Chap 5

Block Diagrams:

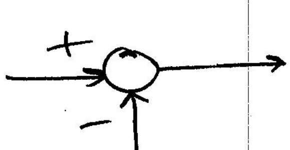
Represent an ODE in the form of a diagram (Block Diagram → also used in Simulink software).

$$\boxed{T.F = \frac{X(s)}{F(s)}}$$

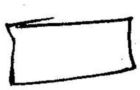
$$\underset{\text{input}}{F(s)} \longrightarrow \boxed{T.F} \longrightarrow \underset{\text{state}}{X(s)}$$

$$F(s) \underbrace{\times}_{\text{multiplication}} [TF] = X(s)$$

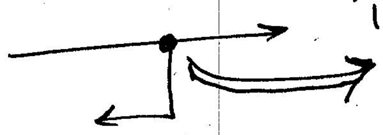
$\longrightarrow$ arrow



summer circle.



→ block



Take-off point.

Multiplier block

$$F(s) \longrightarrow \boxed{K} \longrightarrow X(s)$$

$$F(s)\,K = X(s) \qquad \frac{X(s)}{F(s)} = K$$

Integrator block
(special name)
if the block has $1/s$.

$$F(s) \longrightarrow \boxed{1/s} \longrightarrow X(s)$$

$$F(s)\,\frac{1}{s} = X(s)$$

$$F(s) = s\,X(s)$$

$$s\,X(s) - \cancel{x(0)}^{\,0} = F(s)$$

Taking L^{-1}

$$\dot{x} = f(t)$$

$$\frac{dx}{dt} = f(t) \qquad \boxed{x = \int f(t)\,dt}$$

$$F(s) \longrightarrow \boxed{K_1} \xrightarrow{B(s)} \boxed{K_2} \longrightarrow X(s)$$

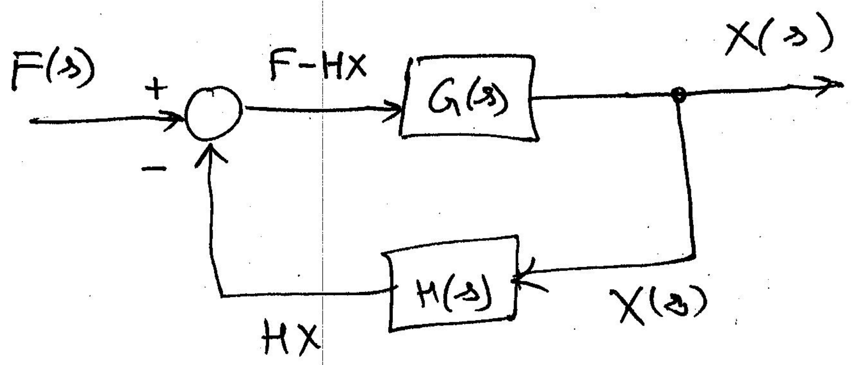
Find $\frac{X}{F}$

$$F K_1 = B$$

$$B K_2 = X$$

$$\underline{\underline{F K_1 K_2 = X}}$$

$$F \longrightarrow \boxed{K_1 K_2} \longrightarrow X \qquad \frac{X}{F} = K_1 K_2$$



Find $\frac{X}{F}$

$$(F - HX)\,G = X$$

$$FG - HGX = X$$

$$FG = X(1 + HG)$$

$$\frac{X}{F} = \frac{G}{1+HG}$$

$$F \longrightarrow \boxed{\frac{G}{1+HG}} \longrightarrow X$$

$$\dot{x} + 7x = f(t)$$

Draw the block Diagram.

By default $x(0) = 0$

Take Laplace on both sides.

