftarrow \ u_j^* = (a - v^* A)_j = 0 \quad \forall\, j = 1, \ldots, m \qquad \text{a)}$$

and (2.3)

$$x_j^* = 0 \ \Rightarrow \ u_j^* = (a - v^* A)_j > 0 \quad \forall\, j = m+1, \ldots, n \qquad \text{b)}$$

Under the assumptions (2.3) both the primal (2.1) and the dual (2.2) LP problems have unique solutions. By splitting the vectors $a = (a_B ; a_N)$ and $u = (u_B ; u_N)$ on basic and nonbasic parts we can rewrite the conditions (2.3) as follows:

$$x_B^* = B^{-1} b > 0^m, \quad x_N^* = (x_{m+1}^*, \ldots, x_n^*) = 0^n \qquad \text{a)}$$

and (2.4)

$$u_B^* = a_B - v^* B = 0^m, \quad u_N^* = a_N - v^* B_N > 0^n \qquad \text{b)}$$

3. The Primal MBF Method

The primal LP problem (2.1) is equivalent to

$$x^* = \operatorname{argmin} \{ (a,x) \mid Ax = b,\ k^{-1} \ln(kx_i + 1) \geq 0,\ i = 1, \ldots, n \} \tag{3.1}$$

for any $k > 0$.

Let $Q = \{ x : Ax = b \}$, $X_k = \{ x = (x_1, \ldots, x_n) : x_i \geq -k^{-1},\ i = 1, \ldots, n \}$, $\mathbb{R}^n_+ = \{ x : x_i \geq 0,\ i = 1, \ldots, n \}$, $\mathbb{R}^n_{++} = \{ x : x_i > 0,\ i = 1, \ldots, n \}$, and $\ln t = -\infty$ for $t \leq 0$.

Let us remove the system $Ax = b$ from the list of constraints, then the primal MBF $P(x,u,k)$: $Q \times \mathbb{R}^n_+ \times R^1_{++} \to \mathbb{R}^1$ is defined by the following formula

$$P(x,u,k) = (a,x) - k^{-1}\sum_{i=1}^{n} u_i \ln(kx_i + 1) \qquad (3.2)$$

Let $U = \mathrm{diag}\,(u_i)_{i=1}^n$, $U_N = \mathrm{diag}\,(u_i)_{m+1}^n$, $\Delta(x,k) = \mathrm{diag}\,(kx_i + 1)_{i=1}^n$

Before we describe the Primal MBF method we would like to recall the basic MBF properties at the primal-dual solution (see Polyak, 1992b).

Due to (2.3) for any $k > 0$ we have

P1. $P(x^*,u^*,k) = (a,x^*) = (b,u^*)$

P2. $\nabla_x P(x^*,u^*,k) = a - u^*$

P3. $\nabla_{xx} P(x^*,u^*,k) = k\,U_N^*$

In view of P2 and (2.4) we have

$$\nabla_x P(x^*,u^*,k) = \mathrm{v}^* A$$

for any $k > 0$.

Due to P3 and (2.4b) the restriction of $P(x,u^*,k)$ to the linear manifold $Q = \{x : Ax - b = 0^m\}$ is a strongly convex function, therefore

$$x^* = \mathrm{argmin}\,\{P(x,u^*,k) / x \in Q\}$$

is unique for any $k > 0$.

Moreover due to $\nabla_{xx} P(x,u,k) = k\,U\,\mathrm{diag}\,((kx_i + 1)^{-2})_{i=1}^n$ the function $P(x,u,k)$ is strongly convex in $x \in X_K$ for any $u \in \mathbb{R}^n_{++}$ and $k > 0$..

Therefore for any $u \in \mathbb{R}^m_{++}$ and $k > 0$ there exists a unique minimizer

$$\hat{x} = \hat{x}(u,k) = \mathrm{argmin}\,\{P(x,u,k) / x \in Q\} \qquad (3.3)$$

It turns out that the minimizer $x(u, k)$ can be used to improve the current approximation of the dual vector $u \in \mathbb{R}^m_{++}$, while the barrier parameter can be fixed.

We will describe the Primal MBF method with a fixed barrier parameter.

The primal MBF method generates a sequence of triples (x^s, u^s, v^s) where x^s is an approximation for x^*, u^s is an approximation for u^*, and v^s is an approximation for v^*.

At the initial phase we set $u^0 = e = (1,1, \ldots ,1) \in \mathbb{R}^n$, choose a sufficiently large fixed $k > 0$ and find $x^0 \in Q \cap X_k$.

Let us assume that $x^s \in Q \cap X_k$, $u^s \in \mathbb{R}^n_{++}$ and $\mathrm{v}^s \in \mathbb{R}^m$ have been found already. The next approximation $(x^{s+1}, u^{s+1}, \mathrm{v}^{s+1})$ we find by solving the problem

$$x^{s+1} = \operatorname{argmin} \{ P(x, u^s, k) / x \in Q \} \tag{3.4}$$

and updating the Lagrange multipliers

$$u^{s+1} = \Delta^{-1}(x^{s+1}, k) u^s \tag{3.5}$$

As a byproduct of solving the problem (3.4) we obtain the approximation v^{s+1} for the dual solution v*. The vector v^{s+1} is a Lagrange multipliers vector, which corresponds to the system $Ax = b$ in problem (3.4). In other words, we have

$$\nabla_x P(x^{s+1}, u^s, k) = a - \Delta^{-1}(x^{s+1}, k) u^s = a - u^{s+1} = v^{s+1} A \tag{3.6}$$

The components of the vector of Lagrange multipliers u^s corresponding to the constraints $k^{-1} \ln(kx_i + 1) \geq 0$, $i = 1, \ldots, n$ are always positive. They define the residuals for the dual problem. Therefore v^s is always a dual feasible vector, while the primal approximation x^s might be infeasible up to the end of the process.

The fundamental difference between the MBF method (3.4) - (3.5) and Interior Point methods which are based on CBFs, is that the convergence of the sequence $\{x^s, u^s, v^s\}$ is due to the Lagrange multipliers (dual residuals) update, while the barrier parameter is fixed. In Interior Point methods the convergence is due to the barrier parameter update no matter whether the IPM is a "pure" primal or a primal-dual.

If P^* and D^* are bounded it follows from [PT95] that (u^s, v^s) converges to (u^*, v^*), while the primal sequence converges to x^* in ergodic sense and both primal and dual sequences converge to the optimal solutions in value.

The strongest so far result about MBF method convergence for LP without nondegeneracy assumption was obtained by M. Powell (see Powell 1995) under the assumption that D is bounded. We will discuss it later, bacause his result relates to the Dual MBF method.

If the dual LP pair (1) and (2) is non-degenerate, i.e. x^* and v^* are unique, then there exists a condition number $c = \operatorname{cond} P(A, a, b) > 0$ for the primal problems such that

$$\max \{ \| x^{s+1} - x^* \|, \| u^{s+1} - u^* \|, \| v^{s+1} - v^* \| \} \leq c k^{-1} \| u^s - u^* \| \tag{3.7}$$

where $\| x \| = \| x \|_\infty = \max_{1 \leq i \leq n} | x_i |$.

The condition number $c > 0$ depends on the input data and the size of the problem, but it is independent on $k \geq k_0$ if $k_0 > 0$ is sufficiently large (see [Polyak 1992b]).

Therefore for any $0 < \gamma < 1$ one can find $k_\gamma > c$ such that the estimation

$$\max \{ \| x^s - x^* \|, \| u^s - u^* \|, \| v^s - v^* \| \} \leq \gamma^s \tag{3.8}$$

holds for any $k \geq k_\gamma$.

Due to (3.7) we can improve the rate of convergence by increasing the barrier parameter, but it might create numerical problems. Therefore we conducted our numerical experiments under the fixed barrier parameter.

It turns out that when $k > 0$ is fixed but reasonably large we can guarantee not only the estimation (3.8) with a reasonable small $0 < \gamma < 1$, but what is even more important after each Lagrange multipliers update the approximation for the current minimizer remains in the Newton area for the problem type (3.4) when u^s is replaced by u^{s+1}.

Therefore only few, and from some point only one Newton step is required for the Lagrange multipliers update.

Strictly speaking every step of the Primal MBF requires an infinite procedure to solve the problem (3.4). Therefore we have to change it to a finite one without compromising the convergence.

We describe such procedure in the next section.

