Location Discrimination in Circular City, Torus Town, and Beyond

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Abstract

Salop's "Circular City" model of spatial competition is generalized to higher dimensions, and to "transportation" costs which are a power of distance. Assuming free entry, mill pricing is compared to location-based price discrimination. For dimensions above one, there is some too little entry below some cutoff power, and too much entry above it. This cutoff cost-power rises with dimension, and is larger under price discrimination. Mill pricing induces more entry for powers of four or less, and less entry for powers of five or more. Overall, too much entry seems a more severe problem, which tends to price discrimination.

Introduction

Salop's "Circular City" model considered firms which are evenly spaced on a circle, and which compete to attract consumers who suffer linear transportaion costs (Salop, 1979). This model, or the infinite line to which it is equivalent, is a favorite starting point in modeling spatial competition (Tirole, 1988; Shy, 1995; Carlton & Perloff, 1994). It is, for example, a common basis for analyzing spatial price discrimination (Hobbs, 1986).

Of course there are many reasons to be cautious about using variations on the Circular City model. One may reasonably doubt that a circle well approximates the bounded product spaces we think of as more typical (Economides, 1993). Also, a full game model where firms enter and choose positions as well as prices has only been worked out for quadratic transportation costs (Economides, 1989). Nonetheless, the simplicity and symmetry of Salop's model commend it, as they allow one to obtain definite interpretable results instead of distracting complexity.

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In this spirit, it seems interesting to generalize the Circular City model to spaces of arbitrary integer dimension, and to transportation costs which are an arbitrary positive power of distance. Most physical spatial competition of interest happens on the two dimensional surface of our planet, and competition in abstract product spaces may typically have even more dimensions. Also, even when fundamental physical transportation costs are linear, effective costs can be sub-linear, such as when transporting liquids via a network of pipes, or super-linear, such as when one considers adding one more stop onto a trip to several other shops (see Appendix for demonstrations). And in more abstract product spaces, the disutility of distance from one's ideal product seems poorly constrained by available data.

Yet even given these reasons to consider other dimensions and powers, I know of only one spatial competition paper that has gone beyond one dimension to consider two (Economides, 1993), and only one paper that has considered a general power law for transportation costs (Economides, 1986). Neither of these papers considered a circular topology.

This paper attempts to fill these gaps. It generalizes the one dimensional Circular City to a two dimensional "Torus Town", to a three dimensional "Boundless Box", and on to arbitrary higher dimensions. This paper also allows "transportation" cost, or disutility of distance, to be arbitrary positive powers of distance. Within this generality, I compare the socially optimal level of entry to free entry under spatial price discrimination and mill, i.e., non-discriminatory pricing.

This generality is gained at the cost of the following simplifications in the model. We assume each consumer wants only one product unit, and has an unbounded value for that unit. We also assume that firms are located in a rectangular array, with a transportation cost "metric" that is of a simple linear form. This form reduces to the usual Euclidean metric for quadratic costs, and to the "city block" metric for linear costs. Note also that we do not consider uniform pricing, where firms charge all customers the same price delivered to their door (Anderson, Palma, & de Thisse, 1992).

Within the scope of this simplified generality, we find that many results which are valid in the Circular City with linear or quadratic costs are no longer valid for other dimensions and cost powers. In the Circular City with linear or quadratic costs, mill pricing induces more entry than price discrimination, and both induce too much entry. In higher dimensions, however, both pricing mechanisms can induce too little entry. And for cost powers above five, mill pricing induces less entry than price discrimination.

More specifically, for all dimensions above one, there is some non-negative cost-power below which there is too little entry and above which there is too much entry. This cutoff power rises with increasing dimension, and is larger under price discrimination. Mill pricing induces more entry for cost-powers of four or less, and less entry for cost-powers of five or more.

The model seems less plausible for very high dimensions and cost powers, however, since then it prefers very few entrants. For moderate dimensions, too much entry seems a more severe problem overall than too little, and so when it applies this model seems to overall favor price discrimination.