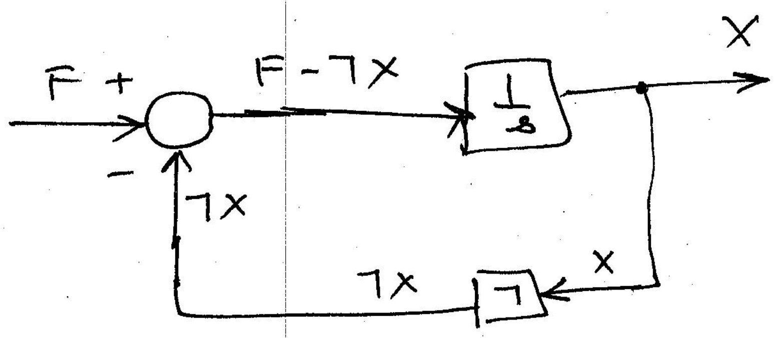
$$s\,X(s) - \cancel{x(0)}^{\,0} + 7\,X(s) = F(s)$$

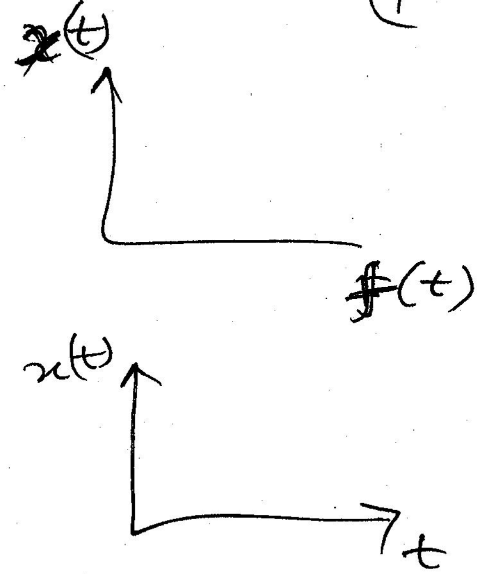
$$s\,X(s) + 7\,X(s) = F(s)$$

Keep higest power of s to the left & all other terms to the right

$$s\,X(s) = F(s) - 7\,X(s)$$

$$X(s) = \frac{1}{s}\left[F(s) - 7X(s)\right]$$





$$\ddot{x} + 7\dot{x} + 10x = f(t)$$

Draw the Block Diagram.

$x(0) = 0$

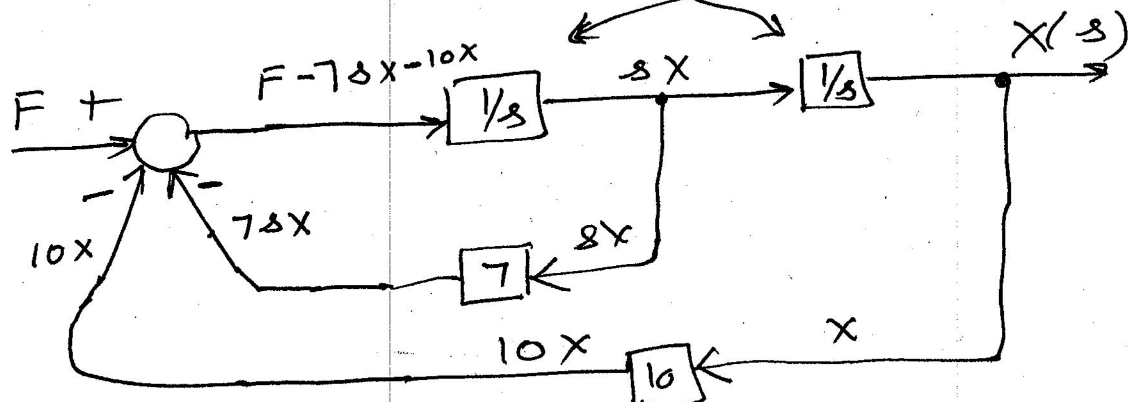
$\dot{x}(0) = 0$

$$s^2X(s) - \cancel{s\,x(0)} - \cancel{\dot{x}(0)} + 7\left(sX(s) - \cancel{x(0)}\right) + 10X(s) = F(s).$$

