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Modified Interior Distance Functions [1]

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ABSTRACT. The purpose of this paper is to introduce and to develop the theory of Modified Interior Distance Functions(MIDFs).

The MIDF is a Classical Lagrangian (CL) for a constrained optimization problem, which is equivalent to the initial one and can be obtained from the latter by monotone transformation both the objective function and constraints, i.e. MIDF is a particular realization of the Nonlinear Rescaling Principle in constrained optimization.

In contrast to the Interior Distance Functions (IDFs), which played a fundamental role in Interior Point Methods (IPMs), the MIDFs are defined on an extended feasible set and along with center, have two extra tools, which control the computational process: the barrier parameter and the vector of Lagrange multipliers.

Our second goal is to develop and analyze Modified Center Methods (MCMs) based on MIDFs theory.

The MCMs find an unconstrained minimizer in primal space and update the Lagrange multipliers, while both the center and the barrier parameter can be fixed or updated at each step. In this paper we will restrict ourselves by considering MCM with fixed both the barrier parameter and the "center".

It was proven that in case of nondegenerate constrained optimization, the MCM produces a primal and dual sequences that converge to the primal-dual solutions with Q-linear rate, when both the center and the barrier parameter are fixed. Moreover, every Lagrange multipliers update shrinks the distance to the primal dual solution by a factor $0 < \gamma < 1$ which can be made as small as one wants by choosing a fixed interior point as a "center" and a fixed but large enough barrier parameter.

Convergence due to the Lagrange multipliers update allows to eliminate the ill-conditioning of the IDFs Hessians and contributes to numerical stability of the MCM.

The MIDFs one can consider as Interior Augmented Lagrangean and MCM as a multipliers method.

1. Introduction

In the mid 60s, P. Huard [BuiH66], [Huar67a] and [Huar67b] introduced Interior Distance Functions (IDFs) and developed Interior Center Methods (ICMs) for solving constrained optimization problems. Later these functions, as well as Interior Center Methods, were intensively studied by A. Fiacco and G. McCormick [FiacM68], K. Grossman and A. Kaplan [GrosK81], R. Mifflin [Miff76], and E. Polak [PolE71], just to mention a few.

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It was found [PolE71] that there are close connections not only between the IDFs and the Barrier Functions [FiacM68], but also between ICMs and methods of feasible directions [Zout60], [ZPP63].

The ICMs consist of finding at each step a central (in a sense) point of the Relaxation Feasible Set (RFS) and updating it in accordance with the objective function level that has already been attained. The RFS is the intersection of the feasible set with the Relaxation (level) set of the objective function at the attained level. In the Classical ICM the "center" is sought as a minimum of the IDF.

Interest in the IDFs, as well as in the Barrier Functions (BFs), grew dramatically in connection with the well known developments in mathematical programming during the last ten years since N. Karmarkar published his projective scaling method [Kar84]. In fact, his potential function is an IDF and his method is a Center Method, which produces centers of spheres, which belong to the feasible polytop. The concept of centers has a long and interesting history.

In the 60s, concurrently with P. Huard's ICM, the Gravity Center Method was independently developed by A. Levin [Lev65] and D. Newman [New65], the Affine Scaling (ellipsoid centers) Method by I. Dikin [Dik67], and the Chebyshev Center Method by S. Zuchovitsky, R. Polyak and M. Pimak [ZPP69]. The Affine Scaling Method, which one can view as a method of feasible direction with special direction normalization, was rediscovered in 1986 independently by E. Barnes [Barn86] and R. Vanderbei, M. Maketon and B. Freedman [VanMF86] as a simplified version of Karmarkar's method.

In the 70s, N. Shor [Sh70] and independently D. Yudin and A. Nemirovsky [YuN76] developed the ellipsoid method, which generates centers of ellipsoids with minimal volume circumscribed around some convex sets. Using this method, L. Khachiyan [Kh79] was the first to prove in 1979 the polynomial complexity of the Linear Programming problem. His result had a great impact on the complexity theory, but numerically the ellipsoid method appeared to be not efficient. It is interesting to note that the rate of convergence, which was established by I. Dikin [Dik74] for the Affine Scaling Method, in case of nondegenerate linear programming problems, is asymptotically much better than the rate of convergence of the ellipsoid method and numerically, as it turned out, the Affine Scaling Method is much more efficient [AdRVK89].

The concept of centers became extremely popular in the 80s. Centering and reducing the cost are two basic ideas that are behind the developments in the Interior Point Methods (IPMs) for the last ten years. Centering means to stay away from the boundary. A successful answer to the main question: how far from the boundary one should stay, was given by Sonnevend [Son85] (see also [JarSS88]) through the definition of the analytic center of a polytop. The analytic center is a unique minimizer of the Interior Distance Function. The central path-curve, which is formed by the analytic centers, plays a very important role in the IPM developments. It was brilliantly shown in the paper by C. Gonzaga [Gon92].

Following the central path J. Renegar [Ren88] obtained the first path-following algorithm with $0(\sqrt{n}L)$ number of iterations against $0(nL)$ of the N. Karmarkar's method.

Soon afterwards, C. Gonzaga [Gon88] and P. Vaidya [Vaid87] described algorithms based on the centering ideas with overall complexity $0(n^3L)$ arithmetic operations, which is the best known result so far.

In the course of the 30 years history of center methods it became clear that both the theoretical importance and the practical efficiency of the center type methods depends very much on the "quality" of the center and on the cost to compute the center or its approximation.

The center was and still is the main tool to control the computational process in a wide variety of center methods in general and in IPMs in particular.

However, still there is a fundamental question, which has to be answered: **how consistent the main idea of center methods - to stay away from the boundary with the main purpose of constrained optimization - to find a solution on the boundary.**

In this paper we will try to address this issue. The purpose of this paper is to introduce the Modified Interior Distance Functions (MIDFs) and to develop their theory. Based on this theory, we are going to develop the Modified Center Methods (MCMs), to investigate their convergence and to establish their rate of convergence.

The MIDFs are particular realizations of the Nonlinear Rescaling Principle (see [Ber82p309], [Pol86], [Teb92], [PolTeb95] and bibliography in it) which consists of transforming a constrained optimization problem into an equivalent one and using the Classical Lagrangean for the equivalent problem for both theoretical analysis and numerical methods. In the case of MIDFs, we transform both the objective functions and the constraints by monotone transformations. The constraints transformation is parametrized by a positive parameter. The MIDFs, which is a Classical Lagrangeans for the equivalent problem have properties that make them substantially different from both IDFs as well as Classical Lagrangeans for the initial problem.

Instead of one tool (the centers), which controls the process in the IDFs, the MIDFs have three tools: the center, the barrier parameter and the vector of Lagrange multipliers. Two extra tools provide the MIDFs with very important properties.

The barrier parameter not only allows to retain the convexity of the MIDFs when both the objective function and the constraints are convex, it also allows to "convexify" the MIDFs in the case when the objective function and/or the constraints are not convex but the second order optimality conditions are satisfied. The barrier parameter is also crucial for the rate of convergence of the MCMs.

The other critical extra tool is the vector of Lagrange multipliers. It allows to attach to the MIDFs nice properties of Augmented Lagrangeans [Ber82], [GolT89], [Hes69], [Man75], [PolT73], [Pow69], [Rock74].

One can consider MIDFs as Interior Augmented Lagrangeans. Moreover, in addition to the nice local Augmented Lagrangean properties, the MIDFs possess important global self concordance properties (see [NesN94]), when the Lagrange multipliers are equal and fixed. It allows to obtain methods with polynomial complexity by fixing the Lagrange multipliers and changing the barrier parameter or the center to approach the solution.

What is most important, the MIDFs are defined and keep smoothness of the order of the initial functions on the extension of the feasible set.

The special MIDFs properties allows to develop MCM, which produces the primal-dual sequences that converge to the primal-dual solution, even when both the center and the barrier parameter are fixed. Moreover, under nondegeneracy assumptions the primal and dual sequences converge to the primal-dual solution with Q-linear rate.

The MIDFs are to IDFs as Modified Barrier Functions (MBFs) (see[Pol92]) are to Classical Barrier Functions. It seems, however, that the differences between MIDFs and MBFs are more fundamental than between Classical Distance and Barrier functions, which are both critical tools in the IPMs developments.

The paper is organized as follows. After the statement of the problem, we discuss the

IDF's properties and introduce the MIDFs. Then we establish the basic MIDF's properties at the primal-dual solution and compare them with the correspondent IDF's properties. Then we prove the basic theorem, which is the foundation for the MCMs and their convergence.

1. Problem Formulation and Basic Assumptions.

Let $f_0(x)$ and $-f_i(x)$, $i = 1,\dots,m$ be convex, C^2-function in $\mathbb{R}^n$ and there exists

$$x^* = \operatorname{argmin}\{f_0(x) / x \in \Omega\} \tag{1.1}$$

where $\Omega = \{x : f_i(x) \geq 0,\ i = 1,\dots,m\}$.
We will assume that Slater condition holds, i.e.

$$\exists x^0 : f_i(x^0) > 0,\ i = 1,\dots,m \tag{1.2}$$

So the Karush-Kuhn-Tucker's (*K-K-T*s) optimality conditions hold true, i.e. there exists a vector $u^* = (u_1^*,\dots,u_m^*) \geq 0^m$ such that

$$L_x'(x^* u^*) = f_0'(x^*) - \sum_{i=1}^{m} u_i^* f_i'(x^*) = 0^n,\ f_i(x^*)u_i^* = 0,\ i = 1,\dots,m, \tag{1.3}$$

where $L(x,u) = f_0(x) - \sum_{i=1}^m u_i f_i(x)$ is the Lagrange function for (1.1) and $f_i'(x) = \operatorname{grad} f_i(x)$, $i = 0, \dots m$, are row-vectors. Let $I^* = \{i : f_i(x^*) = 0\} = \{1,\dots,r\}$ is the active constraints set and $r \leq n$.

We consider the vector-function $f(x) = (f_1(x),\dots,f_m(x))$,the vector-function of active constraints $f_{(r)}(x) = (f_1(x),\dots,f_r(x))$ and the vector-function of passive constraints $f_{(m-r)}(x) = (f_{r+1}(x),\dots,f_m(x))$.
We also consider their Jacobians $f'(x) = J(f(x))$, $f'_{(r)}(x) = J(f_{(r)}(x))$, $f'_{(m-r)}(x) = J(f_{(m-r)}(x))$, diagonal matrices $U = [\operatorname{diag} u_i]_{i=1}^m$, $U_r = [\operatorname{diag} u_i]_{i=1}^r$ with entries u_i, $i = 1, \dots, m$ and Hessians

$$f_i''(x) = \left\| \frac{\partial f_i}{\partial x_s \partial x_t} \right\| \begin{array}{l} s = 1,\dots,n \\ t = 1,\dots,n \end{array} \quad i = 0,1,\dots,m$$

of the objective function and constraints. The sufficient regularity condition

$$\operatorname{rank} f'_{(r)}(x^*) = r,\ u_i^* > 0,\ i \in I^* \tag{1.4}$$

together with the sufficient condition for the minimum x^* to be isolated

$$(L''_{xx}(x^*,u^*)z,z) \geq \lambda(z,z),\ \lambda > 0 \quad \forall z \neq 0 : f'_{(r)}(x^*)z = 0^r \tag{1.5}$$

comprise the standard second order optimality condition, which we will assume in this paper.
We shall use the following assertion, which is a slight modification of the Debreu theorem (see [Pol 92]).

Assertion 1 *Let A be a symmetric $n \times n$ matrix, B be an $r \times n$ matrix, and $U = [\operatorname{diag} u_i]_{i=1}^r : R^r \to R^r$, where $u = (u_1, \dots, u_r) > 0^r$ and let*

$$(Ay, y) \geq \lambda(y, y),\ \lambda > 0,\ \forall y : By = 0^r.$$

Then there exists $k_0 > 0$ such that for any $0 < \mu < \lambda$ the following inequality

$$((A + kB^T UB)x,x) \geq \mu(x,x),\ \forall x \in R^n$$

holds true whenever $k \geq k_0$.