We now consider the general model, the entry level that maximizes social welfare, prices and entry under mill pricing, and prices and entry under discriminatory pricing. Finally, we compare welfare under mill vs. discriminatory pricing.

The Model

Let consumers be uniformly distributed with density ρ on a unit torus in d dimensions. Such a torus can be described by real vectors $(x_1, x_2, \ldots, x_d)$ where $x_i \in [0,1]$ and where we identify the values $x_i = 1$ and $x_i = 0$. (For example, the points on a circle of unit circumference are a one dimensional torus. A unit square where opposite sides are connected is a two-dimensional torus.)

Consider some product produced by independent "shops," also located in this torus, which suffer a fixed cost $f > 0$ to stay in business, and a constant marginal cost m per unit produced. Let each consumer have an unbounded value for a single unit of this product. And let there be a cost c to transport such a unit from a shop at $(y_1, y_2, \ldots, y_d)$ to a consumer at point $(z_1, z_2, \ldots, z_d)$. Assume that this cost c is linear in the sum of some power $\alpha > 0$ of the coordinate differences $x_i = |z_i - y_i|$ between the points. That is, assume

$$c(x_1, x_2, \ldots, x_d) = b + t\sum_{i=1}^{d} x_i^{\alpha},$$

where b, t are positive constants. For $\alpha = 2$, this is quadratic in Euclidian distance. For $\alpha = 1$ and $d = 2$, this is the "city block" metric, describing how many blocks one would have to travel in a grid of city streets to reach a destination.

We seek equilibria where "shops" are distributed in a rectangular array, set a distance $a > 0$ apart along each dimension. We assume a evenly divides 1, so that the total number n of shops in the torus is given by $n = a^{-d} > 1$. We also seek equilibria where the prices that shops choose are also symmetric, so that in equilibrium each consumer frequents the nearest (i.e., lowest cost) shop.

Given this symmetry, we can without loss of generality focus attention on a single shop located at $(0, 0, \ldots, 0)$, on a single competitor shop located at $(a, 0, \ldots, 0)$, and on consumers located in a "wedge" region given by $x_1 \in [0, a/2]$, and $x_i \in [0, x_1]$ for all $i > 1$. These are the customers for which $(0, 0, \ldots, 0)$ is the nearest shop, $(a, 0, \ldots, 0)$ is the second nearest shop, and all $x_i > 0$. There are $d2^d$ similar wedges adjacent to each shop, which together contain all the consumers for which this is the nearest shop.

Social Welfare

A socially optimal arrangement of shops would minimize the loss from the fixed cost f of maintaining shops, and from the variable costs m and c of producing and transporting a unit to customers. This total loss is

$$L(n) \quad = \quad \rho\, m + nf + d2^d n\rho\, t \int_0^{a/2} \int_0^{x_1} \cdots \int_0^{x_1} c(x_1, x_2, \ldots, x_d)\, dx_d \ldots dx_2 dx_1$$

