Table 2.2.1 Table of Laplace transform pairs.

$X(s)$	$x(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse
2. $\frac{1}{s}$	$u_s(t)$, unit step
3. $\frac{c}{s}$	constant, c
4. $\frac{e^{-sD}}{s}$	$u_s(t-D)$, shifted unit step
5. $\frac{n!}{s^{n+1}}$	t^n
6. $\frac{1}{s+a}$	e^{-at}
7. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8. $\frac{b}{s^2+b^2}$	$\sin bt$
9. $\frac{s}{s^2+b^2}$	$\cos bt$
10. $\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
11. $\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
12. $\frac{a}{s(s+a)}$	$1-e^{-at}$
13. $\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$
14. $\frac{s+p}{(s+a)(s+b)}$	$\frac{1}{b-a}[(p-a)e^{-at}-(p-b)e^{-bt}]$
15. $\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{e^{-at}}{(b-a)(c-a)}+\frac{e^{-bt}}{(c-b)(a-b)}+\frac{e^{-ct}}{(a-c)(b-c)}$
16. $\frac{s+p}{(s+a)(s+b)(s+c)}$	$\frac{(p-a)e^{-at}}{(b-a)(c-a)}+\frac{(p-b)e^{-bt}}{(c-b)(a-b)}+\frac{(p-c)e^{-ct}}{(a-c)(b-c)}$
17. $\frac{b}{s^2-b^2}$	$\sinh bt$
18. $\frac{s}{s^2+b^2}$	$\cosh bt$
19. $\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$
20. $\frac{a^2}{s(s+a)^2}$	$1-(at+1)e^{-at}$
21. $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t$
22. $\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right), \phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$
23. $\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$

Table 2.2.2 Properties of the Laplace transform.

$x(t)$	$X(s)=\int_0^\infty f(t)e^{-st}\,dt$
1. $af(t)+bg(t)$	$aF(s)+bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s)-x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2X(s)-sx(0)-\dot{x}(0)$
4. $\dfrac{d^nx}{dt^n}$	$s^nX(s)-\sum_{k=1}^{n}s^{n-k}g_{k-1}$ $g_{k-1}=\left.\dfrac{d^{k-1}x}{dt^{k-1}}\right\vert_{t=0}$
5. $\int_0^t x(t)\,dt$	$\dfrac{X(s)}{s}+\dfrac{g(0)}{s}$ $g(0)=\left.\int x(t)\,dt\right\vert_{t=0}$
6. $x(t)=\begin{cases}0 & t<D\\ g(t-D) & t\geq D\end{cases}$ $=u_s(t-D)g(t-D)$	$X(s)=e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$
8. $tx(t)$	$-\dfrac{dX(s)}{ds}$
9. $x(\infty)=\lim_{s\to 0}sX(s)$	
10. $x(0+)=\lim_{s\to\infty}sX(s)$	

For Entries 2, 3, 4, and 5, if $x \neq 0$ for $t<0$, then replace the initial conditions at $t=0$ with the pre-initial condtions at $0-$.

EXAMPLE 2.2.3 The Sine and Cosine Functions

■ Problem

Derive the Laplace transforms of the exponentially decaying sine and cosine functions, $e^{-at}\sin\omega t$ and $e^{-at}\cos\omega t$, for $t\geq 0$, where a and ω are constants.

■ Solution

Note that from the Euler identity, $e^{j\theta}=\cos\theta+j\sin\theta$, with $\theta=\omega t$, we have

$$e^{-at}(\cos\omega t+j\sin\omega t)=e^{-at}e^{j\omega t}=e^{-(a-j\omega)t} \tag{1}$$

Thus the real part of $e^{-(a-j\omega)t}$ is $e^{-at}\cos\omega t$ and the imaginary part is $e^{-at}\sin\omega t$. However, from the result of Example 2.2.2, with a replaced by $a-j\omega$, we have

$$\mathcal{L}\left[e^{-(a-j\omega)t}\right]=\frac{1}{s+a-j\omega} \tag{2}$$

In this form, we cannot identify the real and imaginary parts. To do so we multiply the numerator and denominator by the complex conjugate of the denominator and use the fact that $(x-jy)(x+jy)=x^2+y^2$ (see Table 2.1.1); that is,

$$\frac{1}{x-jy}=\frac{x+jy}{(x-jy)(x+jy)}=\frac{x+jy}{x^2+y^2}$$

Table 2.3.1 Step response for zero initial conditions.