2. Interior Distance Functions

Let $y \in \operatorname{int}\Omega$ and $\alpha = f_0(y)$, we consider the Relaxation Feasible Set (RFS) on the level $\alpha: \Omega(\alpha) = \Omega \cap \{x : f_0(x) \leq \alpha\}$ and an interval $T = \{\tau : \alpha < \tau < \alpha^* = f_0(x^*)\}$. The Classical IDFs $F(x,\alpha)$ and $H(x,\alpha): \Omega(\alpha) \times T \to \mathbf{R}^1$ are defined by formulas

$$F(x,\alpha) = -m\ln(\alpha - f_0(x)) - \sum_{i=1}^{m} \ln f_i(x);\ H(x,\alpha) = m(\alpha - f_0(x))^{-1} + \sum_{i=1}^{m} f_i^{-1}(x)$$

Let us assume that $\ln t = -\infty$ and $t^{-1} = \infty$ for $t \leq 0$, the Classical Interior Center Methods (ICMs) consists of finding the "center" of the RFS by solving the following unconstrained optimization problem

$$\hat{x} = \hat{x}(\alpha) = \operatorname{argmin}\{F(x,\alpha) / x \in \mathbf{R}^n\}$$

and updating the objective function level α, i.e., replacing α by $\hat{\alpha} = f_0(\hat{x})$. Due to the property $x \to \partial\Omega(\alpha) \to F(x,\alpha) \to \infty$ the new center $\hat{x}(\alpha) \in \operatorname{int}\Omega(\alpha) \subset \Omega$ for any $\alpha \in T$. Moreover, if the IDF possess the self-concordance properties (see [NesN 94]) the central trajectory $\{\hat{x}(\alpha),\ \alpha \in T\}$ has some very special features (see [Ren 88] and [Gon 92]).

Starting at a point close to the central trajectory - "warm" start - for a particular $\alpha \in T$ and using Newton step for solving the system

$$F_x'(x,\alpha) = 0^n$$

in x following by a "careful" α update, one can guarantee that the new approximation will be again a "warm" start and the gap between the current level $\alpha = f_0(x)$ and the optimal level $\alpha^* = f(x^*)$ will be reduced by a factor $0 < q_n < 1$, which is dependent only on the size of the problem.

However along with these nice properties the IDFs have their well known drawbacks. Neither the IDFs $F(x, \alpha)$ and $H(x, \alpha)$ nor their derivatives exist at the solution. Both $F(x, \alpha)$ and $H(x, \alpha)$ grow infinitely when $\hat{x}(\alpha)$ approaches the solution.

All constraints contribute equally to IDFs and one can obtain the optimal Lagrange multipliers only in the limit when $\hat{x}(\alpha) \to x^*$. What is particularly important for nonlinear constrained optimization is the fact that the condition number of the IDF Hessians is increasing unbounded when the process approaches the solution. Let's consider this issue briefly, using $F(x,\alpha)$. Keeping in mind the boundness of the RFS $\Omega(\alpha)$ one can guarantee that the unconstrained minimizer $\hat{x} \in \operatorname{int}\Omega(\alpha)$ exists and

(2.1) $$F_x'(\hat{x},\alpha) = \frac{m}{\alpha - f_0(\hat{x})} f_0'(\hat{x}) - \sum_{i=1}^{m} \frac{f_i'(\hat{x})}{f_i(\hat{x})} = 0^n$$

or

(2.2) $$f_0'(\hat{x}) - \sum_{i=1}^{m} \frac{\alpha - f_0(\hat{x})}{m f_i(\hat{x})} f_i'(\hat{x}) = 0^n$$

We define

$$\hat{u}_i = \hat{u}_i(\alpha) = (\alpha - f_0(\hat{x}))(mf_i(\hat{x}))^{-1}, \; i = 1,...,m$$

and consider the vector of Lagrange multipliers $\hat{u} \equiv \hat{u}(\alpha) = (\hat{u}_i(\alpha), i = 1,...,m)$, then (2.2) can be rewritten as follows:

$$L_x'(\hat{x},\hat{u}) = f_0'(\hat{x}) - \sum_{i=1}^{m} \hat{u}_i f_i'(\hat{x}) = f_0'(\hat{x}) - \hat{u}f'(\hat{x}) = 0^n$$

Also $\hat{u}_i f_i(\hat{x}) = (\alpha - f_0(\hat{x}))m^{-1}$, $i = 1,...,m$. So $\sum_{i=1}^m \hat{u}_i f_i(\hat{x}) = \alpha - f_0(\hat{x})$. Under the uniqueness assumptions (1.4) - (1.5) we have

$$\lim_{\alpha \to \alpha^*} \hat{u}(\alpha) = u^*, \qquad \lim_{\alpha \to \alpha^*} \hat{x}(\alpha) = x^*$$

Let's consider the Hessian $F_{xx}''(x,\alpha)$. We obtain

$$F_{xx}''(\hat{x},\alpha) = f_0''(\hat{x}) - \sum_{i=1}^{m} \frac{\alpha - f_0(\hat{x})}{mf_i(\hat{x})} f_i''(\hat{x}) - \sum_{i=1}^{m}\left[\left(\frac{\alpha - f_0(\hat{x})}{mf_i(\hat{x})}\right)'\right]^T f_i'(\hat{x})$$

Further, for any $i = 1,..., m$, we have $\left(\frac{\alpha - f_0(x)}{mf_i(x)}\right)' = m^{-1}\left[\frac{-f_0'(x)f_i(x) - f_i'(x)(\alpha - f_0(x))}{f_i^2(x)}\right]$

Therefore

$$-\sum_{i=1}^{m}\left[\left(\frac{\alpha - f_0(\hat{x})}{mf_i(\hat{x})}\right)'\right]^T f_i'(\hat{x}) = \sum_{i=1}^{m} \frac{(f_0'(\hat{x}))^T}{mf_i(\hat{x})} f_i'(\hat{x}) + \sum_{i=1}^{m} \frac{\alpha - f_0(\hat{x})}{mf_i(\hat{x})} \frac{1}{f_i(\hat{x})} (f_i'(\hat{x}))^T f_i'(\hat{x})$$

$$= \frac{1}{\alpha - f_0(\hat{x})} (f_0'(\hat{x}))^T \sum_{i=1}^{m} \frac{\alpha - f_0(\hat{x})}{mf_i(\hat{x})} f_i'(\hat{x}) + \sum_{i=1}^{m} \frac{\alpha - f_0(\hat{x})}{mf_i(\hat{x})} \frac{1}{f_i(\hat{x})} (f_i'(\hat{x}))^T f_i'(\hat{x})$$

$$= \frac{1}{\alpha - f_0(\hat{x})} (f_0'(\hat{x}))^T \sum_{i=1}^{m} \hat{u}_i f_i'(\hat{x}) + \sum_{i=1}^{m} \frac{\hat{u}_i}{f_i(\hat{x})} (f_i'(\hat{x}))^T f_i'(\hat{x})$$

Let $D(x) = [\operatorname{diag} f_i(x)]_{i=1}^m$, $U(\alpha) = [\operatorname{diag} u_i(\alpha)]_{i=1}^m$, then for the Hessian $F_{xx}''(\hat{x},\alpha)$ we obtain

$$F_{xx}''(\hat{x},\alpha) = L_{xx}''(\hat{x},\hat{u}) + (f'(\hat{x}))^T U(\alpha) D^{-1}(\hat{x}) f'(\hat{x}) + \frac{1}{\alpha - f_0(\hat{x})} [f_0'(\hat{x})]^T \sum_{i=1}^{m} \hat{u}_i(\alpha) f_i'(\hat{x})$$

In view of $\hat{x} = \hat{x}(\alpha) \to x^*$, $\hat{u} = \hat{u}(\alpha) \to u^*$ we obtain

$$L_{xx}''(\hat{x},\hat{u}) \to L_{xx}''(x^*,u^*), \; [f_0'(\hat{x})]^T \sum_{i=1}^{m} \hat{u}_i f_i'(\hat{x}) \to f_0^T(x^*) f_0'(x^*)$$

$$f'(\hat{x}) \to f'(x^*), \quad \hat{U} \to U^*, \quad D(\hat{x}) \to D(x^*)$$

Therefore

$$F_{xx}''(\hat{x},\alpha) \approx L_{xx}''(x^*,u^*) + (f_{(r)}'(x^*))^T \hat{E}(\alpha) f_{(r)}'(x^*) + (\alpha - f_0(\hat{x}))^{-1} f_0'^T(x^*) f_0'(x^*) \tag{2.3}$$

where $\hat{E}(\alpha) = [\operatorname{diag} \hat{e}_i(\alpha)]_{i=1}^r$ and

(2.4)
$$\lim_{\alpha \to \alpha^*} \hat{e}_i(\alpha) = \lim_{\alpha \to \alpha^*} \hat{u}_i(\alpha) f_i^{-1}(\hat{x}(\alpha)) = +\infty$$

The mineigval $F''_{xx}(\hat{x},\alpha)$ is defined by the first two terms (2.3), therefore in view of (2.4) and due to the Assertion 1 with $A = L''_{xx}(x^*, u^*)$ and $B = f'_{(r)}(x^*)$ there exists $\mu > 0$:

$$\text{mineigval } F''_{xx}(\hat{x},\alpha) = \mu$$

At the same time due to (2.4) we have

$$\text{maxeigval } F''_{xx}(\hat{x},\alpha) \to \infty$$

and the condition number

$$\kappa(\hat{x},\alpha) = \text{maxeigval } F''_{xx}(\hat{x},\alpha)\,(\text{mineigval} F''_{xx}(\hat{x},\alpha))^{-1} \to \infty$$

when $\alpha \to \alpha^*$.

The consequences of the ill-conditioning is much more critical in nonlinear optimization than in Linear Programming. In case of LP the term $L''_{xx}(x,u)$ in the expression for the IDF Hessian disappears and by rescaling one can practically eliminate the ill-conditioning effect, at least, when the problem is not degenerate.

In nonlinear optimization the situation is completely different and the ill conditioning was and still is an important issue both in theory and practice. To eliminate the ill conditioning of the IDF we will introduce the Modified Interior Distance Functions.

3. Modified Interior Distance Functions

We consider a vector $y \in \text{int}\,\Omega$ and $\Delta(y, x) = f_0(y) - f_0(x) > 0$, then the Relaxation Feasible Set (RFS) is defined as follows:

$$\Omega(y) = \{x : f_i(x) \geq 0, i = 1,\dots,m \; ; \; \Delta(y,x) > 0\}$$

The problem (1.1) is equivalent to

(3.1)
$$x^* = \text{argmin}\{f_0(x) / x \in \Omega(y)\}$$

It is easy to see that for any $k > 0$

$$\Omega(y) = \{x : k^{-1}[\ln(kf_i(x) + \Delta(y,x)) - \ln\Delta(y,x)] \geq 0 \quad i = 1,\dots,m; \; \Delta(y,x) > 0\}.$$

Therefore the problem (3.1) is equivalent to the following problem:

(3.2)
$$x^* = \text{argmin}\{-\ln\Delta(y,x) / x \in \Omega(y)\}$$

Assuming $\ln t = -\infty$ for $t \leq 0$ we define the MIDF $F(x, y, u, k)$: $\mathbf{R}^n \times \text{int}\,\Omega \times \mathbf{R}^m_+ \times \mathbf{R}^1_{++} \to \mathbf{R}^1$ as a Classical Lagrangean for the equivalent problem (3.2):

(3.3)
$$F(x,y,u,k) = (-1 + k^{-1}\sum_{i=1}^{m} u_i)\ln\Delta(y,x) - k^{-1}\sum_{i=1}^{m} u_i \ln(kf_i(x) + \Delta(y,x))$$

The MIDF $F(x, y, u, k)$ corresponds to the IDF $F(x, \alpha)$. To define the MIDF, which corresponds to $H(x, \alpha)$, we first note that for any $k > 0$

$$\Omega(y) = \{x : k^{-1}[(kf_i(x) + \Delta(y,x))^{-1} - \Delta^{-1}(y,x)] \le 0\ ,\ i = 1,\dots,m\ ,\ \Delta(y,x) > 0\ \}$$

Therefore the problem (1.1) is equivalent to

$$(3.4) \qquad x^* = \operatorname{argmin}\{\Delta^{-1}(y,x)\,/\,x \in \Omega(y)\}$$

Assuming $t^{-1} = \infty$ for $t \le 0$ we define the MIDF $H(x,y,u,k) : \mathbb{R}^n \times \operatorname{int}\Omega \times \mathbb{R}^m_+ \times \mathbb{R}^1_{++} \to \mathbb{R}^1$ as a Classical Lagrangean for the equivalent problem (3.4):

$$(3.5) \qquad H(x,y,u,k) = (-1 + k^{-1}\sum_{i=1}^m u_i)\,\Delta^{-1}(y,x) + k^{-1}\sum_{i=1}^m u_i(kf_i(x) + \Delta(y,x))^{-1}$$

The MIDF (3.5) corresponds to the P. Huard's IDF $H(x, \alpha)$. Both $F(x, y, u, k)$ and $H(x, y, u, k)$ are Classical Lagrangeans for problems equivalent to (1.1), which we obtained by monotone transformation both the objective function and the constraints.