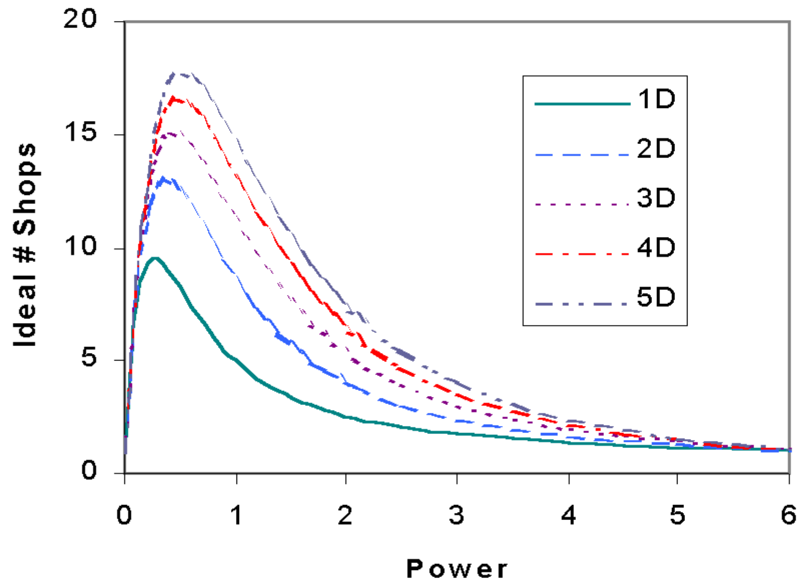

Figure 1: Optimum Number of Shops Given Power, Dimension

$$
\begin{aligned}
&= \rho(m+b) + nf + d2^d n\rho\, t \int_0^{a/2}\int_0^{x_1}\cdots\int_0^{x_1}\left(\sum_{i=1}^{d} x_i^{\alpha}\right) dx_d \ldots dx_2 dx_1 \\
&= \rho(m+b) + a^{-d} f + \frac{d\rho\, t\, a^{\alpha}}{2^{\alpha}(\alpha+1)}.
\end{aligned}
$$

The solution to $\max_n L(n)$ (assuming n could vary continuously) is

$$
n^* = \left(\frac{\alpha}{2^{\alpha}(\alpha+1)}\frac{\rho\, t}{f}\right)^{\frac{d}{\alpha+d}}.
$$

This optimal number is graphed in figure 1 for $\rho\, t/f = 100, d \in [1,5], \alpha \in [0,6]$.

If $m+b=0$, it turns out that the ratio of actual loss $L(n)$ to minimum possible loss $L(n^*)$ depends only on the parameters α, d and the ratio n/n^*. Specifically, we have

$$
\frac{L(n)}{L(n^*)} = \frac{\alpha\left(\frac{n}{n^*}\right) + d\left(\frac{n}{n^*}\right)^{-\alpha/d}}{\alpha+d}.
$$

Mill Pricing

If customers pay for their own transportation, and if shops cannot base their prices on where customers are located, then each shop must set a single "mill" price p for all customers. If $\tilde{p}$

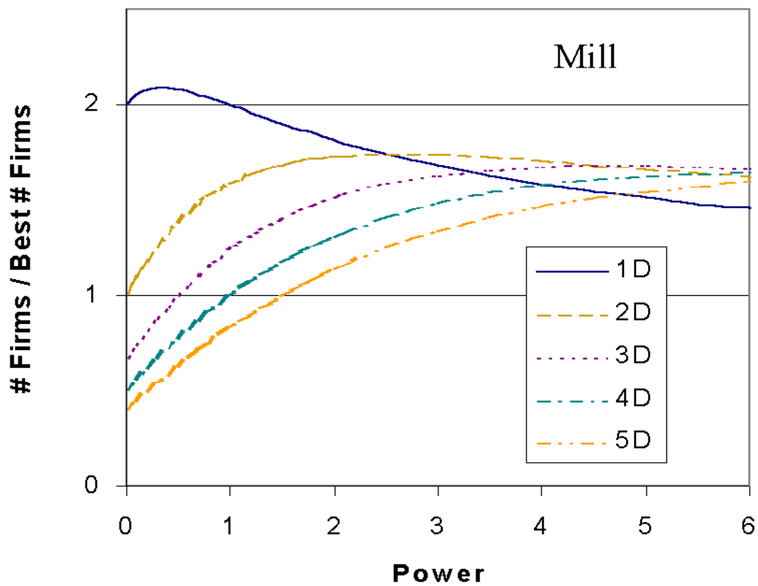

Figure 2: Shop Ratio Given Mill Pricing

is the competitor's price, then a customer is indifferent between the two shops at $(0, 0, \ldots, 0)$ and $(a, 0, \ldots, 0)$ when

$$p + c(x_1, x_2, \ldots, x_d) = \tilde{p} + c(a - x_1, x_2, \ldots, x_d).$$

Given our choice of cost function c, this implies a willingness to pay $p(x_1)$ given by

$$p(x_1) - \tilde{p} = \Delta c(x_1) \equiv t[(a - x_1)^\alpha - x_1^\alpha],$$

where $\Delta c(x_1)$ describes the difference in transportation costs between the two shops. Note that as p varies, only the coordinate x_1 of a customer within our wedge matters. And so if x is the indifference value of x_1, the number of customers who buy from this shop is $q(x) = \rho(2x)^d$, the number of customers in a cube of side $2x$.

