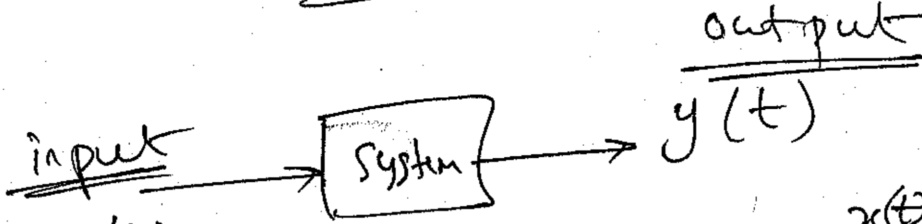
Differential Equations based Modeling

Diff eq



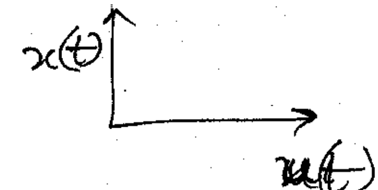
$u(t)$

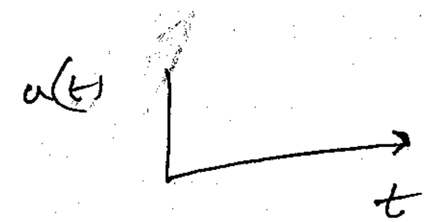
$f(t)$

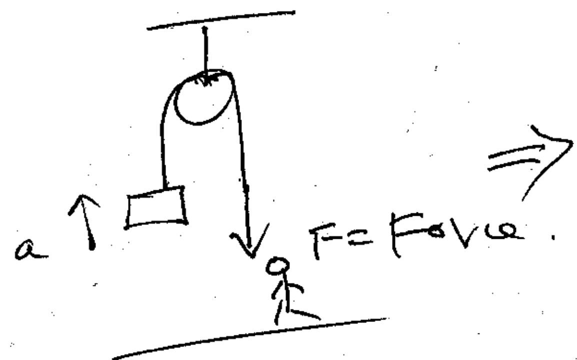
$x(t)$

system variable

$z(t)$







$\Rightarrow$ what is a?

Chap 2 $\rightarrow$ Solving Diff. eqn.

$$\underbrace{\dot{x} + a\,x(t)}_{\text{system variables}} = b \rightarrow 1^{st} \text{ order ODE}$$

ODE: ordinary Differential equation

$\dot{x} = \frac{dx}{dt}$

Right side of ODE is the input to the ODE

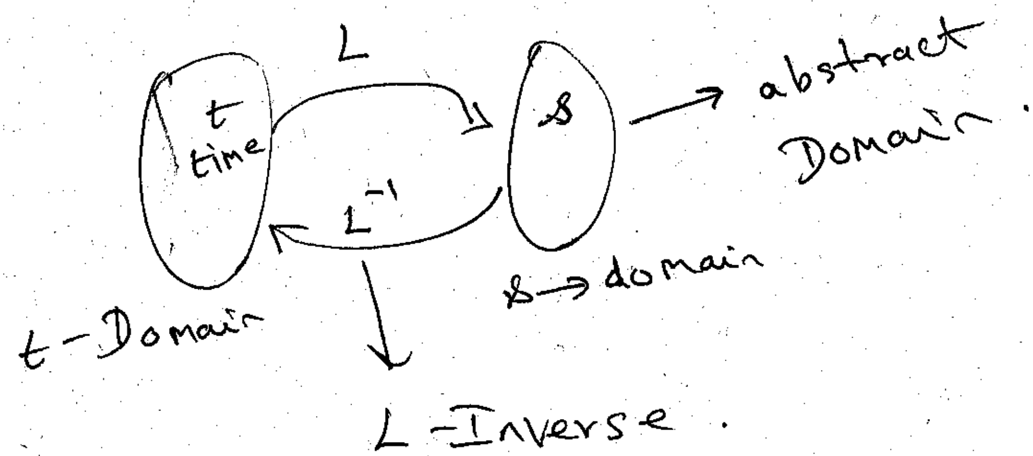
(a, b are constants)

$$\ddot{x} + a\dot{x} + bx = c$$

↑ input to the ODE.

$a, b, c \rightarrow$ constants

Solving ODE with <u>Laplace Transform</u>



What is solving mean for an ODE?

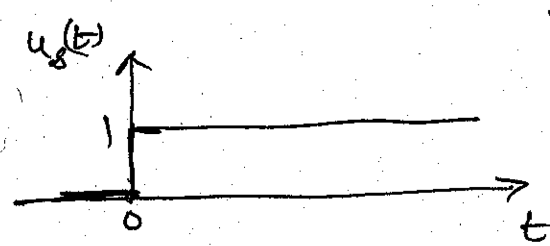
$x(t) = \text{-------}$

Definition of $L(x(t)) = \int_0^\infty x(t)\, e^{-st}\, dt$

$= X(s)$

<u>unit step function</u>

<u>value = 1</u>



$$L(u_s(t)) = \int_0^\infty 1 \times e^{-st}\, dt = \frac{1}{-s}\left[e^{-st}\right]_0^\infty$$

↑ (times multiplication)

$$= \frac{1}{-s}\left[0 - 1\right] = \frac{1}{s}$$

$L^{-1} \rightarrow$ Laplace Inverse.