Finally, the MIDF $Q(x,y,u,k) : \mathbb{R}^n \times \operatorname{int}\Omega \times \mathbb{R}^m_+ \times \mathbb{R}^1_{++} \to \mathbb{R}^1$, which is defined by formula

$$Q(x,y,u,k) = (\Delta(y,x))^{-1+k^{-1}\Sigma u_i} \bullet \prod_{i=1}^m (kf_i(x) + \Delta(y,x))^{-k^{-1}u_i}$$

corresponds to the potential function

$$Q(x,\alpha) = (\alpha - f_0(x))^{-m} \prod_{i=1}^m f_i^{-1}(x).$$

So, we have $F(x, y, u, k) = \ln Q(x, y, u, k)$ and all basic facts about $F(x, y, u, k)$ remain true for $Q(x, y, u, k)$, therefore we will not consider the MIDF $Q(x, y, u, k)$ further in this paper.

There is a fundamental difference between the Classical and Modified Interior Distance Functions. First we are going to show the difference at the local level - in the neighborhood of the primal-dual solution. In the next section, we will consider the local MIDFs properties.

4. Local MIDFs Properties

In contrast to the IDFs, the MIDFs are defined at the solution; they do not grow infinitely when the primal approximation approaches the solution and under the fixed optimal Lagrange multipliers, one can obtain the primal solution by solving one smooth unconstrained optimization problem.

Proposition 4.1. *For any* $k > 0$ *and any* $y \in \operatorname{int}\Omega$, *the following relations are taking place.*

(P1) $F(x^*,y,u^*,k) = -\ln\Delta(y,x^*)$ i.e. $f_0(x^*) = f_0(y) - \exp(-F(x^*,y,u^*,k))$

and

$$H(x^*,y,u^*,k) = \Delta^{-1}(y,x^*) \quad \text{i.e.} \quad f_0(x^*) = f_0(y) - H^{-1}(x^*,y,u^*,k)$$

The property P1 follows immediately from the definition of MIDFs and the complementary conditions for the K-K-T's pair (x^*,u^*):

$$u_i^* f_i(x^*) = 0, \quad i = 1,...,m.$$

The fact that the MIDF's value at (x^*, u^*) coincides with the optimal objective function value for the equivalent problem independently on both the center $y \in \text{int}\,\Omega$ and barrier parameter $k > 0$ indicates that one can approach the solution by means other than those, which have been traditionally used in the IPMs.

Proposition 4.2. *For any $k > 0$ and any $y \in \text{int}\,\Omega$, the following relations are taking place.*

(P2)
$$F'_x(x^*, y, u^*, k) = \Delta^{-1}(y, x^*) L'_x(x^*, u^*) = 0$$
and
$$H'_x(x^*, y, u^*, k) = \Delta^{-2}(y, x^*) L'_x(x^*, u^*) = 0$$

The proposition 4.2 immediately follows from the definition of MIDFs and K-K-T's conditions.

Proposition 4.3. *If $k > \Sigma u_i^*$ the unconstrained minimizer of $F(x, y, u^*, k)$ or $H(x, y, u^*, k)$ in x is a solution of the convex programming problem* (1.1), *i.e., the following property is taking place.*

(P3)
$$x^* = \text{argmin}\{F(x,y,u^*,k) / x \in \mathbb{R}^n\} = \text{argmin}\{H(x,y,u^*,k) / x \in \mathbb{R}^n\}$$

In other words, having the optimal Lagrange multipliers one can solve the problem (1.1) by solving one unconstrained optimization problem. Therefore if $F(x, y, u, k)$ is strongly convex in x and we know a good approximation u for the vector u^*, then $\hat{x} = \hat{x}(y, u, k) = \text{argmin}\{F(x,y,u,k) / x \in \mathbb{R}^n\}$ is a good approximation for x^* while both the "center" $y \in \text{int}\,\Omega$ and $k > \Sigma u_i$ are fixed.

If by using $\hat{x}$ we can improve the approximation u, then it is possible to develop a method where the convergence is due to the Lagrange multipliers update rather than to the center or the barrier parameter update.

Our goal is to develop such a method, but first we will try to understand under what conditions for the problem (1.1) the MIDFs $F(x, y, u, k)$ and $H(x, y, u, k)$ will be strongly convex in x when both y and $k > 0$ are fixed.

The following proposition is the first step in this direction.

Proposition 4.4. *If $f_i(x) \in C^2$, $i = 0, 1,..., m$, then for any fixed $y \in \text{int}\,\Omega$, $k > 0$ and any KKT's pairs (x^*, u^*) the following is true:*

(P4)
$$F''_{xx}(x^*, y, u^*, k) = \Delta^{-1}(y, x^*)[L''_{xx}(x^*, u^*) + \Delta^{-1}(y, x^*)(k(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*) - (f'_{(r)}(x^*))^T u_{(r)}^{*T} u_{(r)}^* f'_{(r)}(x^*))]$$
and
$$H''_{xx}(x^*, y, u^*, k) = \Delta^{-2}(y, x^*)[L''_{xx}(x^*, u^*) + \Delta^{-1}(y, x^*)(k(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*) - (f'_{(r)}(x^*))^T u_{(r)}^{*T} u_{(r)}^* f'_{(r)}(x^*))]$$

The proof is given in the Appendix A1. We are now ready to prove the first basic statement.

Theorem 1. *If* $f_i(x) \in C^2$, $i = 0, 1, \dots, m$, *then*

1) *for any fixed* $y \in \operatorname{int}\Omega$ *and* $k > \Sigma u_i^*$ *the function* $F(x, y, u, k)$ *is strongly convex in the neighborhood* x^* *if* $f_0(x)$ *or one of the functions* $-f_i(x)$, $i = 1, \dots, r$ *is strongly convex at* x^* *or sufficient regularity conditions* (1.4) *are taking place and* $r = n$;

2) *if the second order optimality conditions* (1.4) - (1.5) *are fulfilled then there exist* $k_0 > 0$ *large enough that for any fixed* $y \in \operatorname{int}\Omega$ *and any fixed* $k > \Delta(y, x^*)\, k_0 + \Sigma u_i^*$ *there exist such that* $\mu > 0$ *and* $M < +\infty$ *that the following is true* :

(P5) a) *mineigval* $F''_{xx}(x^*, y, u^*, k) \geq \Delta^{-1}(y, x^*)\,\mu$

b) *maxeigval* $F''_{xx}(x^*, y, u^*, k) \leq \Delta^{-1}(y, x^*)\, M$

and the condition number of the Hessian $F''_{xx}(x^*, y, u^*, k)$ *is bounded, i.e.*

$\kappa(x^*, y, u^*, k) \leq M\mu^{-1}$

Proof 1) Using P4 for any $v \in \mathbb{R}^n$ we obtain

$$
\begin{aligned}
&F''_{xx}(x^*, y, u^*, k)v, v) = \Delta^{-1}(y, x^*)[L''_{xx}(x^*, u^*)v, v) + \\
&\Delta^{-1}(y, x^*)(k(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)v, v) - ((f'_{(r)}(x^*))^T u_{(r)}^{*T} u_{(r)}^* f'_{(r)}(x^*)v, v))] \\
&= \Delta^{-1}(y, x^*)[(L''_{xx}(x^*, u^*)v, v) + \Delta^{-1}(y, x^*)(k - \Sigma u_i^*)((f''_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)v, v) \\
&\quad + \Delta^{-1}(y, x^*)((\Sigma u_i^*)(\Sigma u_i^*(f_i'(x^*), v)^2) - (\Sigma u_i^*(f_i'(x^*), v))^2)]
\end{aligned}
$$

Taking into account identity

$$
\begin{aligned}
&(\sum_{i=1}^{m} u_i^*)(\sum_{i=1}^{m} u_i^*(f_i'(x^*), v)^2) - (\sum_{i=1}^{m} u_i^*(f_i'(x^*), v))^2 \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} u_i^* u_j^* (f_i'(x^*) - f_j'(x^*), v)^2 \geq 0
\end{aligned}
\tag{4.1}
$$

we obtain

$$
\begin{aligned}
&(F''_{xx}(x^*, y, u^*, k)v, v) \geq \Delta^{-1}(y, x^*)[(L''_{xx}(x^*, u^*) \\
&+ \Delta^{-1}(y, x^*)(k - \Sigma u_i^*)((f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)v, v)]
\end{aligned}
\tag{4.2}
$$

So for a convex programming problem (1.1) the function $F(x, y, u^*, k)$ is convex in x for any $y \in \operatorname{int}\Omega$ and $k \geq \Sigma u_i^*$.

If one of $f_0(x)$, $-f_i(x)$, $i = 1, \dots, r$ is strongly convex at x^* then due to $u_i^* > 0$, $i = 1, \dots, r$ and $f_i(x) \in C^2$ the Classical Lagrangean $L(x, u^*)$ is strongly convex in the neighborhood of x^* while the matrix $(k - \Sigma u_i^*)\Delta^{-1}(y, x^*)(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)$ is non negative defined for any $y \in \operatorname{int}\Omega$ and $k \geq \Sigma u_i^*$, therefore $F(x, y, u^*, k)$ is strongly convex in the neighborhood of x^*. If $f_0(x)$ and all $-f_i(x)$ are convex then $L(x, u^*)$ is convex in x. If in addition the sufficient regularity condition (1.4) is satisfied and $r = n$, then for any $y \in \operatorname{int}\Omega$ and $k > \Sigma u_i^*$ the matrix $(k - \Sigma u_i^*)\Delta^{-1}(y, x^*)(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)$ is positive defined and again $F(x, y, u^*, k)$ is strongly convex in the neighborhood of x^*.

Note, due to $f_i(x) \in C^2$, $i = 0, 1, \dots, m$ the MIDF $F(x, y, u^*, k)$ will remain strongly convex in x for any $u \in \mathbb{R}_+^m$ close enough to u^*.

2) Now let's consider the case when none of $f_0(x)$ and $-f_i(x)$, $i = 1,\dots, r$ are strongly convex and $r < n$. If $k > \Delta(y,x^*)k_0 + \Sigma u_i^*$, then due to (4.2) we obtain

$$(F''_{xx}(x^*,y,u^*,k)v,v) \geq \Delta^{-1}(y,x^*)([L''_{xx}(x^*,u^*) + k_0(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)]v,v), \ \forall v \in \mathbb{R}^n \tag{4.3}$$

Therefore if the second order optimality condition (1.4) - (1.5) are satisfied, then due to the Assertion 1 with $A = L''_{xx}(x^*,u^*)$ and $B = f'_{(r)}(x^*)$ for $k_0 > 0$ large enough, any "center" $y \in \text{int}\,\Omega$ and any $k > \Delta(y,x^*)k_0 + \Sigma u_i^*$ there exists $\mu > 0$:

$$(F''_{xx}(x^*,y,u^*,k)v,v) \geq \Delta^{-1}(y,x^*)\,\mu\,(v,v), \ \forall v \in \mathbb{R}^n \tag{4.4}$$

It is also clear that for a fixed $y \in \text{int}\,\Omega$ and fixed $k > \Delta(y, x^*)\,k_0 + \Sigma\, u_i^*$ there exists $M < \infty$:

$$(F''_{xx}(x^*,y,u^*,k)v,v) \leq \Delta^{-1}(y,x^*)\,M(v,v), \ \forall v \in \mathbb{R}^n \tag{4.5}$$

Therefore $\kappa(x^*, y, u^*, k) \leq M\mu^{-1}$ and due to $f_i(x) \in C^2$, $i = 0,1, \dots, m$ the condition number remains bounded in the neighborhood of (x^*, u^*) for any fixed "center" $y \in \text{int}\,\Omega$ and any fixed barrier parameter $k > \Delta(y, x^*)\,k_0 + \Sigma\, u_i^*$.

REMARK 1. The second part of Theorem 1 remain true even for nonconvex problem if the second order optimality conditions are satisfied. In other words the barrier parameter k not only allows to retain the convexity in x of the MIDF $F(x, y, u, k)$ but also provide convexification of the $F(x, y, u, k)$ in x in case when the Classical Lagrangean $L(x, u)$ for the initial problem is not convex in $x \in \mathbb{R}^n$.