The shop's profit maximization problem thus becomes one dimensional, namely

$$\begin{aligned}
\max_x \pi(x) &= \max_x q(x)(p(x) - m) - f \\
&= \max_x \rho\,(2x)^d(\tilde{p} - m + \Delta c(x)) - f.
\end{aligned}$$

Setting $\pi_x = 0$, and substituting $x = a/2$ and $p = \tilde{p}$ as symmetry requires, we find the mill (i.e, non-discriminatory) price to be

$$p^M = m + \frac{2^{1-\alpha}\alpha t}{d} a^\alpha.$$

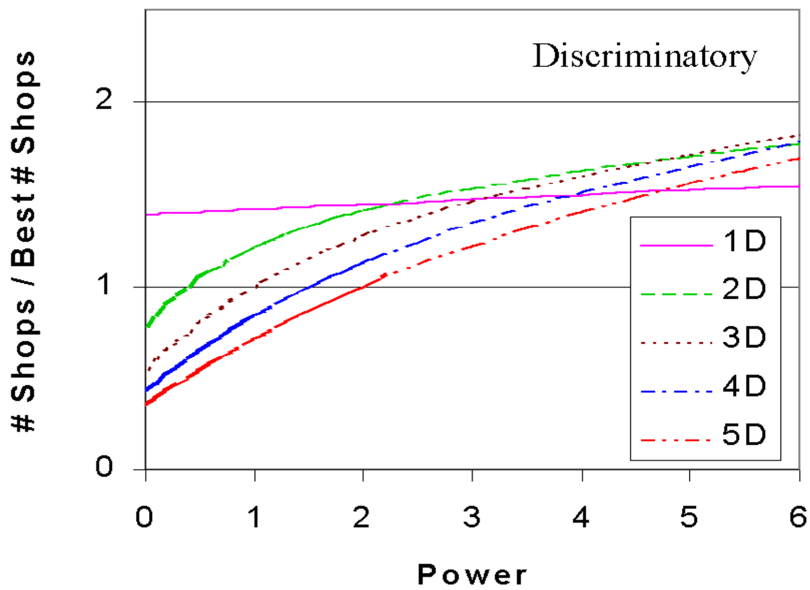

Figure 3: Shop Ratio Given Discriminatory Pricing

The second order condition of $\pi_{xx}(a/2) \leq 0$ is also satisfied, since it reduces to $d \geq -1$. Let us assume m is low enough so that $p^M > 0$.

If we further assume that entry drives profits to zero (and that n could vary continuously), we find that $\pi(a/2) = 0$ implies that the number of shops given mill pricing is

$$n^M = \left(\frac{2^{1-\alpha}\alpha}{d} \frac{\rho\, t}{f} \right)^{\frac{d}{\alpha+d}}.$$

The ratio of this equilibrium shop number to the optimum shop number is the shop ratio

$$\frac{n^M}{n^*} = \left(\frac{2(\alpha+1)}{d} \right)^{\frac{d}{\alpha+d}},$$

This ratio is graphed in figure 2. Note that mill pricing is exactly optimal for $\hat{\alpha}^M(d) = d/2-1$. For $\alpha < \hat{\alpha}^M(d)$ it induces too few shops, while for $\alpha > \hat{\alpha}^M(d)$ it induces too many shops. Note also that while near $\alpha^M(d)$ the shop ratio increases with power α, for large d the shop ratio eventually peaks and then drifts back toward unity.

Discriminatory Pricing

When shops know where customers live and are free to offer prices based on that information, we expect pure price competition between the two nearest shops. The second lowest cost producer should offer a price equal to its marginal cost, while the lowest cost producer should offer a price which makes the consumer indifferent between the two shops.