4. The Primal Newton MBF Method

The numerical realization of the primal MBF method (3.4), (3.5) leads to finding an approximation $\bar{x}^{s+1}$ for x^{s+1} and updating u^s by using $\bar{x}^{s+1}$ instead of x^{s+1} in (3.5). To find the approximation $\bar{x}^{s+1}$ we use Newton method. Newton method for solving (3.4) has been described in (Polyak 1992b). We would like briefly remind that to find the Newton direction ξ_x we have to determine v (x, u, k) by solving the following normal system of equations:

$$(A \Delta^2(x,k) U^{-1} A^T) v = A \Delta(x,k) (U^{-1} \Delta(x,k) a - e) \tag{4.1}$$

Then we calculate the Newton direction $\xi_x = \xi(x,u,k)$:

$$\xi_x = (kU)^{-1} \Delta^2(x,k) (A^T v(x,u,k) - a + \Delta^{-1}(x,k) u) \tag{4.2}$$

Once ξ_x is determined, we update x by

$$x := x + t\,\xi_x \tag{4.3}$$

The stepsize t one can find by solving

$$t = \operatorname{argmin}\{\, P(x + \tau\,\xi_x, u, k) \mid x + \tau\,\xi_x \in X_k \,\} \tag{4.4}$$

or by using Goldstein-Armijo criterion

$$P(x + \tau\,\xi_x, u, k) - P(x, u, k) \le \frac{t}{3}\,(\nabla_x P(x, u, k), \xi_x) \tag{4.5}$$

Starting with $t = 1$, we check (4.5). If (4.5) is satisfied and $x + t\,\xi_x \in X_k$, then we update x as in (4.3). If (4.5) is not satisfied or $x + t\,\xi_x \notin X_k$, then we set $t := t/2$ and check (4.5) again. We repeat this process until (4.5) is satisfied and $x + t\,\xi_x \in X_k$.

We continue to take Newton steps until we get a solution $\bar{x}$ which satisfies:

$$\| \nabla_{xx} P(\bar{x}, u, k)\,\xi_x \| = \| a - \Delta^{-1}(\bar{x}, k)\,u - \mathrm{v}(\bar{x}, u, k)\,A \| \le \varepsilon \tag{4.6}$$

where $\varepsilon > 0$ is small enough. Then we update the Lagrange multipliers

$$\bar{u} = \Delta^{-1}(\bar{x}, k)\,u \tag{4.7}$$

while the barrier parameter $k > 0$ is fixed.

Now we are ready to describe the Primal Newton MBF method. We start with $x^0 \in Q \cap X_k$, $u^0 = e = (1, \ldots, 1) \in \mathbb{R}^n$ and a large enough $k > 0$. We also consider a monotonically decreasing sequence $\{\, \varepsilon_s > 0 \,\}_{s=1}^{\infty}$ such that $\lim_{s \to \infty} \varepsilon_s = 0$.

Let's assume that approximation $(\bar{x}^s, \bar{u}^s, \bar{\mathrm{v}}^s)$ has been found already. To find the next approximation we use the following process:

1. Start with $\bar{x}^s$ and use Newton's method (4.1) - (4.3) to minimize $P(x, \bar{u}^s, k)$ until $\bar{x}^{s+1}$ satisfies:

$$\| a - \Delta^{-1}(\bar{x}^{s+1}, k)\,\bar{u}^s - \mathrm{v}(\bar{x}^{s+1}, \bar{u}^s, k)\,A \| \le \varepsilon_{s+1} \tag{4.8}$$

2. Update the Lagrange multipliers (dual residuals) using the following formula:

$$\bar{u}^{s+1} = \Delta^{-1}(\bar{x}^{s+1}, k)\,\bar{u}^s \tag{4.9}$$

By finding the triple $(\bar{x}^{s+1}, \bar{u}^{s+1}, \bar{v}^{s+1})$ such that $\| a - \bar{u}^{s+1} - \bar{v}^{s+1} A \| \le \varepsilon_{s+1}$ we complete the Primal Newton MBF step.

To describe the stopping criterion we introduce the "merit" function

$$v(w) = v(x, v) = \max \{ \max_{1 \le i \le n} \{ -x_i \}, \max_{1 \le i \le n} \{ (vA - a)_i \}, \sum_{i=1}^{n} | (vA - a)_i | |x_i| \} \qquad (4.10)$$

It is clear that $v(w) \ge 0$ also $v(w) = 0 \iff w = w^*$. In fact, $v(w) = 0 \Rightarrow -x_i \le 0$ or $x_i \ge 0$, $i = 1, \dots, n$, $vA - a \le 0$, i.e. $a - vA \ge 0$ and $(a - vA)_i x_i = 0$, $i = 1, \dots, n$, i.e. x and v are primal and dual feasible and the complementary conditions are satisfied, so $w = w^*$. The second part of the statement, i.e. $w = w^* \Rightarrow v(w) = 0$ follows from the definition (4.10) and LP duality.

If $v(\bar{w}^s)$ is small enough, then $\bar{x}^s$ and $\bar{v}^s$ are "almost" primal-dual feasible and the complementary conditions are "almost" satisfied.

We take $\varepsilon > 0$ small enough ($\varepsilon > 0$ is defined by the desired accuracy) and terminate the process when the couple $(\bar{x}^s, \bar{u}^s)$ satisfy $v(\bar{x}^s, \bar{u}^s) \le \varepsilon$.

In our numerical experience ε is between 10^{-10} to 10^{-14}, therefore for the numerical results obtained the primal-dual gap as well as the constraints violation are of the order of $10^{-10} \div 10^{-14}$ for all problems (see Table 1 and Table 2), while the barrier parameter is fixed between 10^4 to 10^7.

Numerical results obtained with the Primal Newton MBF were not very encouraging and we will explain the reasons later. Therefore in our numerical experiments we focused on the Dual Newton MBF. We will describe first the Dual MBF in the next section.

5. The Dual MBF Method

This method is based on the dual MBF, which is a Classical Lagrangian for a problem that is equivalent to the dual LP (2.2). Let's consider it with more details. First, we transform the inequality constraints of the dual problem (2.2) to obtain the equivalent problem.

The dual feasible set is given by:

$$D = \{ v : a - A^T v \ge 0 \} = \{ v : u_i(v) = (a - A^T v)_i \ge 0, i = 1, \dots, n \} \qquad (5.1)$$
$$= \{ k^{-1} \ln(k u_i(v) + 1) \ge 0, i = 1, \dots, n \}$$

Thus, the problem

$$\mathrm{v}^* = \operatorname{argmax}\{(b,\mathrm{v}) \mid k^{-1}\ln(ku_i(\mathrm{v}) + 1) \geq 0, i = 1, \ldots, n\} \tag{5.2}$$

is equivalent to (2.2) for any $k > 0$. Let $x = (x_1, \ldots, x_n)$ be the vector of Lagrange multipliers which corresponds to the constraints of the dual problem (5.2). We consider an extended dual feasible set $D_k = \{\mathrm{v} : u_i(\mathrm{v}) \geq -k^{-1}, i = 1, \ldots, n\}$. Again, assuming that $\ln t = -\infty$ for $t \leq 0$, we define the Dual MBF as a Classical Lagrange for problem (5.2) which is equivalent to problem (2.2)

$$D(\mathrm{v}, x, k) = (b,\mathrm{v}) + k^{-1}\sum_{i=1}^{n} x_i \ln(ku_i(\mathrm{v}) + 1) \tag{5.3}$$

Let $X = \operatorname{diag}(x_i)_{i=1}^n$, $\Delta(\mathrm{v},k) = \operatorname{diag}(ku_i(\mathrm{v}) + 1)_{i=1}^n$ are diagonal matrices with entries x_i and $(ku_i(\mathrm{v}) + 1), i = 1, \ldots, n$.

Before we describe the Dual MBF method, let's consider the local Dual MBF properties at the dual-primal solution.

First, note that the Dual MBF $D(\mathrm{v}, x, k)$ as well as its gradient

$$\nabla_{\mathrm{v}} D(\mathrm{v}, x, k) = b - A\Delta^{-1}(\mathrm{v}, k)x$$

and Hessian

$$\nabla^2_{\mathrm{vv}} D(\mathrm{v}, x, k) = -kA\Delta^{-2}(\mathrm{v}, k)XA^T$$

exist at the dual-primal solution v^*, x^* for any $k > 0$.

Moreover, the following Dual MBF properties are taking place.

D1. $D(\mathrm{v}^*, x^*, k) = (b, \mathrm{v}^*) = (a, x^*)$

D2. $\nabla_{\mathrm{v}} D(\mathrm{v}^*, x^*, k) = b - Ax^* = 0$

D3. $\nabla_{\mathrm{vv}} D(\mathrm{v}^*, x^*, k) = -kAX^*A^T = -kBX_B^*B^T$

Due to (2.4b) the Dual MBF is strongly concave at the dual-primal solution.

Now we describe the Dual MBF method.

The Dual MBF method generates a sequence of couples (v^s, x^s), where v^s is an approximation for the dual solution v^* and x^s is an approximation for x^*.