REMARK 2. Theorem 1 holds true for the MIDF $H(x, y, u, k)$. For any $y \in \text{int}\,\Omega$ and any fixed $k \geq 0.5\, k_0\, \Delta(y, x^*) + \Sigma\, u_i^*$, there exists $\mu > 0$ and $M < \infty$ that for $\forall$ $v \in \mathbb{R}^n$ the following is true:

a) $H''_{xx}(x^*y,u^*,k)v,v) \geq \Delta^{-2}(y,x^*)([L''_{xx}(x^*y^*) + k_0(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*)]v,v) \geq \Delta^{-2}(y,x^*)\,\mu(v,v)$,

b) $H''_{xx}(x^*y,u^*,k)v,v) \leq \Delta^{-2}(y,x^*)\ M(v,v)$

5. Modified Center Method

It follows from Theorem 1 that to solve a constrained optimization problem for which the second order optimality conditions are fulfilled, it is enough to find a minimizer for a strongly convex and smooth in x function $F(x, y, u^*, k)$ with any fixed $y \in \text{int}\,\Omega$ as a "center" and any fixed $k > \Delta(y, x^*)\,k_0 + \Sigma\, u_i^*$. Due to the strong convexity of $F(x, y, u, k)$ in x to find an approximation to x^* it is enough to find a minimizer

$$\hat{x} = \hat{x}(y,u,k) = \text{argmin}\{\,F(x,y,u,k)\,/\,x \in \mathbb{R}^n\,\} \tag{5.1}$$

for a given Lagrange multipliers vector $u \in \mathbb{R}^m_{++}$ close enough to u^*, when both y and k are

fixed. Moreover, as it turns out, having the minimizer $\hat{x}$ one can find a better approximation $\hat{u}$ for the vector u^* without changing both $y \in \operatorname{int} \Omega$ and $k > 0$.
Let's consider it with more details. Assuming that the minimizer $\hat{x}$ exists, we obtain

$$(5.2) \qquad F_x'(\hat{x},y,u,k) = (1 - k^{-1}\Sigma u_i + k^{-1}\Sigma \hat{u}) f_0'(\hat{x}) - \Sigma \hat{u}_i f_i'(\hat{x}) = 0^n$$

where the components of the new vector of Lagrangean multipliers $\hat{u} \equiv \hat{u}(y,u,k)$ are defined by formulas:

$$\hat{u}_i(y,u,k) = u_i \Delta(y,\hat{x})(k f_i(\hat{x}) + \Delta(y,\hat{x}))^{-1}, \; i=1,\ldots,m$$

Let $d_i(x, y, k) = k f_i(x) + \Delta(y, x)$ then we have the following formula for the Lagrange multipliers update

$$(5.3) \qquad \hat{u}_i(y,u,k) = u_i \Delta(y,\hat{x}) d_i^{-1}(\hat{x},y,k), \; i=1,\ldots,m$$

Formula (5.3) is critical for our further considerations.

First, we have $\hat{u}(u^*,y,k) = u^*$ for any fixed $y \in \operatorname{int} \Omega$ and $k > \Delta(y, x^*) k_0 + \Sigma u_i^*$, i.e. u^* is a fixed point of the map $u \to \hat{u}(u,y,k)$.

Second, we will show later that for the new vector $\hat{u}$ the following estimation:

$$(5.4) \qquad \| \hat{u} - u^* \| \le c k^{-1} \Delta(y,x^*) \| u - u^* \|$$

holds, and $c > 0$ is independent on $y \in \operatorname{int} \Omega$ and $k > 0$, where $\| x \| = \| x \|_\infty = \max_{1 \le i \le n} |x_i|$.

Third, it turns out that the estimation (5.4) is taking place not only for $\hat{u}$ but for the minimizer $\hat{x}$ as well, i.e.

$$(5.5) \qquad \| \hat{x} - x^* \| \le c k^{-1} \Delta(y,x^*) \| u - u^* \|$$

In other words, finding a minimizer $\hat{x}$ and updating the vector $u \in \mathbb{R}^m_{++}$ is equivalent to applying to $u \in \mathbb{R}^m_{++}$ an operator :

$$C_{y,k} : C_{y,k} u = \hat{u}(u,y,k) = \hat{u}$$

Note that $C_{y,k} u^* = u^*$. The operator $C_{y,k}$ is a contractive one if

$$\| C_{y,k} u - u^* \| = \| C_{y,k}(u - u^*) \| < \| u - u^* \|$$

The contractibility of $C_{y,k}$ is defined by

$$Contr\, C_{y,k} = \gamma_{y,k} = c k^{-1} \Delta(y,x^*)$$

The constant $c > 0$ depends on the input data and the size of a given problem and independent on y and k. We will characterize the constant $c > 0$ in the course of proving the basic theorem. So, for a given problem, the contractibility $0 < \gamma_{y,k} < 1$ depends on the "center" $y \in \operatorname{int} \Omega$ and the barrier parameter $k > 0$.

The independence c on y and k makes possible to reduce $\gamma_{y,k} > 0$ to any apriori given level by increasing $k > 0$ under the fixed y, or reducing $\Delta(y, x^*)$ under the fixed k or by changing both the "center" $y \in \text{int}\,\Omega$ and the barrier parameter $k > 0$ in the process of solution.

In particular, for any "center" $y \in \text{int}\,\Omega$ and any given $0 < \gamma < 1$, one can find such a barrier parameter $k > 0$ that the operator $C_{y,k}$ will shrink the distance between current approximation (x, u) and the primal dual solution (x^*, u^*) by a factor $0 < \gamma < 1$. Now we will describe the basic version of the Modified Interior Center Method. The convergence and rate of convergence will be considered later.

We start with $y \in \text{int}\,\Omega$, $u^0 = e_m = (1,...,1) \in \mathbb{R}^m$ and $k > m$. Let's assume that the couple (x^s, u^s) has been found already. Take $k > \Sigma u_i^s$, then the next approximation (x^{s+1}, u^{s+1}) we find by formula:

(5.6) $$x^{s+1} = \text{argmin}\,\{F(x,y,u^s,k)\,/\,x \in \mathbb{R}^n\}$$

(5.7) $$u^{s+1}:\ u_i^{s+1} = u_i^s\,\Delta(y,x^{s+1})\,d_i^{-1}(x^{s+1},y,k)\quad i=1,...,m$$

First, let us consider conditions for the problem (1.1), under which the method (5.6) - (5.7) is executable. To simplify our consideration, we assume without loosing the generality†

(A1) $$\inf_{x \in \mathbb{R}^n} f_0(x) > -\infty$$

We also assume that the set of optimal solutions for the problem (1.1) is not empty and bounded, i.e.

(A2) $$X^* = \text{Argmin}\,\{f_0(x)\,/\,x \in \Omega\} \neq \emptyset$$

is bounded.

Taking into account the Corollary 20 (see [Fiac M68] p 94) and assumptions A1 - A2, we conclude that the set $\Omega_k(y) = \{x : kf_i(x) + \Delta(y,x) \geq 0,\ i=1,...,m;\ \Delta(y,x) > 0\}$ is bounded for any $y \in \text{int}\,\Omega$ and $k > 0$. Also $x \to \partial\Omega_k(y) \Rightarrow F(x,y,u^s,k) \to \infty$,therefore for any $u^s \in \mathbb{R}^m_{++}$, $y \in \text{int}\,\Omega$ and $k > \Sigma u_i^s$ the function $F(x, y, u^s, k)$ is convex in $x \in \Omega_k(y)$, and the minimizer

$$x^{s+1} = \text{argmin}\,\{F(x, y, u^s, k)\,/\,x \in \mathbb{R}^n\}$$

exists. Therefore

(5.8) $$F_x'(x^{s+1},y,u^s,k) = (1 - k^{-1}\Sigma u_i^s + k^{-1}\Sigma u_i^{s+1})f_0'(x^{s+1}) - \Sigma u_i^{s+1} f_i'(x^{s+1}) = 0$$

and $u^s \in \mathbb{R}^m_{++} \Rightarrow u^{s+1} \in \mathbb{R}^m_{++}$.

Hence, starting with a vector $u^0 \in \mathbb{R}_{++}^m$ one can guarantee that the Lagrange multipliers will remain positive up to the end of the process without any particular care about it.

The convergence of the MCM (5.6) - (5.7) will follow from the Basic Theorem, which we will prove in the next section.

---

†If $\text{int} f_0(x) = -\infty$, like in case when $f_0(x)$ is linear, one can take $f_0(x) := e^{f_0(x)}$

6. Basic Theorem

The Basic Theorem establishes the contractibility properties of the operator $C_{y,k}$. We will start by characterizing the domain, where the operator $C_{y,k}$ is defined and possesses these properties. Let's consider a small enough number $\tau > 0$, a fixed y_0: $f_i(y_0) > \tau$ and a subset $\Omega_\tau = \{x : f(x) \geq \tau e\} \cap \{x : \Delta(y_0, x) > 0\}$ of the RFS $\Omega(y_0)$. Note that due to A1 - A2 and the Corollary 20 (see [FiacM68]) the set Ω_τ is bounded. We will choose the "center" y from Ω_τ. For any $y \in \Omega_\tau$ we have $\Delta(y, x^*) > 0$. Along with the fixed $\tau > 0$ we consider two fixed small numbers $\varepsilon > 0$ and $\delta > 0$ and a large fixed number $k_0 > 0$. In the course of proving the Basic Theorem it will become clear what "small" and "large" mean.

To characterize the domain, where the operator $C_{y,k}$ is defined, we will consider few sets, in the dual space. The sets are dependent on the fixed parameters τ, ε, δ and k_0. To simplify the notations we will omit the parameters in the notations.

We consider two types of sets. The first type

$$U^i_{y,k} = \{ u_i : u_i \geq \varepsilon,\ |u_i - u^*| \leq \delta\, \Delta^{-1}(y, x^*) k \},\ i = 1, \dots, r$$

is related to the active constraints set.

The second type

$$U^i_{y,k} = \{ u_i :\ 0 \leq u_i \leq \delta\, \Delta^{-1}(y, x^*) k \},\ i = r+1, \dots, m$$

is associated with the passive constraints.
If $y \in \Omega_\tau$ and $k \geq k_0 \Delta(y, x^*) + \Sigma u_i^*$ are fixed then the set $U_{y,k} = U^1_{y,k} \times \cdots, \times U^r_{y,k} \times \cdots \times U^m_{y,k}$ is the feasible set for the Lagrange multipliers.

We will prove in the Basic Theorem that the operator $C_{y,k}$ is a contractive one on the set

$$D_{k_0} = \{ u \in U_{y,k},\ k \geq k_0 \}$$

Before we turn to the Basic Theorem, let's briefly describe the main idea of the proof. In view of A1-A2 for any $u \in \mathbb{R}^m_{++}$, $y \in \operatorname{int} \Omega_\tau$ and $k > \Sigma u_i$, there exists the MIDF's minimizer $\hat{x} = \hat{x}(y, u, k)$ and

$$F'_x(\hat{x}, y, u, k) = f'_0(\hat{x}) - \sum_{i=1}^{r} \hat{u}_i f'_i(\hat{x}) - h(\hat{x}, y, u, k) + g(\hat{x}, y, u, k) = 0 \tag{6.1}$$

where

$$\hat{u}_i = u_i \Delta(y, \hat{x})\, d_i^{-1}(\hat{x}, y, k),\ i = 1, \dots, r, \tag{6.2}$$

$$h(\hat{x}, y, u, k) = \sum_{i=r+1}^{m} u_i \Delta(y, \hat{x})\, d_i^{-1}(\hat{x}, y, k) f'_i(\hat{x})$$

and

$$g(\hat{x}, y, u, k) = k^{-1} \sum_{i=1}^{m} u_i (-1 + \Delta(y, \hat{x})\, d_i^{-1}(\hat{x}, y, k)) f'_0(\hat{x})$$

Considering (6.1) and (6.2) as a system of equations for $\hat{x}$ and $\hat{u}_{(r)}$, it is easy to verify that $\hat{x} = x^*$ and $\hat{u}_{(r)} = u^*_{(r)}$ satisfy the system for any $y \in \Omega_\tau$, $k > \Sigma u_i^*$, and $u = u^*$ Moreover for any triple $(y, u, k) \in D_{k_0}$, the system (6.1)-(6.2) can be solved for $\hat{x}$ and $\hat{u}_{(r)}$.

Having the solution $\hat{x} = \hat{x}(y, u, k)$ and $\hat{u}_{(r)} = \hat{u}_{(r)}(y, u, k)$ one can find the Jacobians

$\hat{x}_u^{/}(\bullet) = J_u(\hat{x}(y,u,k))$ and $\hat{u}_{(r)u}^{/}(\bullet) = J_u(\hat{u}_{(r)}(y,u,k))$ and estimate $\| \hat{x}_u^{/}(y,u^*,k) \|$ and $\| \hat{u}_{(r)u}^{/}(y,u^*,k) \|$.

It turns out that under second order optimality condition, there is such $k_0 > 0$ that for any $y \in \Omega_\tau$ and $k \geq k_0 \Delta(y,x^*) + \Sigma\, u_i^*$ the following estimation

$$(6.3) \qquad \max\{ \| \hat{x}_u^{/}(y,u^*,k) \| \ , \ \| u_{(r)u}^{/}(y,u^*,k) \| \} \leq c$$

takes place and $c > 0$ is independent on y and k.

Due to the continuity $\hat{x}_u^{/}(\bullet)$ and $\hat{u}_{(r)u}^{/}(\bullet)$ in u the estimation (6.3) is taking place in the neighborhood of u^*.