Thus within our wedge,

$$p(x_1, x_2, \ldots, x_d) + c(x_1, x_2, \ldots, x_d) = m + c(a - x_1, x_2, \ldots, x_d).$$

Again we find that our choice of c implies that only x_1 matters, so that profit per customer is

$$\pi(x_1, x_2, \ldots, x_d) = p^D(x_1) - m = \Delta c(x_1).$$

Total shop profit is then found by integrating over customers, as in

$$\begin{aligned}
\pi + f &= d2^d \rho \int_0^{a/2} \int_0^{x_1} \cdots \int_0^{x_1} \pi(x_1, x_2, \ldots, x_d)\, dx_d \ldots dx_2 dx_1 \\
&= d2^d \rho \int_0^{a/2} x^{d-1} \Delta c(x) dx \\
&= d2^d \rho\, t \int_0^{a/2} [x^{d-1}(a-x)^\alpha - x^{\alpha+d-1}] dx \\
&= d2^d \rho\, t \left(\frac{a}{2}\right)^{\alpha+d} I(\alpha, d),
\end{aligned}$$

where the integral $I(\alpha, d)$ is defined to be

$$\begin{aligned}
\frac{1}{\alpha+d} + I(\alpha, d) &\equiv \int_0^1 x^{d-1}(2-x)^\alpha dx = \int_1^2 (2-u)^{d-1} u^\alpha du \\
&= \int_1^2 \sum_{k=0}^{d-1} \frac{(d-1)!}{k!(d-k-1)!} 2^{d-1-k} (-1)^k u^{\alpha+k} du \\
&= (d-1)! 2^d \sum_{k=0}^{d-1} \frac{(-1)^k (2^\alpha - 2^{-k-1})}{k!(d-k-1)!(\alpha+k+1)}
\end{aligned}$$

If we again assume that entry drives profits to zero (and that n can vary continuously), we find that $\pi = 0$ implies that the number of shops given discriminatory pricing is

$$n^D = \left(d 2^{-\alpha} I(\alpha, d) \frac{\rho\, t}{f}\right)^{\frac{d}{\alpha+d}}.$$

and the shop ratio, i.e., the ratio of this shop number to the optimal number, is

$$\frac{n^D}{n^*} = \left(\frac{d}{\alpha}(\alpha+1) I(\alpha, d)\right)^{\frac{d}{\alpha+d}},$$

This ratio is graphed in figure 3. Note that increasing power α always increases the shop ratio, while increasing the dimension d again increases the cutoff power $\hat{\alpha}^D(d)$ which induces optimal entry.