$$X(s) = \frac{9s+2}{s(s+8)} \quad \text{Find } x(t)$$

Partial Fraction

Method 1 $\rightarrow$

$$\frac{9s+2}{s(s+8)} = \frac{c_1}{s} + \frac{c_2}{s+8}$$

$$\frac{9s+2}{s(s+8)} = \frac{c_1(s+8) + s(c_2)}{s(s+8)}$$

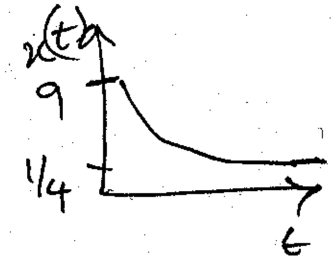
Equate the numerator.

$$9s+2 = c_1 s + 8c_1 + s c_2$$

$s \rightarrow$ terms $\rightarrow$ $\quad 9s = c_1 s + s c_2$

Constants $\rightarrow$ $\quad 2 = 8c_1$

$$c_1 = 1/4$$



$$9 = c_1 + c_2$$

$$c_2 = 9 - c_1 = 9 - \frac{1}{4} = \frac{35}{4}$$

$$X(s) = \frac{9s+2}{s(s+8)} = 1/4\left[\frac{1}{s}\right] + \frac{35}{4}\left[\frac{1}{s+8}\right]$$

$$L^{-1} = 1/4 \times 1 + \frac{35}{4} e^{-8t} = \frac{1}{4} + \frac{35}{4} e^{-8t} = x(t)$$

input $\longrightarrow$ system $\dfrac{9s+2}{s(s+8)}$ $\longrightarrow$ output

__Step1__ : Look for highest power of s in the __denominator__ to determine the number of partial fractions.

__Step2__ : Make sure the coefficient of the highest power of s in the __denominator__ is 1.

input $\longrightarrow$ $x(t) = \frac{1}{4} + \frac{35}{4} e^{-8t}$ $\longrightarrow$ output

Method 2 → partial fractions

$$X(s) = \frac{9s+2}{s(s+8)} = \frac{C_1}{s} + \frac{C_2}{s+8}$$

$$C_1 = \lim_{s \to 0} s X(s) = \lim_{s\to 0} \cancel{s}\left(\frac{9s+2}{\cancel{s}(s+8)}\right) = \frac{2}{8} = \frac{1}{4}$$

↑ Root

$$C_2 = \lim_{s \to -8} (s+8) X(s) = \lim_{s\to -8} \cancel{(s+8)}\left(\frac{9s+2}{s\cancel{(s+8)}}\right)$$

↑ Root

$$= \lim_{s \to -8} \frac{9s+2}{s} = \frac{-70}{-8} = 35/4$$

$$X(s) = \frac{1}{4}\left[\frac{1}{s}\right) + \frac{35}{4}\left(\frac{1}{s+8}\right]$$

$$\boxed{x(t) = \frac{1}{4} + \frac{35}{4} e^{-8t}}$$

$$X(s) = \frac{5}{s^2(3s+12)} \qquad \text{Find } x(t)$$

$$= \frac{5}{3s^3+12s^2} = \frac{5/3}{s^3+4s^2} = \frac{5/3}{s^2(s+4)}$$

$$X(s) = \frac{5/3}{s^2(s+4)} = \underbrace{\frac{C_1}{s^2} + \frac{C_2}{s}}_{\text{2 terms for } s^2} + \frac{C_3}{s+4}$$

$$C_1 = \lim_{s\to 0} s^2 X(s) = \lim_{s\to 0} \cancel{s^2}\left[\frac{5/3}{\cancel{s^2}(s+4)}\right] = 5/12$$

$$C_2 = \lim_{s\to 0} \frac{d}{ds}\left[s^2 X(s)\right] = \lim_{s\to 0} \frac{d}{ds} \cancel{s^2}\left[\frac{5/3}{\cancel{s^2}(s+4)}\right]$$

$$= \lim_{s\to 0}\left[\frac{0-(5/3)(1)}{(s+4)^2}\right] = \frac{-5/3}{16} = \frac{-5}{48}$$

$$C_3 = \lim_{s\to -4} (s+4)X(s) = \lim_{s\to -4}\left[\cancel{(s+4)}\frac{5/3}{s^2\cancel{(s+4)}}\right] = 5/48$$

$$X(s) = \frac{5/3}{s^2(s+4)} = \frac{5}{12}\left[\frac{1}{s^2}\right] + \left(\frac{-5}{48}\right)\left(\frac{1}{s}\right) + \frac{5}{48}\left[\frac{1}{s+4}\right]$$

Take $\mathcal{L}^{-1}$

$$x(t) = \frac{5}{12}\left[t^2\right] - \frac{5}{48} + \frac{5}{48}e^{-4t}$$

$$X(s) = \frac{8s + 13}{s^2 + 4s + 53} \qquad \text{find } x(t)$$

$$X(s) = \frac{8s+13}{s^2+4s+2^2+7^2} = \frac{8s+13}{(s+2)^2+7^2}$$

$$X(s) = \frac{8s+13}{(s+2)^2+7^2} = \frac{C_1(s+2)}{(s+2)^2+7^2} + \frac{C_2(7)}{(s+2)^2+7^2}$$

Equate the numerators.

$$8s+13 = C_1(s+2) + 7C_2$$

s terms $\Rightarrow$ $8s = C_1 s$ $\quad \boxed{C_1 = 8}$

Constants $\Rightarrow$ $13 = 2C_1 + 7C_2$

$$\boxed{C_2 = -3/7}$$

$$X(s) = 8\left[\frac{(s+2)}{(s+2)^2+7^2}\right] - \frac{3}{7}\left[\frac{7}{(s+2)^2+7^2}\right]$$

$$x(t) = 8\left[e^{-2t}\cos 7t\right] - \frac{3}{7}\left[e^{-2t}\sin 7t\right]$$

$x(0) = 8 - 0 = 8$

$x(\infty) = 0$