$$s^2X(s) + 7sX(s) + 10X(s) = F(s)$$

$$s^2X(s) = F(s) - 7sX(s) - 10X(s)$$

$$X(s) = \frac{1}{s}\left[\frac{1}{s}\left[F - 7sX - 10X\right]\right]$$

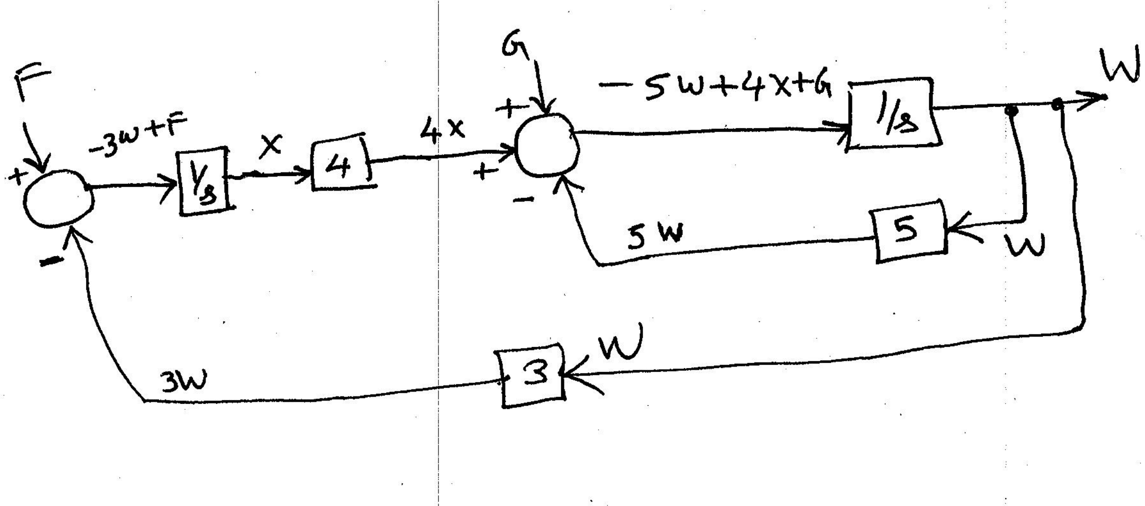


$$\dot{x} = -3w + f(t)$$
$$\dot{w} = -5w + 4x + g(t)$$

W(s) is the final state
X(s) is some intermediate state.
f(t) & g(t) are inputs.
Draw the B.D.

$x(0) = 0$
$w(0) = 0$

$$sX = -3W + F$$
$$sW = -5W + 4X + G$$
$$W = \frac{1}{s}\left[-5W + 4X + G\right]$$
$$X = \frac{1}{s}\left[-3W + F\right]$$