In view of $x^* = \hat{x}(y,u^*,k)$ and $u^* = \hat{u}(y,u^*,k)$ and using (6.3) one can estimate $\| \hat{x} - x^* \|$ and $\| \hat{u} - u^* \|$ through $\| u - u^* \|$.

The independence $c > 0$ on y and k makes possible to prove that for any fixed $y \in \Omega_\tau$ there exists $k_0 > 0$ such that for any $k \geq k_0 \Delta(y,x^*) + \Sigma\, u_i^*$ the operator $C_{y,k}$ is a contractive one, i.e. $0 < \gamma_{y,k} < 1$, therefore $u \in U_{y,k} \rightarrow C_{y,k}\, u = \hat{u} \in U_{y,k}$.

In the course of proving the Basic Theorem we will assume $\min\{ f_i(x^*) / \ i = r+1,...,m \} = \sigma > 0$, $\min\{ \Delta(y,x^*) / y \in \Omega_\tau \} = \tau_0 > 0$, $O^{p,q}$ be the $p \times q$ zero matrix; I^r be the $r \times r$ identity matrix, $S(a,\varepsilon) = \{ x \in \mathbb{R}^n : \| x - a \| \leq \varepsilon \}$. We remind that $d_i(x,y,k) = (k f_i(x) + \Delta(y,x))$ and introduce three diagonal matrices $d(x, y, k) = [\mathrm{diag}\, d_i(x,y,k)]_{i=1}^m$, $d_{(r)}(x,y,k) = [\mathrm{diag}\, d_i(x,y,k)]_{i=1}^r$, $d_{(m-r)}(x,y,k) = [\mathrm{diag}\, d_i(x,y,k)]_{i=r+1}^m$.

Theorem 2. 1) *If* A1 - A2 *are taking place, then for any* $y \in \Omega_\tau$, $u \in \mathbb{R}^m_{++}$ *and* $k > \Sigma\, u_i$ *there exists* $\hat{x} = \hat{x}(y,u,k) = \mathrm{argmin}\{ F(x,y,u,k) / x \in \mathbb{R}^n \} : F_x^{/}(\hat{x},y,u,k) = 0^n$.

2) *If* $f_i(x) \in C^2$, $i = 0,...,m$ *and standard second order optimality conditions* (1.4)-(1.5) *are taking place then*:

a) *for any triple* $(y,u,k) \in D_{k_0}$ *the minimizer* $\hat{x} = \hat{x}(y,u,k)$ *exists,* $F_x^{/}(\hat{x},y,u,k) = 0^n$ *and for the pair* $\hat{x}$ *and* $\hat{u} = \hat{u}(y,u,k)$ *the following estimate*

$$(6.4) \qquad \max\{ \| \hat{x} - x^* \| \ , \ \| \hat{u} - u^* \| \} \leq c k^{-1} \Delta(y,x^*) \| u - u^* \|$$

holds and $c > 0$ *is independent on* y *and* k.

b) *for any fixed* $y \in \Omega_\tau$ and $k > k_0 \Delta(y,x^*) + \Sigma\, u_i^*$ *,the MIDF* $F(x, y, u, k)$ *is strongly convex in the neighborhood of* $\hat{x}$ *and there exists* $\hat{\mu} > 0$ *and* $\hat{M} < \infty$ *independent on* $u \in U_{y,k}$ *that*

$$(6.5) \qquad \text{mineigval} \ \ F_{xx}^{//}(\hat{x},y,u,k) \geq \Delta^{-1}(y,x^*)\, \hat{\mu}$$

$$(6.6) \qquad \text{maxeigval} \ \ F_{xx}^{//}(\hat{x},y,u,k) \leq \Delta^{-1}(y,x^*)\, \hat{M}$$

Proof 1) In view of the assumptions A1 - A2 and the Corollary 20 (see [Fiac M68] p94) the set $\Omega_k(y) = \{ x : k f_i(x) + \Delta(y,x) \geq 0, i = 1,...,m; \Delta(y,x) > 0 \}$ is bounded for any $y \in \Omega_\tau$ and $k > 0$. Also $x \rightarrow \partial\, \Omega_k(y) \rightarrow F(x, y, u^s, k) \rightarrow \infty$, therefore for any $y \in \Omega_\tau$, $u \in \mathbb{R}^m_{++}$ and $k > \Sigma\, u_i$ the function $F(x, y, u, k)$ is convex in $x \in \Omega_k(y)$ and $\hat{x} = \hat{x}(y,u,k)$ is an unconstrained minimizer of $F(x, y, u, k)$, i.e. $F_x^{/}(\hat{x},y,u,k) = 0^n$.

2) For technical reasons, we introduce a vector $t = (t_1,...,t_m)$, $t_i = k^{-1} \Delta(y,x^*)(u_i - u_i^*)$ instead of the vector of Lagrange multipliers u,then $u = u^* \rightarrow t = 0^m$. Such transformation

translates the neighborhood of u^* into the neighborhood $S(0,\delta)=\{t: |t_i| \le \delta,\ i=1,...,m\}$ of the origin of dual space.

We will split the vector $\hat{u}$ on two parts, which correspond to the active and passive constraints.

Let $\hat{u}_{(r)}=(\hat{u}_i,\ i=1,...,r)$ is a vector of Lagrange multiplier, which corresponds to the active constraints, while $\hat{u}_{(m-r)}=\hat{u}_{(m-r)}(x,y,t,k)=(\hat{u}_i(x,y,t,k),\ i=r+1,...,m)$ is the vector of Lagrange multipliers, which corresponds to the passive contraints.

We have $\hat{u}_i(x,y,t,k)=k\Delta^{-1}(y,x^*)t_i\Delta(y,x)d_i^{-1}(x,y,k),\ i=r+1,...,m,\ \hat{u}=(\hat{u}_{(r)},\hat{u}_{(m-r)})$ and for the vector-function $h(x, y, u, k)$, $g(x, y, u, k)$ we will have the following replacement.

$$h(x,y,t,k)=\sum_{i=r+1}^{m}\hat{u}_i(x,y,t,k)\,(f_i'(x))^T=(\hat{u}_{(m-r)}(x,y,t,k)f'_{(m-r)}(x))^T,$$

$$g(x,y,t,k)=k^{-1}\left\{\sum_{i=1}^{m}(kt_i\Delta^{-1}(y,x^*)+u_i^*)\,[-1+\Delta(y,x)\,d_i^{-1}(x,y,k)]\right\}(f_0'(x))^T$$

So for any $k>0$, small enough $\varepsilon_0>0$ and $y\in\Omega_\tau$ the vector functions $h(x, y, t, k)$ and $g(x, y, t, k)$ are smooth in $x\in S(x^*,\varepsilon_0)$ and $t\in S(0,\delta)$. Then we have $h(x^*,y,0,k)=0^n$, $g(x^*,y,0,k)=0^n$, $h_x'(x^*,y,0,k)=0^{n,n}$, $g_x'(x^*,y,0,k)=-\Delta^{-1}(y,x^*)f_0'^T(x^*)f_0'(x^*)$, also $h'_{\hat{u}_{(r)}}(x^*,y,0,k)=0^{n,r}$ and $g'_{\hat{u}_{(r)}}(x^*,y,0,k)=0^{n,r}$. On $S(x^*,\varepsilon_0)\times S(u^*_{(r)},\varepsilon_0)\times\Omega_\tau\times S(0,\delta)\times(0,+\infty)$ we consider the map $\Phi(x,\hat{u}_{(r)},y,t,k):\mathbf{R}^{2n+r+m+1}\to\mathbf{R}^{n+r}$ defined as follows:

$$\Phi(x,\hat{u}_r,y,t,k)=\{f_0'^T(x)-\sum_{i=1}^{r}\hat{u}_i f_i'^T(x)-h(x,y,t,k)+g(x,y,t,k);$$
$$k^{-1}\Delta(y,x^*)\,[\,(k\Delta^{-1}(y,x^*)t_i+u_i^*)\,\Delta(y,x)\,d_i^{-1}(x,y,k)-\hat{u}_i\,]\ ,\ i=1,...,r\}$$

Taking into account (1.3) and $h(x^*, y, 0, k)=g(x^*, y, 0, k)=0^n$ we obtain

$$\Phi(x^*,u^*_{(r)},y,0,k)=0^{n+r}\ \text{ for any } k>0 \text{ and } y\in\Omega_\tau .$$

Let $\Phi'_{x\hat{u}_{(r)}}\equiv\Phi'_{x\hat{u}_{(r)}}(x^*,u^*_{(r)},y,0,k)$, $L''_{xx}=L''_{xx}(x^*,u^*)$, $f'=f'(x^*)$, $f'_{(r)}\triangleq f'_{(r)}(x^*)$, $U_r^*=[\,\mathrm{diag}\,u_i^*\,]_{i=1}^{r}$, $u^*_{(r)}=(u_i^*,\ i=1,...,r)$.

In view of $h_x'(x^*,y,0,k)=0^{n,n}$, $h'_{\hat{u}_{(r)}}(x^*,y,0,k)=0^{n,r}$, $g_x'(x^*,y,0,k)$ $=-\Delta^{-1}(y,x^*)f_0'^T(x^*)f_0'(x^*)$, $g'_{\hat{u}_{(r)}}(x^*,y,0,k)=0^{n,r}$ we obtain

$$\Phi_{(y,k)}\equiv\Phi'_{x\hat{u}_{(r)}}(x^*,u^*_{(r)},0,y,k)=\begin{bmatrix} L''_{xx}-\Delta^{-1}(y,x^*)f_0'^Tf_0' & -f'^T_{(r)}\\ -U_r^*f'_{(r)} & -k^{-1}\Delta(y,x^*)I^r\end{bmatrix}$$

Now we will prove the nondegeneracy of the matrix $\Phi_{(y,k)}$ for any $y\in\operatorname{int}\Omega_\tau$ and any $k\ge k_0\Delta(y,x^*)+\sum u_i^*$. Let us consider $w=(z,v)\in R^{n+r}$, then the system $\Phi_{(y,k)}w=0^{n+r}$ can be rewritten as follows:

$$L''_{xx}z-\Delta^{-1}(y,x^*)f_0'^Tf_0'z-f'^T_{(r)}v=0^n \tag{6.7}$$

$$U_r^*f'_{(r)}z-k^{-1}\Delta(y,x^*)v=0^r \tag{6.8}$$

We find v from (6.8) and substitute in (6.7). Taking into account the K-K-T's condition $f'_0 = u^*_{(r)} f'_{(r)}$ we obtain

$$L''_{xx} z + \Delta^{-1}(y,x^*)\,[\,k f'^T_{(r)} U^*_r f'_{(r)} - (f'^T_{(r)} u^{*T}_{(r)} u^*_{(r)} f'_{(r)})\,]\, z = 0^n$$

i.e.

$$(L''_{xx} z, z) + \Delta^{-1}(y,x^*)\,[\,k(U^*_r f'_{(r)} z, f'_{(r)} z) - (u^*_{(r)} f'_{(r)} z)^2\,] = 0$$

The inequality

$$k \geq k_0 \Delta(y,x^*) + \Sigma\, u^*_i$$

implies

$$\begin{aligned}&(L''_{xx} z, z) + \Delta^{-1}(y,x^*)\,[\,k(U^*_r f'_{(r)} z, f'_{(r)} z) - (u^*_{(r)} f'_{(r)} z)^2\,]\\ &\geq (L''_{xx} z, z) + \Delta^{-1}(y,x^*)\,[\,k_0 \Delta(y,x^*)\,(U^*_r f'_{(r)} z, f'_{(r)} z)\\ &\quad + (\Sigma\, u^*_i)((L''_{xx} z, z) + \Delta^{-1}(y,x^*)\,[\,k(U^*_r f'_{(r)} z, f'_{(r)} z) - (u^*_{(r)} f'_{(r)} z)^2\,]\\ &= (L''_{xx} z, z) + k_0 (f'^T_{(r)} U^*_{(r)} f'_{(r)} z, z) + \Delta^{-1}(y,x^*)\,[\,(\Sigma u^*_i)(\Sigma u^*_i (f'_i, z)^2) - (\Sigma u^*_i (f'_i, z))^2\,]\end{aligned}$$

Due to identity (4.1) we obtain $(\Sigma u^*_i)(\Sigma u^*_i (f'_i, z)^2) - (\Sigma u^*_i (f'_i, z))^2 \geq 0$. Therefore taking into account Assertion 1, we have $0 = ((L''_{xx} + k_0 f'^T_{(r)} U^*_{(r)} f'_{(r)}) z, z) \geq \mu(z,z)$, $\mu > 0$, i.e. $z = 0^n$, hence from (6.7) we obtain $f'^T_{(r)} v = 0^n$, so due to (1.4) we have v = 0, i.e. $\Phi_{(y,k)} w = 0^{n+r} \rightarrow w = 0^{n+r}$, i.e. the matrix $\Phi_{(y,k)}$ is a nonsingular matrix.