We start with $x^0 = e = (1, \ldots, 1) \in \mathbb{R}^n$, pick a fixed and large enough $k > 0$ and $\mathrm{v}^0 \in D_k$. The Dual MBF method consists of finding the dual-primal sequence $\{\mathrm{v}^s, x^s\}_{s=0}^{\infty}$ using the following formulas:

$$v^{s+1} = \operatorname{argmax} \{ D(v, x^s, k) \mid v \in \mathbb{R}^n \} \tag{5.4}$$

$$x^{s+1} = \Delta^{-1}(v^{s+1}, k) x^s \tag{5.5}$$

Vector v^{s+1} is an unconstrained maximizer of $D(v, x^s, k)$.

$$\nabla_v D(v^{s+1}, x^s, k) = b - A\Delta^{-1}(x^{s+1}, k)x^s = b - Ax^{s+1} = 0$$

Vector v^{s+1} exists for any x^s and any $k > 0$, because $x^s \in \mathbb{R}^n_{++}$ and $D(v, x^s, k)$ is strongly concave. Moreover if $x^s \in \mathbb{R}^n_{++}$ then due to (5.5) vector $x^{s+1} \in \mathbb{R}^n_{++}$ and $Ax^s = b$, therefore x^s is always primal feasible, while the dual approximation v^s might be infeasible up to the end of the process.

The Dual MBF method converges to the dual solution for any $k > 0$ if D is bounded (see Powell 95). Moreover, M. Powell proved that the sequence $\{v^s\}_{s=0}^{\infty}$ converges to the Chebushev center on the dual optimal face D^* whether both the primal and dual LP degenerate or not, and the objective function converges to the optimal value and the constraints violation tend to zero with R-linear rate.

Note that convergence $\{x^s\}$ to $x^* \in D^*$ under the only assumption that D^* is bounded is a consequence of results obtained in (see Polyak and Teboulle, 1995).

If the dual pair LP problems (2.1)-(2.2) is non-degenerate i.e. both the primal and the dual LP problem have unique solutions, then there exists a condition number $c = \operatorname{cond} D(A, a, b)$ for the Dual LP problem such that the sequence $\{v^s, x^s\}_{s=0}^{\infty}$ converges linearly to the dual and primal solutions and the following estimation is satisfied:

$$\max \{ \| v^{s+1} - v^* \|, \| x^{s+1} - x^* \| \} \le c k^{-1} \| x^s - x^* \| \tag{5.6}$$

The condition number $c = \operatorname{cond} D(A, a, b)$ depends on the input data and the size of the problem but it is independent of $k \ge k_0$, as long as $k_0 > 0$ is sufficiently large. Thus, for any $0 < \gamma < 1$ one can find $k_\gamma > c$ such that

$$\max \{ \| v^s - v^* \|, \| x^s - x^* \| \} \le \gamma^s$$

for any $k \ge k_\gamma$ (see Polyak 1992b).

The Dual MBF method requires solving an unconstrained optimization problem (5.4) at each step. To make the Dual MBF practical, we have to change the infinite procedure of finding the dual maximizer v^s for a finite one without compromizing the convergence. In the next section we describe

such method.

6. The Dual Newton MBF Method

We will use Newton's method to find an approximation $\bar{v}^{s+1}$ for v^{s+1} and update x^s by (5.5) using $\bar{v}^{s+1}$ instead of v^{s+1}. To find the Newton direction ξ_v we have to solve the following normal system of equations

$$k A \Delta^{-1}(v,k) X A^T \xi_v = (b - A \Delta^{-1}(v,k) x) \tag{6.1}$$

Once ξ_v is found we update v by:

$$v := v + t \xi_v \tag{6.2}$$

The step size t one can find by maximizing $D(v + t\xi_v, x, k)$ on $t > 0 : v + t\xi_v \in V_k$ or by using Goldstein-Armijo criterion:

$$D(v + t\xi_v, x, k) - D(v, x, k) \geq \frac{t}{3} (\nabla_v D(v, x, k), \xi_v) \tag{6.3}$$

Starting with $t = 1$, we check (6.3). If (6.3) is satisfied and $v + t\xi_v \in V_k$, then we update v as in (6.2). If (6.3) is not satisfied or $v + t\xi_v \notin V_k$ then we set $t := t/2$ and check (6.3) again. We repeat this process until (6.3) is satisfied and $v + t\xi_v \in V_k$. We continue to perform Newton's steps until we get a solution v which satisfies:

$$\| \nabla_v D(\bar{v}, x, k) \| = \| b + A \Delta^{-1}(\bar{v}, k) x \| \leq \varepsilon \tag{6.4}$$

where $\varepsilon > 0$ is small enough. Then we update the Lagrange multipliers

$$\bar{x} = \Delta^{-1}(\bar{v}, k) x \tag{6.5}$$

Now we are ready to describe the Dual Newton MBF method (see the flowchart, Fig. 2). Let's consider a monotonically decreasing sequence $\{ \varepsilon_s > 0 \}_{s=1}^{\infty}$ such that $\lim_{s \to \infty} \varepsilon_s = 0$. We start with $v^0 \in V_k$, $x^0 = e = (1, \ldots, 1) \in \mathbb{R}^n$ and a large enough $k > 0$.

Let's assume that the approximation $(\bar{v}^s, \bar{x}^s)$ has been found already. To find the next approximation $(\bar{v}^{s+1}, \bar{x}^{s+1})$ we use the following process:

1. Start with $\bar{v}^s$ and use Newton's method (6.1) - (6.3) to maximize $D(v, \bar{x}^s, k)$ until $\bar{v}^{s+1}$

satisfies:

$$\| \nabla_v D(\bar{v}^{s-1}, \bar{x}^s, k) \| = \| b - A\Delta^{-1}(\bar{v}^{s-1}, k)\bar{x}^s \| \le \varepsilon_s \tag{6.6}$$

2. Update the Lagrange multipliers (primal variables) by:

$$\bar{x}^{s-1} = \Delta^{-1}(\bar{v}^{s-1}, k)\bar{x}^s \tag{6.7}$$

We are using the same stopping criterion as in the Primal Newton MBF but with a different "merit" function. Let $\mu(w) = \mu(v,x) = \max\{ \| b - Ax \|, \max_{1\le i\le n}\{-u_i(v)\}, \sum_{i=1}^{n} | u_i(v) | x_i \}$. It is easy to see that $\mu(v, x) \ge 0$. Also $\mu(v, x) = 0 \Longleftrightarrow v = v^*, x = x^*$ because if $\mu(v, x) = 0$, then x and v are primal and dual feasible and complementary slackness conditions are satisfied. It is also clear that $\mu(v^*, x^*) = 0$ we terminate the process at the point $(\bar{v}^s, \bar{x}^s)$: $\mu(\bar{v}^s, \bar{x}^s) \le \varepsilon$, where a small enough $\varepsilon > 0$ is defined by the desired accuracy. In our numerical experiments $\varepsilon = 10^{-10} \div 10^{-14}$.

The main purpose of our numerical experiments was to observe in practice the "hot start" phenomenon, which has been predicted for primal and dual nondegenerate LP in (see Polyak 1992b). We will briefly remind the notion of "hot start", using the dual LP (2.2) and the Dual Newton MBF method. For exact definition of the "hot start" and related to this notion complexity issues (see Melman and Polyak, 1995).

We remind that one step of Dual Newton MBF requires solving approximately the system

$$\nabla_v D(v, \bar{x}^s, k) = b - A\Delta^{-1}(v, k)\bar{x}^s = 0 \tag{6.8}$$

in v and updating x^s by (6.7) using the approximate solution $\bar{v}^{s-1}$.

We recall that vector v is "well" defined for the system (6.8) if starting from v the Newton method can be realized and the correspondent sequence converges to the solution quadratically (see Smale 1986).

Due to the convergence $\bar{x}^s$ to x^* under the fixed barrier parameter, the condition number of the Dual MBF Hessian $\nabla^2_{vv} D(v, \bar{x}^s, k)$ is stable when x approaches x^* and so is the Newton area, where v is "well" defined for the system (6.8).

We call an approximation $\bar{v}^s$ a "hot start" if $\bar{v}^s$ is "well" defined for the system

$$\nabla_v D(v, x^s, k) = 0$$

implies $\bar{v}^{s+1}$ is well defined for the system

$$\nabla_v D(v, x^{s+1}, k) = 0$$

From the "hot start" on to find a Dual MBF maximizer (or a minimizer in v of $-D(v, x^s, k)$) with accuracy $\varepsilon > 0$ one has to perform $0(\ln \ln \varepsilon^{-1})$ Newton steps. So only $0(\ln \ln \varepsilon^{-1})$ Newton steps are required for the Lagrange multipliers update, which in case of nondegenerate dual pair LP shrinks the distance to the primal dual solution by a factor $0 < \gamma = ck^{-1} < 1$ and the new approximation $\bar{v}^s$ will stay again in the Newton area for the following system $\nabla_v D(v, x^{s+1}, k) = 0$ (see Fig.1).

Moreover, after each Lagrange multipliers update we have a reduction in the number of Newton steps up to the point when only one Newton step is enough to find an approximation for the MBF maximizer.

In case of Dual Newton MBF method the "hot start" depends on the condition number $C = \text{cond } D(A, a, b)$ of the dual problem (see Polyak 1992b, Melman and Polyak, 1995).

The "hot start" phenomenon has been steadily observed for all LP problems. It is worse to mention that this phenomenon has been observed when MBF type methods were applied for solving nonlinear optimization problems (see Ben-Tal, Yuzelovich and Zubylevsky, 1992, Breinfeld and Shanno, 1994).