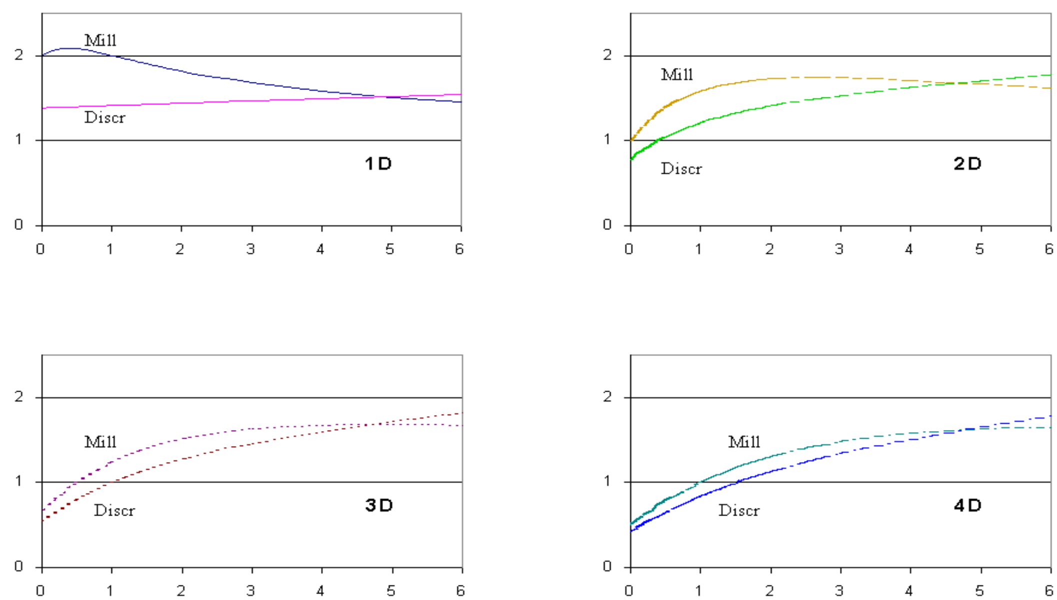

Figure 4: Comparing Shop Ratios for both Pricing Methods and Dimensions 1-4

Comparing Pricing Methods

Figure 4 compares mill and discriminatory pricing in dimensions d of $1, 2, 3,$ and 4. Note that mill pricing always induces more shops at powers below 4.5, while discriminatory pricing induces more shops at powers above 5. (For dimensions d of $(1, 2, 3, 4, 5)$, the exact transition point is at powers α of $(4.90, 4.66, 4.67, 4.75, 4.85)$.)

For dimensions d of $(2, 3, 4, 5)$, the power $\hat{\alpha}^M(d)$ at which mill pricing is optimal is $(0, 0.5, 1, 1.5)$ while power $\hat{\alpha}^D(d)$ at which discriminatory pricing is optimal is $(.4067, 1, 1.5158, 2)$. Thus for $d > 1$ we have $\hat{\alpha}^D(d) \approx \hat{\alpha}^M(d + 1)$.

Figure 5 shows how the loss ratio $L(n)/L(n^*)$ varies with dimension, power, and pricing method. For the moderate dimensions considered, too many shops is clearly a more severe problem than too few. The maximum loss from too little entry is 1.6%, for discriminatory pricing in dimension $d = 5$ at a power of $\alpha = 1/2$. In contrast, the loss from too much entry can exceed 30%, e.g., for mill pricing at $d = 1$ and $\alpha \in [2, 3]$.

As figure 1 shows, the optimal and equilibrium number of shops is near for very low power or high powers (e.g., $\alpha \notin [.1, 5]$ for $\rho t/f = 100$). Since this is in conflict with our assumption of competition. Also, for very high dimensions, it becomes difficult to justify the torus model as an approximation to the interior of some Euclidean region with a border, since in high dimensions almost all the volume is near the border.