Let k_1 be large enough. We consider a compact

$$K = \{0^n\} \times \{(y,k) : y \in \Omega_\tau,\; k_1 \geq k \geq k_0 \Delta(y,x^*) + \Sigma\, u^*_i\}$$

Since $\Phi(x^*, \hat{u}_{(r)}, y, 0, k) = 0^{n+r}$, the matrix $\Phi_{(y,k)} = \Phi_{x\hat{u}_{(r)}}(x^*, u^*_{(r)}, y, 0, k)$ is nonsingular, $f_i(x) \in C^2$, $i = 1,...,m$ and K is compact, it follows from the second implicit function theorem (see[Ber82]p.12) that there is a small enough $\delta > 0$ that in the neighborhood $S(K,\delta) = \{(y,t,k) : |t_i| \leq \delta,\ i = 1,...,m, y \in \Omega_\tau, k \in [k_0 \Delta(y,x^*) + \Sigma u^*_i, k_1]\}$ of the compact K there exist unique continuously differentiable vector-functions $x(\bullet) = x(y,t,k) = (x_1(y,t,k),...,x_n(y,t,k))$ and $\hat{u}_{(r)}(\bullet) = \hat{u}_{(r)}(y,t,k) = (\hat{u}_1(y,t,k),...,\hat{u}_r(y,t,k))$ such that $x(y,0,k) = x^*, \hat{u}_{(r)}(y,0,k) = u^*_{(r)}$ and for any triple $(y, t, k) \in S(K, \delta)$ there is $\varepsilon_0 > 0$ that

(6.9) $$\max\,\{\,\| x(y,t,k) - x^* \|\,,\ \| \hat{u}_{(r)}(y,t,k) - u^*_{(r)} \|\,\} \leq \varepsilon_0$$

The identity

(6.10) $$\Phi(x(y,t,k), \hat{u}_{(r)}(y,t,k), y, t, k) = \Phi(x(\bullet), \hat{u}_{(r)}(\bullet), \bullet) \equiv 0^{n+r}$$

holds true for all $(y, t, k) \in S(K, \delta)$.

So we obtain

(6.11) $$\begin{aligned}&f'^T_0(x(\bullet)) - \sum_{i=1}^{r} \hat{u}_i(\bullet) f'^T_i(x(\bullet)) - h(x(\bullet),\bullet) + g(x(\bullet),\bullet)\\ &\equiv \Delta(y, x(\bullet)) F'_x(x(\bullet), y, u, k) \equiv 0^n\end{aligned}$$

which due to $\Delta(y, x(\bullet)) > 0$ is the necessary optimality condition for the vector $x(\bullet)$ to be a minimizer of the function $F(x,y,u,k)$ in x under the fixed (y,u,k). Also from (6.10) we obtained the identities

$\hat{u}_i(\bullet) \equiv (k\Delta^{-1}(y,x^*)t_i + u_i^*)\,\Delta(y,x(\bullet))d_i^{-1}(x(\bullet),y,k),\ i=1,...,r$ for the Lagrange multipliers that corresponds to the active contraints. After multiplying both sides by $k^{-1}\Delta(y, x^*)$, it can be rewritten as follows:

$$(6.12)\quad (t_i + k^{-1}\Delta(y,x^*)u_i^*)\,\Delta(y,x(\bullet))\,d_i^{-1}(x(\bullet),y,k) - k^{-1}\Delta(y,x^*)\hat{u}_i(\bullet) = 0\ \ i=1,...,r$$

The Lagrange multipliers that correspond to the passive constraints , we can rewrite in the following way:

$$(6.13)\qquad \hat{u}_i(x(\bullet),\bullet) \equiv \hat{u}_i(\bullet) = k\Delta^{-1}(y,x^*)t_i\,\Delta(y,x(\bullet))\,d^{-1}(x(\bullet),y,k),\ \ i=r+1,...,m$$

Let $\hat{u}_{(m-r)}(\bullet) = (\hat{u}_i(\bullet),\ i=r+1,...,m)$ and $\hat{u}(\bullet) = (\hat{u}_{(r)}(\bullet);\ \hat{u}_{(m-r)}(\bullet))$. To prove the sufficient optimality condition for the vector $x(\bullet)$ to be a minimizer of the function $F(x,y,u,k)$ in x under fixed (y,u,k) we will show later that the function $F(x,y,u,k)$ is strongly convex in the neighborhood of $x(\bullet)$ for any $(y, u, k) \in D_{k_0}$. But first of all we will ascertain the estimation (6.4).

First, let us prove that for small enough $\delta > 0$ and large enough k_0 there exists $\rho > 0$ such that the inequality

$$(6.14)\qquad \| (\Phi'_{x,\hat{u}_{(r)}}(x(\bullet),\hat{u}_{(r)}(\bullet),\bullet))^{-1} \| \le \rho$$

holds true for all $(t, y,k) \in S(K, \delta)$.
We consider the matrix

$$\Phi_{(y,\infty)} = \Phi'_{x,\hat{u}_{(r)}}(x^*, u^*_{(r)}, 0, y, \infty) = \begin{bmatrix} L''_{xx} - \Delta^{-1}(y,x^*)f'^T_0 f'_0 & -f'^T \\ -U^*_r f'_{(r)} & 0^{r,r} \end{bmatrix}$$

The matrix $\Phi_{(y,\infty)}$ is nonsingular for any $y \in \Omega_\tau$. In fact, for a vector $w = (z, v) \in \mathbf{R}^{n+r}$ the system

$$\Phi_{(y,\infty)}\,w = 0^{n+r}$$

can be rewritten in the following way

$$(6.15)\qquad L''_{xx}z - \Delta^{-1}(y,x^*)f'^T_0 f'_0 z - f'^T_{(r)}v = 0^n$$

$$(6.16)\qquad -U^*_r f'_{(r)}z = 0^r$$

Because of $u^*_{(r)} > 0^r$ from (6.16) we obtain $f'_{(r)}z = 0^r$, i.e. $(f'_i, z) = 0,\ i=1,...,r$, therefore $(\sum_{i=1}^r u_i^* f'_i, z) = (f'_0, z) = 0$. Multiplying (6.15) by z we obtain

$$(6.17)\qquad (L''_{xx}z, z) - \Delta^{-1}(y,x^*)(f'_0,z)^2 - (v, f'_{(r)}z) = 0$$

i.e.

$$(L''_{xx}z, z) = 0,\ \forall z : f'_{(r)}z = 0^r$$

so due to (1.5) we have $z = 0^n$ then from (6.14) one obtains $f'^T_{(r)}v = 0^n$, which due to (1.4) implies $v = 0^r$.

Therefore, $\Phi_{(y,\infty)}w = 0^{n+r}$ implies $w = 0^{n+r}$ for any $y \in \Omega_\tau$ i.e. the matrix $\Phi_{(y,\infty)}$ is nonsingular, so there exists a constant $\rho > 0$ independent of k and $y \in \Omega_\tau$ such that

$$\| \Phi^{-1}_{(y,\infty)} \| \le \rho_0 .$$

Hence, for the Gram matrix $G_{(y,\infty)} = \Phi^T_{(y,\infty)} \Phi_{(y,\infty)}$ we have mineigval $G_{(y,\infty)} = \mu_0 > 0$. Then there exists a large enough $k_0 > 0$ such that for any $y \in \Omega_\tau$ and $k \ge k_0\Delta(y,x^*) + \Sigma u_i^*$ i.e. $k^{-1}\Delta(y,x^*) < k_0^{-1}$ we obtain for the Gram matrix $G_{(y,k)} = \Phi^T_{(y,k)} \Phi_{(y,k)}$ the inequality

$$\text{mineigval } G_{(y,k)} \ge ½\, \mu_0$$

and $\mu_0 > 0$ is independent of $y \in \Omega_\tau$ and $k \in [\, k_0\Delta(y,x^*) + \Sigma u_i^*, k_1 \,]$. Therefore $\Phi_{(y,k)}$ is not only nonsingular, but there exists a constant $\rho > 0$ independent of $y \in \Omega_\tau$ and $k \ge k_0\Delta(y,x^*) + \Sigma u_i^*$ such that

$$(6.18) \qquad \| \Phi^{-1}_{(y,k)} \| = \| \Phi'^{-1}_{x,\hat u_{(r)}} (x(y,0,k)\,;\, \hat u_r(y,0,k)) \| \le \rho$$

The last inequality implies (6.14) if $\delta > 0$ is small enough. Now we will prove estimation (6.4). First, let us estimate the norm $\| \hat u_{(m-r)}(\bullet) \|$.
Due to (6.9) for any small enough $\delta > 0$ there exists such small enough $\varepsilon_0 > 0$ that for $\forall\, (y, t, k) \in S(K,\delta)$ $\max\{ |f_0(x(y,t,k)) - f_0(x(y,0,k))|, |f_i(x(y,t,k)) - f_i(x(y,0,k))|,$ $i = r+1,\dots,m \} \le \varepsilon_0$.
Therefore, in view of $f_i(x^*) \ge \sigma > 0$ for the passive constraints, we obtain $f_i(x(y,t,k)) \ge \frac{\sigma}{2} > 0$, $i = r+1,\dots,m$ and for the Lagrange multipliers, that correspond to the passive constraints we have

$$\begin{aligned}
u_i(y,t,k) &= u_i(f_0(y) - f_0(x(y,t,k)))\,(kf_i(x(y,t,k)) + f_0(y) - f_0(x(y,t,k)))^{-1} \\
&= u_i(f_0(y) - f_0(x^*) + f_0(x^*) - f_0(x(y,t,k))[kf_i(x^*) - k(f_i(x^*) - f_i(x(y,t,k))) \\
&\qquad + f_0(y) - f_0(x^*) - (f_0(x(y,t,k) - f_0(x^*))]^{-1} \\
&\le u_i(f_0(y) - f_0(x^*) + \varepsilon_0)(kf_i(x^*) - (k+1)\varepsilon_0 + f_0(y) - f_0(x^*))^{-1} \\
&\le \frac{2u_i}{k}(f_0(y) - f_0(x^*) + \varepsilon_0)\,(f_i(x^*) - k^{-1}(k+1)\varepsilon_0 + k^{-1}(f_0(y) - f_0(x^*)))^{-1}.
\end{aligned}$$

Hence, for small enough $\varepsilon_0 < \frac{\sigma}{2} k(k+1)^{-1}$ we obtain

$$u_i(\bullet) \equiv u_i(y,t,k) \le \frac{2u_i\Delta(y,x^*)}{k[\sigma - \frac{\sigma}{2} + \Delta(y,x^*)k^{-1}]} \le \frac{4u_i\Delta(y,x^*)}{k\sigma}\ , \quad i = r+1,\dots,m$$

So we have

$$\| u_{(m-r)}(y,t,k) - u^*_{(m-r)} \| \le \frac{4}{\sigma} k^{-1}\Delta(y,x^*) \| u_{(m-r)} - u^*_{(m-r)} \|$$

Now we will show that the estimation (6.4) holds for $\hat x(y,t,k)$ and $\hat u_{(r)}(y,t,k) = (\hat u_i(y,t,k),\, i = 1,\dots,r)$. To this end we differentiate the identities (6.11) and (6.12) with respect to t.
Let $x'_t(\bullet) = J_t(x(\bullet)) = (x'_{j,t}(\bullet), j = 1,\dots,n)$, $\hat u'_{(r),t}(\bullet) = J_t(u_{(r)}(\bullet)) = (\hat u'_{i,t}(\bullet), i = 1,\dots,r)$ are the Jacobians of the vector functions $x(\bullet)$ and $\hat u_{(r)}(\bullet)$. Also $L(x(\bullet), \hat u_r(\bullet)) = f_0(x(\bullet)) - \Sigma^r_{i=1} \hat u_i(\bullet) f_i(x(\bullet))$ and let $J_t(h(x(\bullet),\bullet))$ and $J_t(g(x(\bullet),\bullet))$ are the Jacobians of the

vector functions $h(x(\bullet),\bullet)$ and $g(x(\bullet),\bullet)$. Then differentiating (6.11) with respect to t we obtain:

$$\bar{L}''_{xx}(x(\bullet),\hat{u}_{(r)}(\bullet))x'_t(\bullet) - (f'_{(r)}(x(\bullet))^T\hat{u}'_{(r),t}(\bullet) - J_t(h(x(\bullet),\bullet)) + J_t(g(x(\bullet),\bullet) \equiv 0 \tag{6.19}$$