Before commenting on the numerical results, we would like to discuss some numerical difficulties which we faced.

7. Numerical Aspects of the Newton MBF Methods

The crucial part of the primal Newton MBF is the Newton step. To find the Newton direction one has to solve system (4.1). By taking $C = A\Delta(x,k)U^{-1/2}$, one can rewrite the left hand side of (4.1) as $CC^T v$. Thus if C is a full rank matrix (i.e. rank $C = m$), then there exists a very well developed technique for solving such systems (see Forrest and Tomlin, 1992, Lustig, Marsten and Shanno, 1992, 1994). In our case, in spite of the fact that matrix A is a full rank matrix (i.e. rank $A = m$) the matrix CC^T might be nearly rank deficient even in the case when the dual pair is nondegenerate. In other words, even in the case where the set of columns of matrix A defining the primal optimal face has a full rank, system (4.1) might be very unstable. In fact, when the process

approaches the solution, the diagonal elements $\Delta_i^2(x,u)u_i^{-1}$ that correspond to $x_i^* > 0$ become, unbounded because $u_i \to u_i^* = (p - v^* A)_i = 0$.

Even for a moderate $k > 0$, $x_i > 0$ and $u_i^{-1} > 0$ the diagonal elements of $\Delta^2(x,k)U^{-1}$ may become very large and the system becomes numerically unstable. For example with $k = 10^4$, $x_i = 10^3$, $u_i = 10^{-6}$ we have $\Delta_i^2(x,k)u_i^{-1} = 10^{20}$. At the same time other diagonal elements may be substantially smaller. Therefore, after scaling the matrix A by $\Delta(x,k)U^{-1/2}$ the normal system $CC^T v = C(U^{-1/2}\Delta(x,k)a - U^{1/2}e)$ may be practically rank deficient.

When the problem is primal degenerate, this effect exagerates because the number of positive x_i^* and the number of zero u^* is $r < m$. Moreover in such case it is sometimes difficult to distinguish between the diagonal elements $\Delta_i^2(x,k)u_i^{-1}$, which correspond to $x_i > 0$ and those for which $u_i^* = x_i^* = 0$ because for the last one $\Delta_i^2(x,k)u_i^{-1} \to u_i^{-1} \to \infty$.

Due to these difficulties, we faced some severe numerical problems when the process approaches the solution. In particular, we have difficulties in finding the Newton direction which is in the null space $\{\zeta : A\zeta = 0\}$.

The numerical problems can start at the point when the relative duality gap $\nabla(x,v) = \frac{|(a,x) - (b,v)|}{\max\{1, |(a,x)|\}}$ is quite large, i.e. $\nabla(x, v) = 10^{-3}$-10^{-5}. For some problems these numerical difficulties made it impossible to decrease the gap. It is also very hard for some problems to drive down the infeasibilities because of the fact that normally the infeasibilities decrease substantially only when the gap is small enough.

Our implementation of the dual MBF approach is much more successful and allows us to obtain much better numerical results than those of the primal MBF approach. The main difference between the primal and dual approaches lies in the normal system of equations for finding the Newton direction. The normal system of equation (6.1) remains more stable than the system (4.1) when the process approaches the solution.

Let us consider system (6.1) at the primal-dual solution. If the dual pair of LP is nondegenerate, then taking into account

$$\Delta^{-2}(v^*,k) = \begin{array}{c} \\ m \\ n-m \end{array} \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \begin{bmatrix} I^{m,m} & 0^{m,n-m} \\ 0^{n-m,m} & \Delta_N^{-2}(v^*,k) \end{bmatrix} \end{array}$$

and

$$X^* = \begin{array}{c} \\ m \\ n-m \end{array} \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \begin{bmatrix} X_B^* & 0^{m,n-m} \\ 0^{n-m,m} & 0^{n-m,n-m} \end{bmatrix} \end{array}$$

we obtain

$$A\Delta^{-2}(v^*,k)X^*A^T = BX_B^*B^T.$$

In the nondegenerate case B and X_B^* are full rank matrices. So $BX_B^*B^T$ is a full rank matrix too and system (6.1) is stable in the neighborhood of (v^*, x^*).

System (6.1) can be made stable even in case when the vector $x^* = (x_1^*, \ldots, x_n^*)$ has only $r < m$ non-zero components, i.e. the dual solution v^* is not unique. In such case the matrix $A\Delta^{-2}(v,k)XA^T$ becomes practically rank deficient when the process approaches the solution.

To find one of the optimal dual solutions we can fix $(m-r)$ dual variables and simultaneously take special care of $(m-r)$ redundant primal equality constraints. Then the dual pair of LP problems becomes nondegenerate and at the same time the number of primal linear independent constraints and basic dual variables is $r < m$. This motivates the following procedure. When both the diagonal elements of $\Delta_i^{-2}(v,k)x_i$ and the corresponding right hand side $(b - A\Delta^{-1}(v,k)x)_i$ of system (6.1) are close to zero, we consider the equations $(Ax)_i = q_i$ as redundant for the current iteration.

Thus by setting for example $\zeta_{v_i} = 0$ for each redundant row i, we obtain from (6.1) a nondegenerate normal system of equations from which we determine the rest of the components for the Newton direction ζ_v.

In the next sections we discuss some numerical results.

8. Numerical Results

In this section we analyze the numerical results obtained by using the Dual Newton MBF method. In our implementation we used the computational kernels developed for OSL (see Forrest, Tomlin, 1992).

We start with a good approximation for the dual solution and an approximation for the primal

which is normally far from the primal solution. Usually we took the unit vector for the initial Lagrange multipliers (primal approximation).

Then we find an approximation for the Dual Modified Barrier Function maximizer. Such an approximation is in 0 (k^{-1}) neighborhood of the dual solution. Using this approximation, we update the Lagrange multipliers.

The approximation for the dual MBF maximizer and the updated Lagrange multipliers (the approximation for the primal vector) serve as initial approximation for the MBF procedure. In other words, we consider such a pair as the initial "hot start".

Tables 1-2 show the results between the initial "hot start" and the solution. Tables 3-5 show the dynamics of the primal and dual objective functions, the gap and the infeasibility from the initial "hot start" to the point where the process has been terminated.

In Tables 1 and 2 we present numerical results of the Dual Newton MBF method for over 80 LP problems from NETLIB and other applications. The problem ranges between $m \times n = 10 \times 11$ and $m \times n = 22513 \times 99185$.

The dual-primal approximation (x, v) we characterize by the relative duality gap $\nabla(x,v)$ and the dual infeasibility $I(v) = \max \{ \vdash u_i(v) \mid i = 1, \ldots, n\}$. It is clear that $\max \{\nabla(x^*, v^*), I(v^*)\} = \mu(w^*, k) = 0$.

Note that in the Primal or Dual MBF methods the current primal or dual vector might be infeasible up to the end of the process, therefore the well-known inequality $(a, x) \geq (b, v)$, which is true for any primal-dual feasible couple is not necessarily true for a current primal-dual vector, produced by MBF methods.

We terminate the solution process when both $\nabla(x, v)$ and $I(v)$ are sufficiently small, in particular, between 10^{-10} and 10^{-14}.

In Table 1 we list the size of each problem, the fixed penalty parameter k, the number of Lagrange multiplier updates, the final objective function value, the relative gap and the infeasibility upon termination.

In Table 2 we concentrate on the steps between the first Lagrange multipliers update and the final solution.

The gap after the first Lagrange multipliers update is shown in the second column. In the

third column we indicate the number of Newton steps required for the first update. In the fourth column we show the final gap. The fifth column indicates the total number of Newton steps. Then the next column shows the factor by which the gap has been shrinked between the first update - "hot start" and the final solution. Then we show the number of Newton MBF steps required to achieve such an improvement. Finally "beta" is the factor by which the gap is shrinked per one Newton step.

From Tables 1 and 2 we observe the following:

1) We terminated the process for almost each problem when the gap and infeasibility are between 10^{-10} and 10^{-14}.

2) The number of Newton steps required to achieve the mentioned level of accuracy grows slowly with the problem size.

3) The number of Lagrange multipliers update as well as the dynamics of the number of Newton steps between two sequential updates is empirically better than the theory predicts. In particular the number of Lagrange multiplier updates is less than 10 for almost each problem in the set. The exception is 80 bau 3b - 18 updates and four other problems, for which we have not more that 12 updates.

The number of Newton steps between two Lagrange multipliers updates, beginning at the "hot start", is estimated as $0(\ln n)$. The observed number of Newton steps between two successive Lagrange multipliers update decreases from step to step until one Newton step is sufficient to find the MBF maximizer with a reasonable level of accuracy.

4) All results are obtained by the Dual Newton MBF method with a fixed penalty parameter k in the range 10^4 - 10^7. Since the MBF method enlarges the feasible set to D_k the potential infeasibility is on the order of $k^{-1} = 10^{-4}$ - 10^{-7}. The Lagrange multipliers, however, drive the solution to the original feasible set. Therefore, we are able to obtain an approximation that is practically feasible.