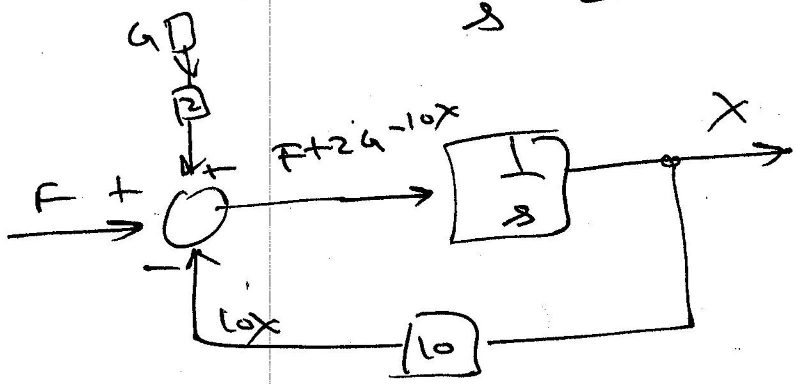
$$\dot{x} + 10x = f(t) + 2g(t)$$

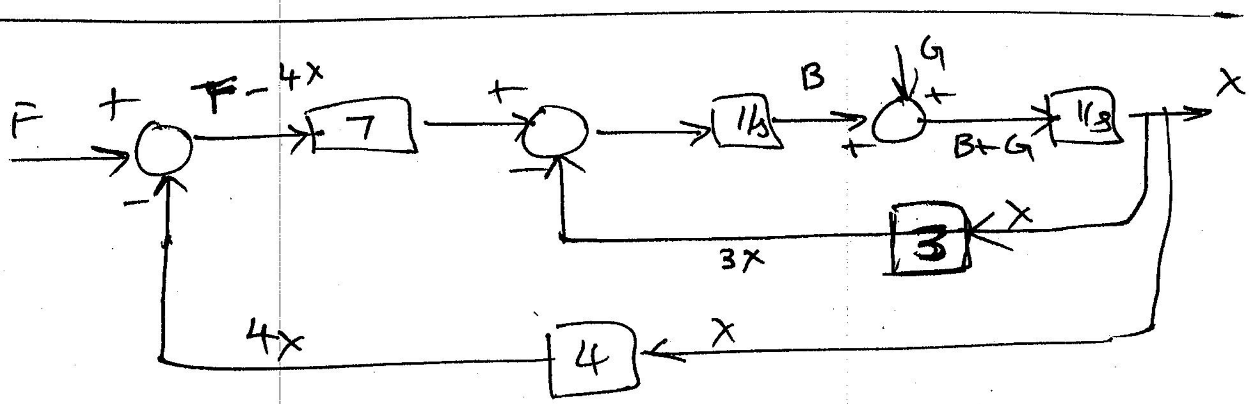
Draw the B.D

$$sX(s) + 10X = F + 2G$$

$$sX = F + 2G - 10X$$

$$X = \frac{1}{s}\left[F + 2G - 10X\right]$$





$$\left\{(F-4X)7 - 3X\right\}\frac{1}{s} = B$$

$$(B+G)\frac{1}{s} = X \qquad \text{Eliminate } B.$$

$$\left[\left\{(F-4X)7 - 3X\right\}\frac{1}{s} + G\right]\frac{1}{s} = X$$

$$\left\{\left[7F - 28X - 3X\right]\frac{1}{s} + G\right\}\frac{1}{s} = X$$

$$\left[7F - 31X\right]\frac{1}{s^2} + \frac{G}{s} = X$$

$$7F - 31X + Gs = s^2 X$$

$$s^2 X + 31X = 7F + Gs$$

$$\ddot{x} + 31x = 7f(t) + \dot{g}(t)$$

Converting higher order ODE into <u>coupled</u>

1st ORDER ODE

<u>Ex</u> $5\ddot{z} + 7\dot{z} + 4z = f(t)$

Convert into 1st order ODE

New state variables
$$\begin{cases} x_1 = z \Rightarrow \dot{x}_1 = \dot{z} = x_2 \Rightarrow \boxed{\dot{x}_1 = x_2} \\ x_2 = \dot{z} \Rightarrow \dot{x}_2 = \ddot{z} \end{cases}$$

$$\boxed{5\dot{x}_2 + 7x_2 + 4x_1 = f(t)}$$

Matrix format
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-4}{5}x_1 - \frac{7}{5}x_2 + \frac{f(t)}{5}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-4}{5} & \frac{-7}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} f(t)$$