Let $F_{(r)}(x(\bullet)) = r\begin{bmatrix} f'_0(x(\bullet)) \\ \cdots \\ f'_0(x(\bullet))\end{bmatrix}^{n}$, $\Psi(x(\bullet),y,t,k) = [\,\mathrm{diag}(t_i + k^{-1}u_i^*\Delta(y,x^*))]_1^r \times$

$$[-d_{(r)}^{-1}(x(\bullet),y,k)F_{(r)}(x(\bullet)) - k\Delta(y,x(\bullet))d_{(r)}^{-2}(x(\bullet),y,k)f'_{(r)}(x(\bullet)) + \Delta(y,x(\bullet))\bullet d_{(r)}^{-2}(x(\bullet),y,k)\,F_{(r)}(x(\bullet))]].$$

Differentiating (6.12) with respect to t we obtain

$$\Psi(x(\bullet),y,t,k)x'_t(\bullet) - k^{-1}\Delta(y,x^*)\hat{u}'_{(r),t}(\bullet) \equiv -[\Delta(y,x(\bullet))\bullet d_{(r)}^{-1}(x(\bullet),y,k)\,;0^{r,m-r}] \equiv S(x(\bullet),\bullet) \tag{6.20}$$

Now let us consider the Jacobians

$$J_t(h(x(\bullet),\bullet)) = h'_x(x(\bullet),\bullet)x'_t(\bullet) + h'_t(x(\bullet),\bullet) = N(x(\bullet),\bullet)x'_t(\bullet) + q(x(\bullet),\bullet)$$

and

$$J_t(g(x(\bullet),\bullet)) = g'_x(x(\bullet),\bullet)x'_t(\bullet) + g'_t(x(\bullet),\bullet) = G(x(\bullet),\bullet)x'_t(\bullet) + p(x(\bullet),\bullet)$$

We also consider the matrix

$$\Phi'_{x,\hat{u}_{(r)}}(x(\bullet),\hat{u}_{(r)}(\bullet),\bullet) \equiv \Phi'_{x,\hat{u}_{(r)}}(\bullet) =$$

$$\begin{array}{c} \\ n \\ r \end{array}\begin{bmatrix} \overset{n}{L''_{xx}(x(\bullet),\hat{u}_{(r)}(\bullet)) + G(x(\bullet),\bullet) + N(x(\bullet),\bullet)} & \overset{r}{-f'^T_{(r)}(x(\bullet))} \\ \Psi(x(\bullet),\bullet) & -k^{-1}\Delta(y,x^*)I^r \end{bmatrix}$$

Then combining (6.19) and (6.20) we obtain

$$\begin{array}{c} n \\ r \end{array}\begin{bmatrix} \overset{m}{x'_t(\bullet)} \\ \hat{u}'_{(r),t}(\bullet)\end{bmatrix} = (\Phi'_{x,\hat{u}_{(r)}}(\bullet))^{-1} \begin{array}{c} n \\ r \end{array}\begin{bmatrix} \overset{m}{q(x(\bullet),\bullet) - p(x(\bullet),\bullet)} \\ S(x(\bullet),\bullet)\end{bmatrix} = \Phi'_{x,\hat{u}_{(r)}}(\bullet))^{-1}R(x(\bullet),\bullet) \tag{6.21}$$

Now we consider the system (6.21) for $t = 0^m$. Taking into account $x(y,0,k) = x^*$, $\hat{u}_{(r)}(y,0,k) = u^*_{(r)}$, $\hat{u}_{(m-r)}(y,0,k) = 0^{m-r}$

$$\bar{L}''_{xx}(x(y,0,k),\hat{u}_{(r)}(y,0,k) = L''_{xx}(x^*,u^*),\ G(x(y,0,k),y,0,k)$$
$$= -(f_0(y) - f_0(x^*))^{-1} f'^T_0(x^*) f'_0(x^*),\ N(x(y,0,k),y,0,k) = 0^{n,n}$$

$$\Psi(x(y,0,k);y,0,k) = -U^*_r f'_{(r)}(x^*),\ q(x(y,0,k);y,0,k) = q(x^*,y,0,k)$$
$$= k^{-1}[0^{n,r}; f'^T_{(m-r)}(x^*)\, d^{-1}_{(m-r)}(x^*,y,k)],\ p(x(y,0,k),y,0,k) = p(x^*,y,0,k) =$$
$$[0^{n,r}; F^T_{(m-r)}(x^*)[\operatorname{diag} f_i(x^*) d_i^{-1}(x^*,y,k)]^m_{r+1}],$$
$$S(x(y,0,k),y,0,k) = S(x^*,y,0,k) = (-I^r; 0^{r,m-r})$$

we obtain the following system

$$(6.22)\quad \begin{bmatrix} x'_t(y,0,k) \\ \hat{u}'_{(r),t}(y,0,k) \end{bmatrix} = \Phi^{-1}_{(y,k)} \begin{bmatrix} q(x^*,y,0,k) - p(x^*,y,0,k) \\ S(x^*,y,0,k) \end{bmatrix} = \Phi^{-1}_{(y,k)} \bullet \mathbf{R}(x^*,y,0,k).$$

Therefore

$$(6.23)\qquad \max\{\| x'_t(y,0,k)\|, \| \hat{u}'_{(r),t}(y,0,k)\|\} \le \|\Phi^{-1}_{(y,k)}\|\, \|R(x^*,y,0,k)\|.$$

Taking into account $\min\{\Delta(y,x^*)/\ y \in \Omega_\tau\} = \tau_0 > 0$,

$$\|[\operatorname{diag}(f_i(x^*) + k^{-1}\Delta(y,x^*))^{-1}]^m_{r+1}\| \le \sigma^{-1},$$
$$\|[\operatorname{diag}(f_i(x^*)(f_i(x^*) + k^{-1}\Delta(y,x^*))^{-1}]^m_{r+1}\| \le 1,$$

one obtains

$$\|q(x^*,t,0,k)\| \le \sigma^{-1}\|f'^T_{(m-r)}(x^*)\|,\ \|p(x^*,y,0,k)\| \le \tau_0^{-1}\|F^T_{(m-r)}(x^*)\|,$$
$$\|S(x^*,0,y,k)\| \le 1 \text{ and } \|R(x^*,y,0,k)\| \le \sigma^{-1}\|f'^T_{(m-r)}(x^*)\| + \tau_0^{-1}\|F^T_{(m-r)}(x^*)\| + 1.$$

In view of (6.18) and (6.23) we have

$$\max\{\|x'_t(y,0,k)\|, \|\hat{u}'_{(r),t}(y,0,k)\|\} \le \rho(1 + \sigma^{-1}\|f'^T_{(m-r)}(x^*)\| + \tau_0^{-1}\|F^T_{(m-r)}(x^*)\|) = c_0$$

So there exists a small enough $\delta > 0$ such that for any $(y,\ t,\ k) \in S(K,\ \delta)$ the inequality

$$(6.24)\quad \begin{aligned} &\|(\Phi'_{x,\hat{u}_{(r)}}(x(y,\alpha t,k), \hat{u}_{(r)}(y,\alpha t,k); y,\alpha t,k))^{-1} R(x(y,\alpha t,k); y,\alpha t,k)\| \\ &\le 2\rho(1 + \sigma^{-1}\|f'^T_{(m-r)}(x^*)\| + \tau_0^{-1}\|F_{(m-r)}(x^*)\|) = c_0 \end{aligned}$$

holds true for any $0 \le \alpha \le 1$. Also we have

$$(6.25)\qquad \begin{bmatrix} x(y,t,k) - x(y,0,k) \\ \hat{u}_r(y,t,k) - \hat{u}_r(y,0,k) \end{bmatrix} =$$

$$\int_1^0 \Phi'^{-1}_{x,\hat{u}_{(r)}}(x(y,\alpha t,k), \hat{u}_{(r)}(y,\alpha t,k), y,\alpha t,k) \bullet R(x(y,\alpha t,k); y,\alpha t,k)[t]\, d\alpha.$$

From (6.24) and (6.25) we obtain

$$\max\{\|x(y,t,k) - x^*\|, \|\hat{u}_{(r)}(y,t,k) - u^*_{(r)}\|\} \le c_0\|t\| = c_0 k^{-1}\Delta(y,x^*)\|u - u^*\|.$$

Let $\hat{x}(y,u,k)=x(y,k^{-1}\Delta(y,x^*)(u-u^*),k), \hat{u}(y,u,k)=(u_{(r)}(y,k^{-1}\Delta(y,x^*)(u-u^*),k),$ $\hat{u}_{(m-r)}(y,k^{-1}\Delta(y,x^*)(u-u^*),k))$ and $c=\max\{c_0,4\sigma^{-1}\}$, then $\max\{\|\hat{x}(y,u,k)-x^*\|,\|\hat{u}(y,u,k)-u^*\|\}\le ck^{-1}\Delta(y,x^*)\|u-u^*\|$.

So we ascertained the estimation (6.4). Also $\hat{x}(y,u^*,k)=x^*$ and $\hat{u}(y,u^*,k)=u^*$ follows from (5.3) for any triple $(y, u^*, k)\in D_{k_0}$ i.e. u^* is the fixed point of the mapping $u\to\hat{u}(y,u,k)$.

3) Now we will prove that $F(x, y, u, k)$ is strongly convex in a neighborhood of $\hat{x}=\hat{x}(y,u,k)$ for any $(y,u,k)\in D_{k_0}$.

Using the formula for $F''_{xx}(\hat{x},y,u,k)$ (see Appendix A2) and taking into account the estimation (6.4) we obtain for a small enough δ and for any triple $(y, u, k)\in D_{k_0}$ that

$$F''_{xx}(\hat{x},y,u,k)\approx\Delta^{-1}(y,x^*)[L''_{xx}(x^*,u^*)+\Delta^{-1}(y,x^*)(kf'^T_{(r)}(x^*)U^*_rf'_{(r)}(x^*)-f'^T_0(x^*)f'_0(x^*))$$
$$+(k\Delta(y,x^*))^{-1}(\Sigma(u_i^*-u_i))f'^T_0(x^*)f'_0(x^*)+k^{-1}(\Sigma(u_i^*-u_i))f''_0(x^*)].$$

For any triple $(y, u, k)\in D_{k_0}$ we have $k\ge k_0\ \Delta(y,x^*)+\ \Sigma u_i^*$, and $k^{-1}|u_i-u_i^*|\le$ $\delta\Delta^{-1}(y,x^*)$, $i=1,\dots,m$. Keeping in mind min $\{\Delta(y, x^*)\mid y\in\Omega_\tau\}=\tau_0>0$ for any v $\in\mathbb{R}^n$ we obtain

$$(F''_{xx}(\hat{x},y,u,k)\mathrm{v},\mathrm{v})\ge\Delta^{-1}(y,x^*)[((L''_{xx}(x^*,u^*)+k_0f'^T_{(r)}U^*_rf'_{(r)}(x^*))\mathrm{v},\mathrm{v})]$$
$$\Delta^{-1}(y,x^*)(((\Sigma u_i^*)f'^T_{(r)}U^*_rf'_{(r)}(x^*)-f'^T_{(r)}(x^*)u^{*T}_{(r)}u^*_{(r)}f'_{(r)}(x^*))\mathrm{v},\mathrm{v})$$
$$-(k\Delta(y,x^*))^{-1}\Sigma|u_i^*-u_i|(f'_0(x^*),\mathrm{v})^2-k^{-1}\Sigma|u_i^*-u_i|(f''_0(x^*)\mathrm{v},\mathrm{v})$$
$$\ge\Delta^{-1}(y,x^*)[((L''_{xx}(x^*,u^*)+k_0f'^T_{(r)}(x^*)U^*_rf'_{(r)}(x^*))\mathrm{v},\mathrm{v})$$
$$-\delta m\tau_0^{-1}(\tau_0^{-1}(f'_0(x^*),\mathrm{v})^2+(f''_0(x^*)\mathrm{v},\mathrm{v}))].$$

So due to Assertion 1, there exists a k_0 large enough such that

$$(F''_{xx}(\hat{x},y,u,k)\mathrm{v},\mathrm{v})\ge\Delta^{-1}(y,x^*)[\mu(\mathrm{v},\mathrm{v})-\delta\tau_0^{-1}m(\tau_0^{-1}(f'_0(x^*),\mathrm{v})^2+(f''_0(x^*)\mathrm{v},\mathrm{v}))].$$

So for small enough $\delta>0$ and any triple $(y, u, k)\in D_{k_0}$ there exists $0<\hat{\mu}<\mu$:

$$F''_{xx}(\hat{x},y,u,k)\mathrm{v},\mathrm{v})\ge\Delta^{-1}(y,x^*)\hat{\mu}(\mathrm{v},\mathrm{v}),\quad\forall\,\mathrm{v}\in\mathbb{R}^n$$

i.e. for $\forall\,(y, u, k)\in D_{k_0}$. We have

$$\text{mineigval } F''_{xx}(\hat{x},y,u,k)\ge\Delta^{-1}(y,x^*)\hat{\mu}$$

To complete the proof we note that for any triple $(y, u, k)\in D_{k_0}$ we have $k^{-1}\Sigma u_i\le(\Sigma u_i^*)(k_0\Delta(y,x^*)+\Sigma u_i^*)^{-1}+\delta m\Delta^{-1}(y,x^*)\le(\Sigma u_i^*)(k_0\tau_0+\Sigma u_i^*)^{-1}+\delta m\tau_0^{-1}$. Therefore if $0<\delta<\frac{\tau_0}{m(1+(\Sigma u_i^*)(k_0\tau_0)^{-1})}$ then $k^{-1}\Sigma u_i<1$. So for small enough $\delta>0$ the function $F(x, y, u, k)$ is strongly convex in $x\in\Omega_k(y)$ for any $(y, u, k)\in D_{k_0}$. Hence the vector $\hat{x}=\hat{x}(y,u,k)$ is a unique minimum of the function $F(x,y,u,k)$ in $\Omega_k(y)$, i.e.