5) The results in Table 2 show that we consistently obtain the approximations $\bar{x}$ and $\bar{v}$ for primal and dual solutions with gap 10^{-10} - 10^{-15} starting with a gap of 10^{-1} - 10^{-6} (after the first Lagrange multiplier update). In other words for almost all problems that are solved, the last 9 - 10 digits of accuracy for the primal-dual gap and the constraints violation are obtained by fewer than 50 Newton steps and between 5 and 7 Lagrange multiplier updates. More specifically the dual MBF

method on average decreases the objective functions gap by factor of $\beta = 0.472244$ per Newton step (see Table 2).

6) For every LP we observe the so called "hot start" phenomenon.

The "hot start" begins when both the primal and dual variables are "well defined". The dual approximation is "well defined" in terms of S. Smale's Theorem (see Smale 1986), i.e. the dual variables are in the Newton area for the system $\nabla_v D(v, x, k) = 0$, when the primal vector x and the penalty parameter $k > 0$ are fixed. The primal approximation is "well defined" in terms of the basic MBF Theorem (see Polyak 1992). From this point on it takes only $0(\ln \ln \varepsilon^{-1})$ Newton steps to find an approximation for the dual MBF maximizer with accuracy $\varepsilon > 0$. In other words if $\varepsilon > 0$ is small enough and $k \geq k_0 = 0(\gamma^{-1} \text{cond } D(A, p, q))$ then it takes few Newton steps to find an approximation for dual MBF maximizer and to update the Lagrange multipliers. That leads to the improvement of the current approximation by a factor $0 < \gamma < 1$ (see Fig. 1).

Moreover, both the dual maximizer and the new primal approximation are "well defined" again. Therefore it takes again $0(\ln \ln \varepsilon^{-1})$ Newton steps to improve the current primal-dual gap by a factor of $0 < \gamma < 1$. Assuming that the elements of the matrix A as well as the elements of the vectors a and b can be represented by $l \leq n$ bits we have $\varepsilon = 2^{-L} \leq 2^{0(n^3)}$, where L is the input length of the LP, so $0(\ln \ln \varepsilon^{-1}) = 0(\ln n)$. In fact, as we mentioned already the number of Newton steps decreases after each Lagrange multiplier update.

Theoretically the "hot start" depends on the condition number cond $D(A, p, q)$ of the dual LP problems and the condition number is bounded if the primal and dual LP are nondegenerate (see Polyak 1992b). In practice we observe the "hot start" phenomenon for all problems that were solved.

In Tables (3-6) we show results for four LP problems with more details.

The problem "*ISRAEL*" is quite difficult in spite of its modest size ($m \times n = 176 \times 142$). The "hot start" is observed after 18 Newton steps. After the first update the gap and the infeasibility is 10^{-4}, the final gap is 10^{-15} and the infeasibility is 7.10^{-14}. The average improvement per Newton step is by a factor $\beta = 0.3$.

Problem p30 is quite large ($m \times n = 30090 \times 57000$) but sparse. The "hot start" was observed after 11 Newton steps. The gap after the first update is 10^{-7} and the infeasibility is 10^{-4}. The final gap

is 10^{-14}. Although we use $k = 10^4$, which means that the potential infeasibility might be up to 10^{-4}, the actual final infeasibility is 4.10^{-10}.

Problem Fit2D whose size is $m \times n = 25 \times 10500$ is solved with a fixed penalty parameter $k = 10^4$. In 25 Newton steps the gap was reduced from 10^{-7} to 10^{-14} and the infeasibility from 10^{-4} to 4.10^{-10}.

We observe the "hot start" phenomenon after the first update which happened after 10 Newton steps. The number of Newton steps decreases after each update and we are finally able to update the Lagrange multipliers after each Newton step. Actually in the last 4 Newton steps the Lagrange multipliers were updated 4 times. As a result, the infeasibility is brought from 10^{-6} to 10^{-10}. On average the gap is improved at every Newton step by a factor $\beta = 0.51$.

The same general behavior we observed for the LP problem "*GREENBEA*" with a size $m \times n = 2392 \times 5405$. This problem is solved with $k = 10^4$. The "hot start" is observed after the first update, which happened after 18 Newton steps. In the next 36 Newton steps the gap is reduced from 10^{-6} to 10^{-13} and the infeasibility from 10^{-7} to 5.10^{-13}.The average improvement per one Newton step is by a factor $\beta = 0.63$.

As we mentioned above, although we enlarge the dual feasible set, which potentially might lead to dual infeasibility, the Lagrange multipliers drive the solution to the original dual feasible region. In case of Dual MBF the Lagrange multipliers are just the primal vector, which is always primal feasible.

All examples show that the MBF trajectory is quite different from the CBF trajectory.

The dual variables converge to the solution, not from int Ω, but from outside the feasible set. The convergence is guaranteed not by infinitely increasing the penalty parameters as in the Classical Barrier Method, but due to the Lagrange multipliers updating while the penalty parameter remain fixed.

9. Comparison To Other Interior Point Methods

The difference between the MBF and CBF approaches is based on the fundamental differences between the MBF and CBF properties. This matter is discussed in detail in (Polyak 1992a, 1992b, Jensen, Polyak 1994, Powell 1995). The difference between the MBF and CBF

properties leads to the differences in the trajectories of corresponding methods. We already mentioned that the only parameter that controls the numerical process in the CBF approach (Fiacco, McCormick 1968) is the penalty parameter. In the MBF methods, along with the penalty parameter, the vector of Lagrange multipliers plays an important role. Moreover, when the barrier parameter is fixed the vector of Lagrange multipliers became the only driver which controls the process.

The CBF method has an arithmetic rate of convergence under the infinite increase of the barrier parameter, whereas under the same strategy for the penalty parameter and Lagrange multipliers update the MBF method has a super linear rate of convergence for any nondegenerate dual pair LP.

When the barrier parameter is fixed, the CBF does not converge to the solution at all while the MBF method has a linear rate of convergence if $k > 0$ is large enough. Moreover for any given ratio $0 < \gamma < 1$ one can find a fixed parameter $k \geq k_0$ that the estimate (3.8) or (5.6) hold true.

In this section we will compare our results with those that were obtained by the CBF approaches. For all problems that are solved (see Tables 1 and 2) the accuracy and feasibility obtained by the MBF is at least as good as those obtained by CBF.

For some problems the MBF results are much better than the CBF results in both accuracy and primal-dual feasibility. Let us consider some examples. For the problem *"GREENBEA"* we obtained only one digit of accuracy for the objective function by using CBF with the barrier parameter k up to 10^{12} while the MBF approach gave 11 digits of accuracy with a fixed $k = 10^7$. Moreover, the primal infeasibility $\| Ax - q \|_2$ is 407.6 for the CBF and 10^{-7} for the MBF. The level of primal infeasibility is attained in spite of the fact that the MBF solves the dual problem and does not enforce the primal feasibility. We are able to attain such a level of accuracy because of the formula for the primal update and the fact that we nearly optimize the Dual MBF.

One can see from Table 7 that even by increasing the CBF parameter significantly (for example, in case of *"ISRAEL"* up to $k = 10^{21}$ or in case of *"KI"* up to $k = 10^{22}$) one cannot drive the infeasibility down. Moreover, when we increase k the primal infeasibility for both problems increases indicating that the process is unstable. At the same time the dual MBF method with a fixed barrier parameter $k = 10^7$ is able to reduce the dual infeasibility for the problem *"KI"* from 10^{-4} to 10^{-13} and primal infeasibility to 10^{-7}.

For most problems the number of Newton steps per one digit of accuracy for the MBF method is at least as good as the same index for the CBF and often much better. For example, it took 41 Newton steps for the CBF method instead of 20 Newton steps for the dual MBF to get the same accuracy for the problem *"grow 22"*.

In the following we will briefly compare the dual MBF method with the results obtained by a primal-dual PC code (see Lustig, Marsten, Shanno, 1992, 1994, Forrest, Tomlin 1992) which is recognized as the best interior point method for linear programming.

From the theoretical point of view the Primal-Dual PC method (see Mehrotra 1992) is different from the Projected Newton CBF (see Gill et. al. 1986) in spite of the fact that both methods are based on CBF. For all of the problems that were solved the total number of Newton steps, the number of Newton steps per digit of accuracy as well as the factor of the gap improvement per one Newton step by Dual Newton MBF, is competitive with corresponding indices for the Primal-Dual PC method. In general the accuracy achieved for every problem that was solved by Dual Newton MBF, the final gap as well as primal and dual feasibility, are not worse than those obtained by primal-dual PC.

Moreover, for few problems (see Table 8) the results obtained by Dual Newton MBF are better in both the objective function accuracy and primal and dual feasibility.