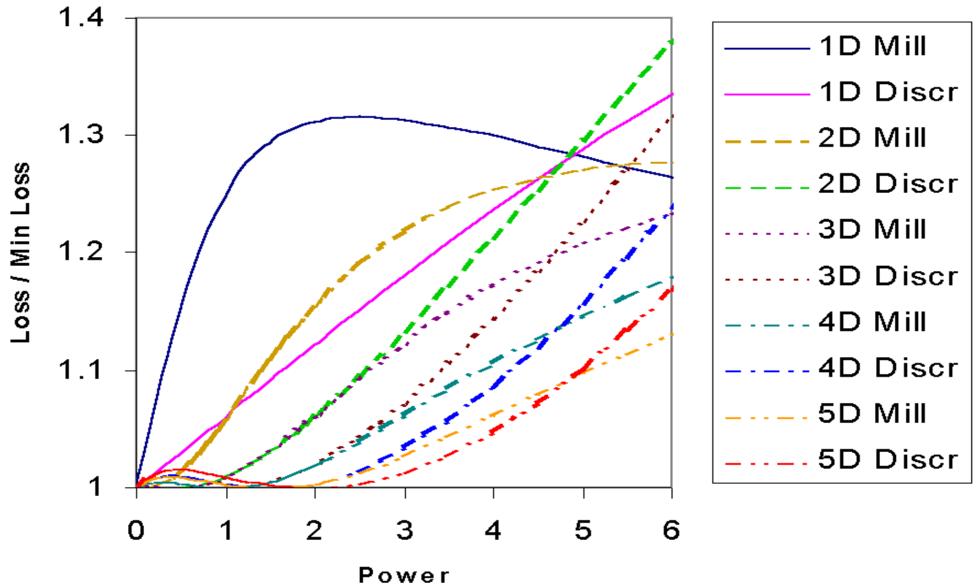

Figure 5: Comparing Loss Ratios

We thus expect this model to be relevant mainly for moderate dimensions. In such cases it is hard to imagine a "transportation" cost power of over five, but if such a case should arise our model favor mill pricing. Mill pricing is also favored, though only slightly, if one knows enough to say that the power is near or below $\hat{\alpha}^M(d)$. (The exact transition powers below which mill pricing is better are, for dimensions d of $(2,3,4,5)$, powers α of $(0.17, 0.70, 1.29, 1.70)$). In other cases where the model seems likely to be relevant, or if one is uncertain about which relevant case applies, discriminatory pricing would seem to minimize expected social losses.

Conclusion

An old proverb says that the three most important features of a small business are "location, location, location." More abstract product positioning is similarly a central issue for large businesses. Also, information technologies are making it easier for businesses to learn more about their customers, and are hence making price discrimination more feasible. Consider, for example, the growth of grocery "club cards." We should thus expect more firms to use "spatial" price discrimination. Is this a good thing?

Most of our models of spatial competition and price discrimination are in one dimension with linear or quadratic "transportation" costs. Most competition in physical space, however, takes place in two dimensions, while most product differentiation is in even higher dimensions. Similarly, the disutility of distance from one's ideal can deviate greatly from linear or quadratic. As shown in the Appendix, a fundamentally linear cost can be transformed into an effectively quite sub-linear cost, via a hierarchical network of pipes or roads, or into an effectively super-linear cost, when trips to several local shops or customers can be combined into one.

Arguably the simplest standard model of spatial competition is Salop's "Circular City" (Salop, 1979). To learn more about how spatial competition works in higher dimensions and with more diverse transportation cost powers, I have generalized this simplest model in these ways.

I find that in two dimensions, discriminatory pricing improves welfare by reducing average prices and hence entry, unless the relevant transportation cost is best approximated as varying with distance by a power of 0.17 or less. For competition in more abstract product spaces, the popular modeling choice of a quadratic loss implies that price discrimination is preferred for dimensions up to and including five, while mill pricing is preferred in higher dimensions. Also, since the losses from too little entry are much less than from too much entry, uncertainty also seems to favor price discrimination.

All of these conclusions could, of course, be overturned by more exact modeling of relevant issues. Such issues include finite bounds on and uncertainty regarding consumer value (i.e. individual demand curves), strongly varying consumer density, "uniform" (delivered) pricing, and alternative metrics by which transportation costs depend on location.

Appendix

The following two sections illustrate how effectively non-linear transportation costs can naturally arise, even when the fundamental transportation costs are linear.

Transportation Costs With Pipes

Consider customers distributed uniformly in space, with each customer occupying a unit square and demanding a unit flow of some liquid (e.g., water or oil). Consider collecting these customers into groups of four adjacent customers which share a single corner. Then collect four such adjacent groups into a larger group containing sixteen customers. Continue to build a hierarchy of by collecting groups of four into a larger group, until one stops at n levels of hierarchy of with one has a square of side $X = 2^n$.

To serve all these customers, we can construct a fractal network of pipes organized by this hierarchy of. For each group, a large pipe connects a corner of that group to its center. That center is the corner of four smaller groups, each of which have a smaller pipe leading from that corner to their center. At the lowest level, each customer has a pipe from a corner to the center of their unit-square property.

The total cost of the pipe network is found by summing over the levels l of the hierarchy of.

$$C(X) \propto \sum_{l=0}^{n} N(l)L(l)C(D(F(l))),$$

where $N(l) = (X2^{-l})^2$ is the number of level l units that fit in the L^2 area, $L(l)$ is the length of each pipe at level l, $C(D)$ is the cost per unit length of a pipe of diameter D, $D(F)$ is the diameter of pipe needed to support a flow F, and $F(l)$ is the liquid flow needed by a pipe at level l.