$F_x^{\prime}(\hat{x},y,u,k) = 0$.

Due to the definition of $F(x, y, u, k)$ we obtain

$$\hat{x} = \operatorname{argmin}\{ F(x,y,u,k) \mid x \in \mathbb{R}^n \}.$$

Using the formula for $F_{xx}^{\prime\prime}(\hat{x},y,u,k)$ one can find $\hat{M}$ such that for any triple $(y, u, k) \in D_{k_0}$ the estimate (6.6) is taking place.

We completed the proof of the basic theorem.

The following assertion is direct consequence of the Basic Theorem.

Assertion 2 *If the standard second order optimality conditions* (1.4) - (1.5) *hold, then for any fixed "center"* $y \in \Omega_\tau$ *and any fixed barrier parameter* $k \geq k_0 \Delta(y,x^*) + \Sigma u_i^*$ *the MCM*(5.6)-(5.7) *generates a primal dual sequence* $\{ (x^s, u^s)\}_{s=1}^{\infty}$ *such that* $\{ u^s\}_{s=1}^{\infty} \subset U_{y,k}$ *and*

$$\max\{ \| x^{s+1} - x^* \| , \| u^{s+1} - u^* \| \} \leq c k^{-1} \Delta(y,x^*) \| u^s - u^* \|$$

and $c > 0$ *is independent on y and k.*

Corollary *For any given* $0 < \gamma < 1$ *and any fixed center* $y \in \Omega_\tau$ *there exists* $k_0 > 0$ *that for any fixed* $k > k_0 \Delta(y,x^*) + \Sigma u_i^*$ *the following is true.*

$$\max\{ \| x^s - x^* \| , \| u^s - u^* \| \} \leq \gamma^s \tag{6.26}$$

REMARK 3. All statements of Theorem 2 remain true for the MIDF $H(x, y, u, k)$. To prove it we consider instead of $\Phi(x,\hat{u}_{(r)},y,t,k)$ the mapping $\Phi_H(x,\hat{u}_{(r)},y,t,k) : \mathbb{R}^{2n+r+m+1} \to \mathbb{R}^{n+r}$ defined by

$$\Phi_H(x,\hat{u}_r,y,t,k) = (f_0^{\prime T}(x) - \sum_{i=1}^{r} \hat{u}_i f_i^{\prime T}(x) - h(x,y,t,k) + g(x,y,t,k)$$

$$k^{-1}\Delta(y,x^*)[(k\Delta^{-1}(y,x^*)t_i + u_i^*)\Delta^2(y,x)\, d_i^{-2}(x,y,k) - \hat{u}_i],\ i = 1,...,r)$$

where

$$h(x,y,t,k) = \sum_{i=r+1}^{m} \hat{u}_i(x,y,t,k)(f_i^{\prime}(x))^T$$

$$g(x,y,t,k) = k^{-1}\{ \sum_{i=1}^{m} (kt_i\Delta^{-1}(y,x^*) + u_i^*)[-1 + \Delta^2(y,x)\, d_i^{-2}(x,y,k)]\}(f_0^{\prime}(x))^T$$

and

$$\hat{u}_i(x,y,t,k) = k\Delta^{-2}(x,y^*)\, t_i\, \Delta^2(y,x)\, d_i^{-2}(x,y,k),\ i = r+1,...,m$$

7. Concluding Remarks

The MCM (5.6) - (5.7) converges due to the Lagrange multipliers update when both the "center" $y \in \Omega_\tau$ and the barrier parameter $k \geq k_0\Delta(y,x^*) + \Sigma u^*$ are fixed. It makes the MCM different from both Classical Center Methods (see [Huard67a], [Huard67b], [FiacM68]) and modern IPM (see [Gon92], [Ren88]).

However the algorithm as stated is not practical in the sense that it requires the exact optimum to the MIDF at each step. A suitable relaxation of such requirement which allow to retain the Q-linear convergence is the first issue, which has to be addressed.

The other important issue is the MCM convergence in the absence of the non-degeneracy assumptions.

Due to the properties P1-P5 one can expect that the Newton MCM will exhibit the "hot" start phenomenon (see [Pol92], [MelP]), i.e., after each Lagrange multipliers update the approximation for the primal minimizer remains "well" defined (see [Sm86]) for the updated MIDF.

To find the conditions for the "hot" start is the third question which we are going to consider.

Duality issues, which are coming up in connection with MIDFs and MCM, in particular, the relations between the method (5.6) - (5.7) and Prox methods with Entropy-like Distances for the dual problem (see [JenP94][PolTeb96]) is another area which deserves attention.

Finally, the MIDFs have some features which are typical for MBFs (see [Pol92]), however, in many ways they are substantially different which can be seen even on small examples, which we are going to provide.

All these issues we hope to discuss in another paper.

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9. Appendix

A1. Proof of the Proposition 4. We consider the Hessian $F''_{xx}(x^*,y,u,k)$ for a fixed $y \in \operatorname{int}\Omega$ and $k>0$.

$$\begin{aligned}
F''_{xx}(x,y,u^*,k) &= [(1-k^{-1}\Sigma u_i^*)(f_0(y)-f_0(x))^{-1}f_0'(x) - k^{-1}\Sigma u_i^*(kf_i(x)+f_0(y)-f_0(x))^{-1}(kf_i'(x) - \\
&\quad f_0'(x))]'_{|x=x^*} \\
&= [(1-k^{-1}\Sigma u_i^*)(f_0(y)-f_0(x))^{-2}(f_0'(x))^T f_0'(x) + (1-k^{-1}\Sigma u_i^*)(f_0(y)-f_0(x))^{-1}f_0''(x) + \\
&\quad k^{-1}\Sigma u_i^*(kf_i(x)+f_0(y)-f_0(x))^{-2}(kf_i'(x)-f_0'(x))^T(kf_i'(x)-f_0'(x)) - k^{-1}\Sigma u_i^*(kf_i(x)+ \\
&\quad f_0(y)-f_0(x))^{-1}(kf_i''(x))]_{|x=x^*} \\
&= [(1-k^{-1}\Sigma u_i^*)(f_0(y)-f_0(x^*))^{-2}(f_0'(x^*))^T f_0'(x^*) + (1-k^{-1}\Sigma u_i^*)(f_0(y)-f_0(x^*))^{-1}f_0''(x^*) + \\
&\quad k^{-1}\Sigma u_i^*(f_0(y)-f_0(x^*))^{-2}(kf_i'(x^*)-f_0'(x^*))^T(kf_i'(x^*)-f_0'(x^*)) - \\
&\quad k^{-1}\Sigma u_i^*(f_0(y)-f_0(x^*))^{-1}(kf_i''(x^*)-f_0''(x^*))] \\
&= (f_0(y)-f_0(x^*))^{-1}[(f_0(y)-f_0(x^*))^{-1}(f_0'(x^*))^T f_0'(x^*) - k^{-1}(\Sigma u_i^*)(f_0(y)-f_0(x^*))^{-1} \bullet \\
&\quad (f_0'(x^*))^T f_0'(x^*) + f_0''(x^*) - k^{-1}(\Sigma u_i^*)f_0''(x^*) + (f_0(y)-f_0(x^*))^{-1}(k\Sigma u_i^*(f_i'(x^*))^T f_i'(x^*) \\
&\quad - (f_0'(x^*))^T(\Sigma u_i^* f_i'(x^*)) - (\Sigma u_i^* f_i'(x^*))^T f_0'(x^*)) + k^{-1}(\Sigma u_i^*)(f_0'(x^*))^T f_0'(x^*)) \\
&\quad - \Sigma u_i^* f_i''(x^*) + k^{-1}(\Sigma u_i^*)f_0''(x^*)]
\end{aligned}$$

Taking into account the K-K-Ts relation $f_0'(x^*) = \Sigma u_i^* f_i'(x^*)$ we obtain

$$\begin{aligned}
F''_{xx}(x^*,y,u^*,k) &= (f_0(y)-f_0(x^*))^{-1}[L''_{xx}(x^*,u^*) \\
&\quad + k(f_0(y)-f_0(x^*))^{-1}(f'_{(r)}(x^*))^T U_r^* f'_{(r)}(x^*) - (f_0(y)-f_0(x^*))^{-1}(f_0'(x^*))^T f_0'(x^*)] \\
&= (f_0(y)-f_0(x^*))^{-1}[L''_{xx}(x^*,u^*) + \\
&\quad (f_0(y)-f_0(x^*))^{-1}(kf'^T_r(x^*)\,U_r^* f'_{(r)}(x^*) - f'^T_{(r)}(x^*)\,u^{*T}_{(r)} u_r^* f'_{(r)}(x^*)]
\end{aligned}$$

Using the same considerations we obtain

$$H''_{xx}(x^*,y,u^*,k) = (f_0(y)-f_0(x^*))^{-2}[(L''_{xx}(x^*,u^*))$$
$$+2(f_0(y)-f_0(x^*))^{-1}(k(f'_{(r)}(x^*))^T U^*_r f'_{(r)}(x^*) - (f'_{(r)}(x^*))^T u^{*T}_{(r)} u^*_{(r)} f'_{(r)}(x^*))].$$

A.2. Formula for the MIDF Hessian $F''_{xx}(\hat{x},y,u,k)$

$$F''_{xx}(x,y,u,k)_{x=\hat{x}} = (1-k^{-1}\Sigma u_i)(f_0(y)-f_0(\hat{x}))^{-2} f'^T_0(\hat{x}) f'_0(\hat{x}) + (1-k^{-1}\Sigma u_i)\bullet$$
$$(f_0(y)-f_0(\hat{x}))^{-1} f''_0(\hat{x}) - k^{-1}\Sigma u_i (kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-1}(kf''_i(\hat{x})-f''_0(\hat{x})) +$$
$$k^{-1}(\Sigma u_i (kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-2}(kf'_i(\hat{x})-f'_0(\hat{x}))^T (kf'_i(\hat{x})-f'_0(\hat{x}))$$
$$= (f_0(y)-f_0(\hat{x}))^{-1}[(f_0(y)-f_0(\hat{x}))^{-1}(f'^T_0(\hat{x})f'_0(\hat{x}) - k^{-1}(\Sigma u_i)f'^T_0(\hat{x})f'_0(\hat{x})) +$$
$$f''_0(\hat{x}) - k^{-1}(\Sigma u_i)f''_0(\hat{x}) - \Sigma u_i (f_0(y)-f_0(\hat{x}))(kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-1} f''_i(\hat{x})) +$$
$$k^{-1}\Sigma u_i (f_0(y)-f_0(\hat{x}))(kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-1} f''_0(\hat{x}) + k^{-1}\Sigma u_i (f_0(y)-f_0(\hat{x}))\bullet$$
$$(kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-1}(kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-1}(k^2 f'^T_i(\hat{x})f'_i(\hat{x}) -$$
$$kf'^T_0(\hat{x})f'_i(\hat{x}) - kf'^T_i(\hat{x})f'_0(\hat{x}) + f'^T_0(\hat{x})f'_0(\hat{x}))]$$
$$= (f_0(y)-f_0(\hat{x}))^{-1}[(f_0(y)-f_0(\hat{x}))^{-1}(f'^T_0(\hat{x})f'_0(\hat{x}) - k^{-1}(\Sigma u_i)f'^T_0(\hat{x})f'_0(\hat{x}) +$$
$$f''_0(\hat{x}) - k^{-1}(\Sigma u_i)f''_0(\hat{x}) - \Sigma \hat{u} f''_i(\hat{x}) + k^{-1}(\Sigma \hat{u}_i)f''_0(\hat{x}) +$$
$$k^{-1}\Sigma \hat{u}_i (kf_i(\hat{x})+f_0(y)-f_0(\hat{x}))^{-1}(k^2 f'^T_i(\hat{x})f'_i(\hat{x}) - kf'^T_0(\hat{x})f'_i(\hat{x}) -$$
$$kf'^T_i(\hat{x})f'_0(\hat{x}) + f'^T_0(\hat{x})f'_0(\hat{x}))].$$

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