10. Concluding Remarks

1. The purpose of this work was to check the behavior of the MBF methods for LP in practice. The preliminary numerical results obtained by the Dual Newton MBF are very much in the spirit of the MBF theory. In some cases the observed behavior of the Dual Newton MBF is even better than was predicted by the theory.

2. Although sometimes the theoretical analysis requires nondegeneracy for both primal and dual LP, the numerical results show that the behavior of the Dual Newton MBF method are typical for all LP problems which have been solved.

3. In particular, the "hot start" phenomenon was observed for all LP problems independently of their degeneracy, which was the main purpose of this work.

4. The numerical results show that a code based on the MBF approach has the potential of

being competitive with the best available interior point codes for LP problems.

5. Although the MBF methods converge with a fixed barrier parameter, it is in no way our recommendation. A reasonable increase of the barrier parameter from step to step can improve the convergence without compromising the numerical stability. However, the increase of the barrier parameter shrinks the domain, where MBF is defined, therefore the current approximation may not belong to the new domain.

Therefore to find a version of the MBF insensitive to the barrier parameter increase is an important issue.

Some encouraging results in this direction have been reported recently (see Ben-Tal, Yuzefovich, Zibulevsky, 1992, Breitfeld, Shanno, 1994, Nash, Polyak, Sofer, 1994) where different versions of Truncated MBF method have been successfully applied for solving large scale nonlinear programming problems.

6. The primal-dual approach might be another direction for improving efficiency of the MBF type methods.

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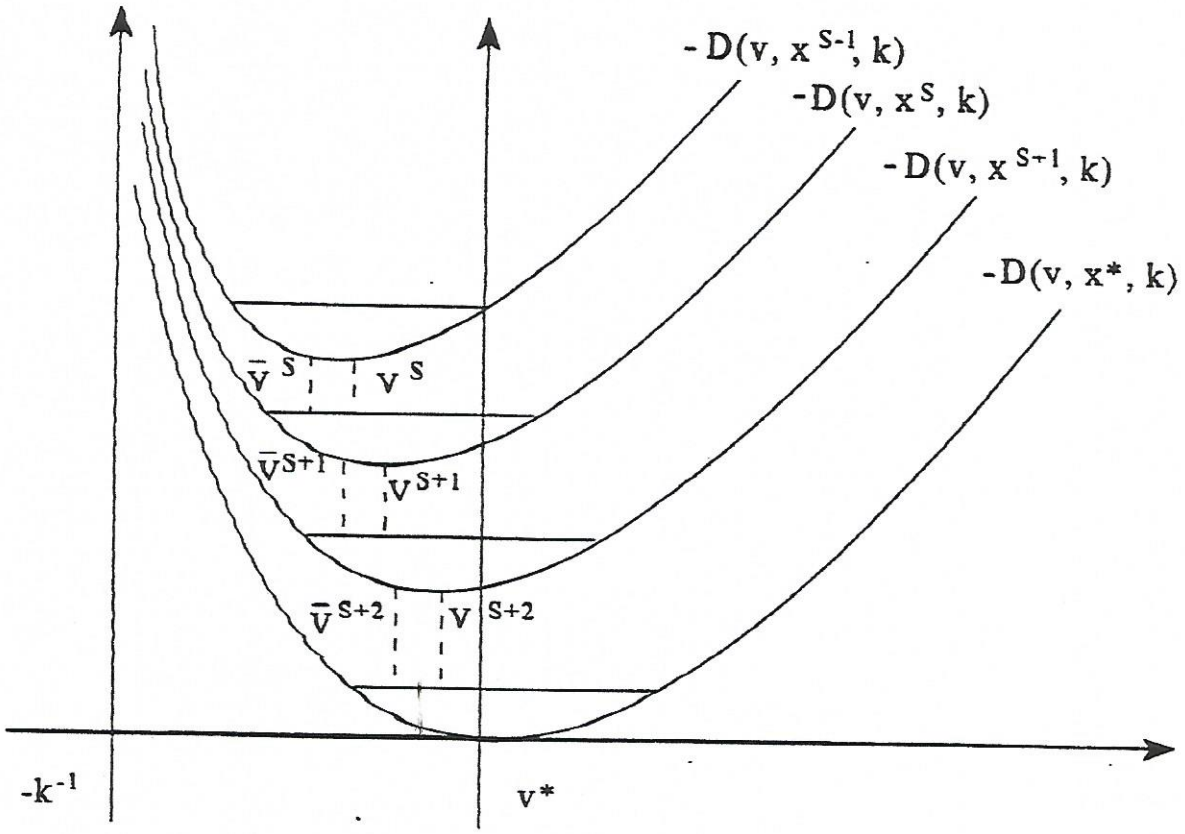