	ODE	Roots	$x(t)$
1.	$\dot{x} + ax = Mu_s(t)$	$s = -a$	$x(t) = \frac{M}{a}\left(1 - e^{-at}\right)$
2.	$\ddot{x} + (a+b)\dot{x} + abx = Mu_s(t)$	$s = -a,\ -b$ $a \neq b$	$x(t) = M\left[\frac{e^{-at}}{a(a-b)} + \frac{e^{-bt}}{b(b-a)} + \frac{1}{ab}\right]$
3.	$\ddot{x} + 2a\dot{x} + a^2x = Mu_s(t)$	$s = -a,\ -a$	$x(t) = \frac{M}{a^2}\left[1 - (at+1)\,e^{-at}\right]$
4.	$\ddot{x} + b^2x = Mu_s(t)$	$s = \pm bj \quad b > 0$	$x(t) = \frac{M}{b^2}\,(1 - \cos bt)$
5.	$\ddot{x} + 2a\dot{x} + (a^2+b^2)x = Mu_s(t)$	$s = -a \pm bj \quad b > 0$	$x(t) = \frac{M}{a^2+b^2}\left[1 - \left(\frac{a}{b}\sin bt + \cos bt\right)e^{-at}\right]$

■ Solution

Transforming the equation gives

$$[s^2X(s) - x_0s - \dot{x}_0] + a[sX(s) - x_0] + bX(s) = 0$$

$$\left(s^2 + as + b\right) X(s) = x_0s + \dot{x}_0 + ax_0$$

$$X(s) = \frac{x_0s + \dot{x}_0 + ax_0}{s^2 + as + b}$$

We cannot go any further until we have numerical values for the constants a and b, because without them we do not know the type of roots (distinct real, complex, etc.). In addition, nonzero values for the initial conditions x_0 and $\dot{x}_0$ introduce a term $cs + d$ in the numerator, where c and d are nonzero constants. The entries in Table 2.3.1 do not generate such a numerator, and so we must perform additional algebra, as was done in Example 2.3.4.

For first and second-order linear equations with constant coefficients and a constant input, the forms of the free and forced responses can be summarized as in Table 2.3.2. For such cases, if the initial conditions are not zero, it may be easier to solve the problem using this table rather than applying the Laplace transform. To obtain the free response with this table, set the constant $c = 0$.

Table 2.3.2 Solution forms for a constant input.

Equation	Solution form
First order: $\dot{x} + ax = b \quad a \neq 0$	$x(t) = \frac{b}{a} + Ce^{-at}$
Second order: $\ddot{x} + a\dot{x} + bx = c \quad b \neq 0$	
1. $(a^2 > 4b)$ distinct, real roots: s_1, s_2	$x(t) = C_1e^{s_1t} + C_2e^{s_2t} + \frac{c}{b}$
2. $(a^2 = 4b)$ repeated, real roots: s_1, s_1	$x(t) = (C_1 + tC_2)e^{s_1t} + \frac{c}{b}$
3. $(a = 0, b > 0)$ imaginary roots: $s = \pm j\omega$, $\omega = \sqrt{b}$	$x(t) = C_1 \sin \omega t + C_2 \cos \omega t + \frac{c}{b}$
4. $(a \neq 0, a^2 < 4b)$ complex roots: $s = \sigma \pm j\omega$, $\sigma = -a/2, \omega = \sqrt{4b - a^2}/2$	$x(t) = e^{\sigma t}(C_1 \sin \omega t + C_2 \cos \omega t) + \frac{c}{b}$