The cost per length of pipes is roughly linear in their diameter, and for a given pressure the flow through a pipe of diameter D goes as D^3. The flow at each level is given by the number of customers within a unit at that level, i.e., 2^{2l}. Thus we can write the cost per customer as

$$
\begin{aligned}
C(X)/X^2 &\propto X^{-2} \sum_{l=0}^{\log_2 X} (X2^{-l})^2 2^l (2^{2l})^{1/3} \\
&= \sum_{l=0}^{\log_2 X} 2^{-\frac{1}{3}l} \\
&= \frac{1 - 2^{-\frac{1}{3}\log_2 X}}{1 - 2^{-1/3}} \\
&\propto (1 - X^{-1/3}).
\end{aligned}
$$

This is substantially sub-linear in distance X, and quickly approaches a constant cost. (For an array of pipes serving a three dimensional set of customers, we instead get per person cost proportional to $1 - X^{-1}$.)

Similar effects come from travel via hierarchical networks of roads, such as the familiar dense network of slow roadways connected to a sparse network of fast freeways.

Adding Trips

Consider a person who will make a trip to several local shops, and is considering adding a stop at one more shop. The cost of adding this additional shop is the cost of the shortest trip possible to all the shops, minus the cost of the shortest trip to the other shops, without including this additional shop. (The same analysis applies for computing the marginal cost to a shop of delivering a product to a customer, given that the delivery van is already scheduled to make several other stops.)

Figure 6 illustrates how the marginal cost of adding another stop can be more than linear in distance over relevant ranges. It considers trips in two dimensions, where the cost of a trip to a set of shops is the sum of the Euclidian lengths of the shortest path that passes through all the shops and starts and ends at the origin. Each line describes a different number of other shops on the trip.

The top line describes the case of zero other shops on the trip, and so of course describes linear cost with distance. The second line down considers one other shop placed at a radius of 1.5 from the origin, and averages over all the possible relative angles between the two shops. The third line considers two shops at radii of 1 and 2, while the forth line considers shops at radii $1, 1.5$ and 2. (Again we average over all possible relative angles.)

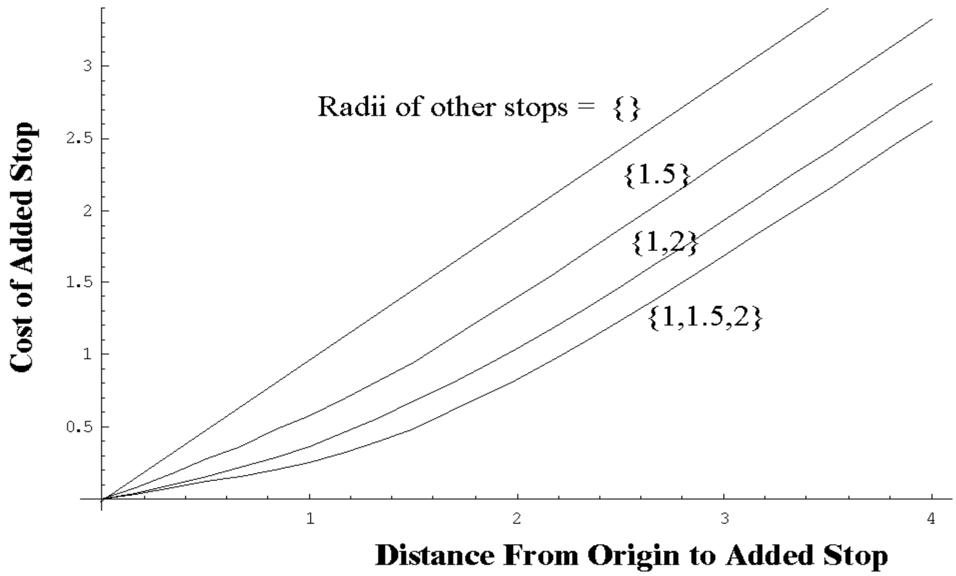

Figure 6: Super-Linear Cost of Adding Stop to Trip

For very small and very large distances, the cost of an additional stop is linear in the distance from the origin to the added stop. For distances comparable to the other shops, however, the cost curve is more than linear, and seems roughly quadratic.

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