Figure 1 - The "Hot Start" Phenomenon

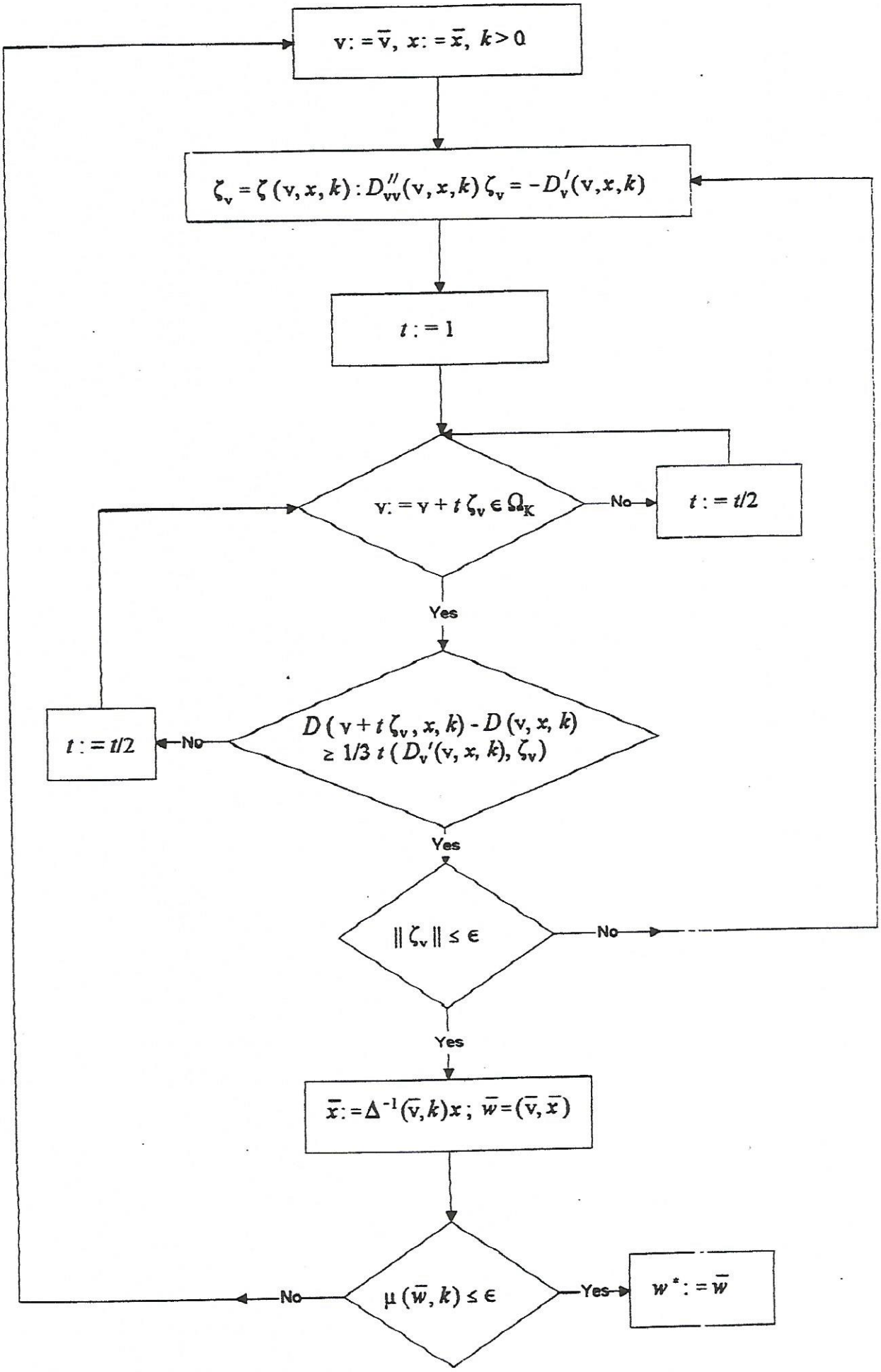

Figure 2 - Flowchart for the Dual Newton MBF Method

Name	Rows	Columns	K	Updates	Objective	Gap	Infeas.
25fv47	821	1571	4	6	5501.8458882	.1E-11	.1E-13
80bau3b	2262	9799	4	18	987224.192409	.1E-12	.1E-09
aadt	30	6461	4	6	.000000000	.1E-09	.1E-18
afiro	27	32	4	10	-464.75314286	.1E-11	.1E-15
agg	488	163	7	4	-35991767.286576	.1E-14	.1E-14
bandm	305	472	7	12	-158.6280184	.1E-10	.1E-12
barahona	515	2487	4	5	241.0000000	.1E-10	.1E-12
bigmip	3421	5153	4	12	.000000000	.1E-13	.1E-13
blend	74	83	4	5	-30.81214985	.1E-10	.1E-11
bm23u	20	27	4	6	20.57092176	.1E-10	.1E-19
bm24u	21	28	4	6	24.57092176	.1E-10	.1E-19
bm25p	28	30	4	5	30.62167731	.1E-10	.1E-15
bm25u	20	30	4	5	29.03261538	.1E-10	.1E-13
bnl2	2324	3489	7	9	1811.236540	.1E-10	.1E-11
boise	371	4468	7	8	31505.12016714	.1E-13	.1E-12
brandy	220	249	4	6	1518.50989649	.1E-12	.1E-11
bronco	2	2	4	7	.000000000	.1E-17	.1E-19
c100150s	1237	1087	7	6	577007.1644758	.1E-13	.1E-14
c90100s	994	894	7	3	389113.7695186	.1E-13	.1E-10
chevd4	4647	5771	4	8	-66798.758854	.1E-11	.1E-11
choix	780	510	4	9	.000000000	.1E-14	.1E-12
czprob	929	3523	5	4	2185196.698856	.1E-13	.1E-13
d2q06c	2171	5167	7	7	122784.21081	.1E-11	.1E-12
example	10	11	4	6	270.8463057	.1E-10	.1E-19
fit2d	25	10500	4	8	-68464.29329383	.1E-13	.1E-09
ford3	196	970	6	12	-741717.7392	.1E-10	.1E-11
ganges	1309	1681	7	6	-109585.7361292	.1E-13	.1E-11
greenbea	2392	5405	7	8	-72555248.1298	.1E-12	.1E-11
grow15	300	645	4	5	-106870941.29357536	.1E-17	.1E-11
grow22	440	946	4	3	-106834336.4825629	.1E-17	.1E-11
grow7	140	301	4	4	-47787811.8147	.1E-12	.1E-12
ipc1	1107	1547	6	6	-82871600.249	.1E-11	.1E-13
ipc4	458	1410	6	4	81360152.642	.1E-11	.1E-19
ipc6	1521	2831	6	5	-14897.8070	.1E-09	.1E-12
ipc7	1157	4051	6	4	-15273.90915	.1E-10	.1E-13
israel	174	142	4	5	-896644.82186305	.1E-14	.1E-12
jay	75	121	4	4	352795498.027	.1E-12	.1E-10
jay2	54	116	4	4	352796092.106	.1E-12	.1E-09
jay3	54	122	4	4	352796092.106	.1E-12	.1E-10

Table 1 - Execution statistics for the dual modified barrier method:

Name	Rows	Columns	K	Updates	Objective	Gap	Infeas.
k1	5203	18994	7	2	131127378.44	.1E-11	.1E-12
k4	12469	52481	7	5	376647161.252	.1E-12	.1E-12
k8	22513	99785	7	3	719891964.95	.1E-11	.1E-12
lps	3459	12811	4	4	26579676.43693	.1E-13	.1E-09
lseu	28	89	4	7	834.68235294	.1E-11	.1E-13
lseusos	28	106	4	7	834.68235294	.1E-11	.1E-13
ltw	67	48	4	3	-3498.09703304	.1E-12	.1E-10
ltw2	67	48	4	3	-3498.09703304	.1E-12	.1E-10
ltw3	67	48	4	3	-3498.09703304	.1E-12	.1E-09
merloni	180	345	4	4	1050:79698955	.1E-12	.1E-12
mod006	29	89	4	7	834.68235294	.1E-11	.1E-13
mod007	755	2756	4	5	2688.7500000	.1E-11	.1E-13
mod008	6	319	4	3	-23554.0000000	.1E-12	.1E-13
mod009	176	548	4	6	315.25490196	.1E-11	.1E-10
mod010	146	2655	4	2	6532.083333	.1E-10	.1E-08
mod011	4480	10958	4	2	-62121982.55188	.1E-13	.1E-11
mod011a	4480	10974	4	2	-62121982.55188	.1E-13	.1E-11
mod013	62	96	4	5	256.01666667	.1E-11	.1E-12
mod014	98	141	4	5	149.58876622	.1E-11	.1E-13
mod015	780	870	7	2	112130.04066	.1E-11	.1E-09
mod016	141	96	7	11	1397.99739	.1E-09	.1E-13
mod017	325	541	4	5	3273.02456699	.1E-12	.1E-10
mod019	952	1209	7	5	18262.6461	.1E-09	.1E-11
mod020	399	236	4	4	.0000000000	.1E-11	.1E-13
mod021	132	173	4	3	465442.58931	.1E-11	.1E-11
modi24	1631	2788	4	5	9494.45850308	.1E-12	.1E-09
p05	5090	9500	4	3	556000.276625	.1E-12	.1E-09
p30	30090	57000	4	4	3345141.94862	.1E-12	.1E-14
shell	536	1775	5	2	1208825346.000	.1E-13	.1E-16
ship12s	1151	2763	4	2	1489236.13440	.1E-12	.1E-09
stocfor1	117	111	7	5	-41131.97621	.1E-10	.1E-14
vienna12	2929	5059	7	3	30397.871707000	.1E-11	.1E-12

Table 1 - Execution statistics for the dual modified barrier method (cont.)

Name	Gap 1st Update	Iter. 1st	Gap Optimum	Iter. Opt.	Ratio of Gaps	Iter. Diff.	Beta
25fv47	.1E-04	14	.1E-11	38	.1E+08	24	.51
25fv47i	.1E-03	23	.1E-12	46	.1E+10	23	.41
80bau3b	.1E-03	20	.1E-12	86	.1E+10	66	.73
aadt	.1E-01	9	.1E-09	23	.1E+09	14	.27
aadti	.1E+00	10	.1E-09	23	.1E+10	13	.20
afiro	.1E-03	7	.1E-11	22	.1E+09	15	.29
afiroi	.1E-03	16	.1E-11	29	.1E+09	13	.24
agg	.1E-04	16	.1E-14	72	.1E+11	56	.66
aggi	.1E-02	46	.1E-12	65	.1E+11	19	.30
bandm	.1E-04	11	.1E-10	33	.1E+07	22	.53
barahona	.1E-02	11	.1E-10	27	.1E+09	16	.32
bigmip	.1E+00	37	.1E-13	69	.1E+14	32	.39
bigmipi	.1E+00	51	.1E-13	83	.1E+14	32	.39
blend	.1E-02	9	.1E-10	26	.1E+09	17	.34
bm23u	.1E-03	10	.1E-10	24	.1E+08	14	.32
bm24u	.1E-03	17	.1E-10	32	.1E+08	15	.34
bm25p	.1E-03	9	.1E-10	23	.1E+08	14	.32
bm25u	.1E-03	11	.1E-10	24	.1E+08	13	.29
bnl2	.1E-05	18	.1E-10	82	.1E+06	64	.84
boise	.1E-03	21	.1E-13	47	.1E+11	26	.41
brandy	.1E-03	26	.1E-12	47	.1E+10	21	.37
bronco	.1E-01	5	.1E-17	14	.1E+17	9	.02
c100150s	.1E-08	16	.1E-13	31	.1E+06	15	.46
c90100s	.1E-05	10	.1E-13	52	.1E+09	42	.64
chevd4	.1E-03	22	.1E-11	60	.1E+09	38	.62
choix	.1E-04	17	.1E-14	52	.1E+11	35	.52
czprob	.1E-06	17	.1E-13	38	.1E+08	21	.46
d2q06c	.1E-07	18	.1E-11	44	.1E+05	26	.70
dantzig	.1E-02	2	.1E-07	10	.1E+06	8	.24
example	.1E-06	8	.1E-10	20	.1E+05	12	.46
fit2d	.1E-06	11	.1E-13	35	.1E+08	24	.51
ford3	.1E-07	12	.1E-10	41	.1E+04	29	.79
ganges	.1E-06	38	.1E-13	76	.1E+08	38	.65
greenbea	.1E-05	19	.1E-12	54	.1E+08	35	.63
grow15	.1E-04	15	.1E-17	40	.1E+14	25	.30
grow22	.1E-04	17	.1E-17	37	.1E+14	20	.22
grow22i	.1E-04	70	.1E-15	92	.1E+12	22	.32
grow7	.1E-04	16	.1E-12	33	.1E+09	17	.34
ipc1	.1E-04	35	.1E-11	66	.1E+08	31	.59
ipc4	.1E-04	20	.1E-11	76	.1E+08	56	.75
ipc6	.1E-07	12	.1E-09	29	.1E+03	17	.76
ipc7	.1E-07	46	.1E-10	67	.1E+04	21	.72

Table 2 - Empirically observed factor of objective function convergence β

Name	Gap 1st Update	Iter. 1st	Gap Optimum	Iter. Opt.	Ratio of Gaps	Iter. Diff.	Beta
israel	.1E-03	19	.1E-14	44	.1E+12	25	.36
israeli	.1E-03	22	.1E-13	46	.1E+11	24	.38
jay	.1E-06	14	.1E-12	35	.1E+07	21	.52
jay2	.1E-08	14	.1E-12	35	.1E+05	21	.64
jay3	.1E-08	14	.1E-12	36	.1E+05	22	.66
k1	.1E-06	15	.1E-11	39	.1E+06	24	.62
k4	.1E-08	19	.1E-12	53	.1E+05	34	.76
k8	.1E-08	29	.1E-11	69	.1E+04	40	.84
lps	.1E-03	22	.1E-13	98	.1E+11	76	.74
lseu	.1E-03	10	.1E-11	31	.1E+09	21	.42
lseusos	.1E-03	10	.1E-11	30	.1E+09	20	.40
ltw	.1E-03	12	.1E-12	27	.1E+10	15	.25
ltw2	.1E-03	12	.1E-12	27	.1E+10	15	.25
ltw3	.1E-03	12	.1E-12	27	.1E+10	15	.25
merioni	.1E-03	13	.1E-12	36	.1E+10	23	.41
mod006	.1E-03	13	.1E-11	34	.1E+09	21	.42
mod007	.1E-01	26	.1E-11	72	.1E+11	46	.61
mod008	.1E-05	8	.1E-12	16	.1E+08	8	.13
mod009	.1E-01	27	.1E-11	79	.1E+11	52	.64
mod010	.1E-04	36	.1E-10	80	.1E+07	44	.73
mod011	.1E-05	21	.1E-13	50	.1E+09	29	.53
mod011a	.1E-05	21	.1E-13	50	.1E+09	29	.53
mod013	.1E-04	8	.1E-11	21	.1E+08	13	.29
mod014	.1E-03	12	.1E-11	33	.1E+09	21	.42
mod015	.1E-08	13	.1E-11	43	.1E+04	30	.79
mod016	.1E-01	8	.1E-09	34	.1E+09	26	.49
mod017	.1E-03	18	.1E-12	56	.1E+10	38	.58
mod019	.1E-06	21	.1E-09	70	.1E+04	49	.87
mod020	.1E+00	23	.1E-11	60	.1E+12	37	.50
mod021	.1E-04	19	.1E-11	58	.1E+08	39	.66
modi24	.1E-01	20	.1E-12	89	.1E+12	69	.69
p05	.1E-03	18	.1E-12	34	.1E+10	16	.27
p30	.1E-04	19	.1E-12	38	.1E+09	19	.38
p30	.1E-04	46	.1E-12	65	.1E+09	19	.38
shell	.1E-07	17	.1E-13	32	.1E+07	15	.40
ship12s	.1E-07	18	.1E-12	36	.1E+06	18	.53
stocfor1	.1E-04	11	.1E-10	27	.1E+07	16	.42
tiny	.1E-07	3	.1E-11	8	.1E+05	5	.16
tinyn1	.1E-04	2	.1E-18	6	.1E+15	4	.00
vienna12	.1E-07	17	.1E-11	52	.1E+05	35	.77

Table 2 - Empirically observed factor of objective function convergence β (cont.)

Problem Name ISRAEL $\mu = \frac{1}{k} = 10^{-4}$ $\beta = 0.3$

0	1	2	3	4	5	6
Update #		Itr #	Primal Obj.	Dual Obj.	Gap	Inf.
0		18	11256.50400000	-896444.83382418		
1		19	-896644.84364246	-896644.83382418	10^{-4}	10^{-4}
	22					
2		41	-896644.82211996	-896644.82190932	10^{-10}	5×10^{-6}
	1					
3		42	-896644.82186296	-896644.82186352	10^{-12}	2×10^{-7}
	1					
4		43	-896644.82186304	-896644.82186306	10^{-14}	2×10^{-12}
	1					
5		44	-896644.82186305	-896644.82186305	10^{-15}	7×10^{-14}

Table 3 -"Hot start" phenomenon in NETLIB problem ISRAEL.

Problem Name P30 $\mu = \frac{1}{k} = 10^{-4}$ $\beta = 0.38$

Update #	Itr #	Primal Obj.	Dual Obj.	Gap	Inf.
0	10	349048.90000	-68463.28633194		
1	11	-68463.24387296	-68463.28633194	10^{-7}	10^{-4}
11					
2	22	-68464.29322304	-68464.29320554	10^{-9}	5×10^{-5}
4					
3	26	-68464.29329846:	-68464.29328029	10^{-10}	2×10^{-5}
3					
4	29	-68464.29329073	-68464.29329175	10^{-10}	2×10^{-5}
2					
5	31	-68464.29329011	-68464.29329359	10^{-10}	2×10^{-6}
1					
6	33	-68464.29329383	-68464.29329381	10^{-13}	4×10^{-8}
1					
7	34	-68464.29329356	-68464.29329383	10^{-12}	4×10^{-9}
1					
8	35	-68464.29329383	-68464.29329383	10^{-14}	4×10^{-10}

Table 4 - "Hot start" phenomenon in NETLIB problem P30.

Problem Name FIT2D $\mu = \frac{1}{k} = 10^{-4} \cdot \beta = 0.51$

Update #	Itr #	Primal Obj.	Dual Obj.	Gap	Inf.
0	10	349048.90000	-68463.28633194		
1	11	-68463.24387296	-68463.28633194	10^{-7}	10^{-4}
11					
2	22	-68464.29322304	-68464.29320554	10^{-9}	5×10^{-5}
4					
3	26	-68464.29329846:	-68464.29328029	10^{-10}	2×10^{-5}
3					
4	29	-68464.29329073	-68464.29329175	10^{-10}	2×10^{-5}
2					
5	31	-68464.29329011	-68464.29329359	10^{-10}	2×10^{-6}
1					
6	33	-68464.29329383	-68464.29329381	10^{-13}	4×10^{-8}
1					
7	34	-68464.29329356	-68464.29329383	10^{-12}	4×10^{-9}
1					
8	35	-68464.29329383	-68464.29329383	10^{-14}	4×10^{-10}

Table 5 - "Hot start" phenomenon in NETLIB problem FIT2D.

Problem Name GREENBEA $\mu = \frac{1}{k} = 10^{-4} \cdot \beta = 0.63$

Update #	Itr #	Primal Obj.	Dual Obj.	Gap	Inf.
0	18	-17025.4606	-72555190.8585		
1	19	-72555248.1227	-72555190.8585	10^{-6}	10^{-7}
29					
2	48	-72555248.1302	-72555248.1298	$\times 10^{-12}$	3×10^{-10}
1					
3	49	-72555248.1299	-72555248.1298	10^{-12}	3×10^{-12}
1					
4	50	-72555248.1293	-72555248.1298	10^{-12}	10^{-12}
1					
5	51	-72555248.1282	-72555248.1298	10^{-11}	5×10^{-13}
1					
6	52	-72555248.1288	-72555248.1298	10^{-11}	5×10^{-13}
1					
7	53	-72555248.1304	-72555248.1298	10^{-11}	5×10^{-13}
1					
8	54	-72555248.1298	-72555248.1298	10^{-13}	5×10^{-13}

Table 6 - "Hot start" phenomenon in NETLIB problem GREENBEA.

	CBF			MBF			
NAME	$\mu = \frac{1}{k}$ (10*)	OBJ.	$\|Ax - q\|_2$	$\mu = \frac{1}{k}$ (10*)	OBJ.	$\|(\tilde{x}, \tilde{v})\|$	$\|Ax - q\|_2$
greenbea	-12	-70396013.	407.6	-7	-7255248.1298	10^{-12}	10^{-7}
israel	-21	-896644.82185	6.10^{-5}	-4	-896644.82186305	10^{-13}	5.10^{-5}
	(-22	-896644.82144	5.10^{-4})				
Jay 2	-18	352796107.5	2.0	-4	352796092.106	10^{-10}	10^{-6}
kl	-11	131127379	10^{-4}	-7	131127378.44	10^{-13}	10^{-7}
	(-22	131150718	24)				
Lseusos	-17	834.6844	0.01	-4	834.68235294	10^{-14}	10^{-11}
mod011a	-19	-62121982.99	10^{-3}	-4	-62121982.55188	10^{-12}	10^{-8}

	PDPC			MBF			
NAME	$\mu = \frac{1}{k}$ (10*)	OBJ.	$\|Ax - q\|_2$	$\mu = \frac{1}{k}$ (10*)	OBJ.	$\|(\tilde{x}, \tilde{v})\|$	$\|Ax - q\|_2$
80bau3b	-6	987223.4	10^{-6}	-4	987224.2	10^{-10}	10^{-6}
bn12	-10	1811.236	0.17	-7	1811.237	10^{-12}	10^{-2}
Lseusos	-18	834.6823	10^{-5}	-4	834.6824	10^{-14}	10^{-8}

Tables 7 and 8 - Comparisons with Classical Barrier Methods

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