$[2\times2]\ [2\times1] \quad [2\times1]\ [1\times1]$

Ex2

$$5\ddot{x}_1 + 12\dot{x}_1 + 5x_1 - 8\dot{x}_2 - 4x_2 = 0$$

$$3\ddot{x}_2 + 8\dot{x}_2 + 4x_2 - 8\dot{x}_1 - 4x_1 = f(t)$$

Express in 1st order ODE

New state variable

$$\left\{\begin{array}{llll} z_1 = \dot{x}_1 & \Rightarrow \dot{z}_1 = \ddot{x}_1 & & \\ z_2 = x_1 & \Rightarrow \dot{z}_2 = \dot{x}_1 = z_1 & \Rightarrow & \boxed{\dot{z}_2 = z_1} \\ z_3 = \dot{x}_2 & \Rightarrow \dot{z}_3 = \ddot{x}_2 & & \\ z_4 = x_2 & \Rightarrow \dot{z}_4 = \dot{x}_2 & \Rightarrow & \boxed{\dot{z}_4 = z_3} \end{array}\right.$$

$$5\dot{z}_1 + 12z_1 + 5z_2 - 8z_3 - 4z_4 = 0$$

$$3\dot{z}_3 + 8z_3 + 4z_4 - 8z_1 - 4z_2 = f(t)$$

$$\dot{z}_1 = -\frac{12}{5}z_1 - \frac{5}{5}z_2 + \frac{8}{5}z_3 + \frac{4}{5}z_4$$

$$\dot{z}_3 = +\frac{8}{3}z_1 + \frac{4}{3}z_2 - \frac{8}{3}z_3 - \frac{4}{3}z_4 + \frac{f(t)}{3}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} & 1 & \frac{8}{5} & 4/5 \\ 1 & 0 & 0 & 0 \\ 8/3 & 4/3 & -8/3 & -4/3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/3 \\ 0 \end{bmatrix} f(t)$$

$$f \quad x \quad y$$

$$\text{input} \longrightarrow \boxed{\text{System}} \longrightarrow \text{output}$$

Standard form of ODE

Also called "State variable" form

s-v form

s-v form

$$\boxed{\begin{aligned} \dot{x} &= A x + B f(t) \\ y(t) &= C x(t) + D f(t) \end{aligned}}$$

$\dot{x} = A x + B f(t) \longrightarrow$ ODE

$y(t) = C x(t) + D f(t) \longrightarrow$ Not an ODE

$y \to$ output $\quad x \to$ state variable

$b, u, f \to$ inputs.

$$\frac{\text{output}}{\text{input}} = \frac{Y(s)}{U(s)} = \frac{5s+3}{s+2}$$

Express in s-v form.

$$\frac{\text{State}}{\text{input}} = \frac{X(s)}{U(s)} = \text{Transfer function.}$$

$$U(s) \longrightarrow \boxed{T.F} \longrightarrow X(s)$$

$$U(s)(T.F) = X(s)$$

$$\frac{\text{output}}{\text{input}} = \frac{Y(s)}{U(s)} = \frac{5s+3}{s+2}$$

Express in s–v form.

Same as saying find A B C & D

Method 1

$$\frac{Y}{U} = \frac{5s+3}{s+2}$$

Make the Highest power of s term to 1 in the denominator

$$\frac{Y}{U} = \frac{5+\frac{3}{s}}{1+\frac{2}{s}}$$

Make this 1

$$Y\left(1+\frac{2}{s}\right) = U\left(5+\frac{3}{s}\right)$$

$$Y = 5U + \frac{3U}{s} - \frac{2Y}{s}$$

$$Y = 5U + \frac{1}{s}[3U - 2Y]$$

State X

$$\frac{1}{s}[3U-2Y] = X$$

$$sX = -2Y + 3U$$

$$\dot{x} = -2y(t) + 3u(t)$$

$$\dot{x} = -2[5u(t) + x(t)] + 3u(t)$$

$$\boxed{\dot{x} = -2x(t) - 7u(t)}$$

$A = -2 \quad B = -7$

$$Y = 5U + X$$

$$y(t) = 5u(t) + x(t)$$

$$\boxed{y(t) = x(t) + 5u(t)}$$

$C = 1 \quad D = 5$

Std format

$$\dot{x} = -2x - 7u$$

$$y = x + 5u$$

Method 2

$$\frac{Y}{U} = \frac{5s+3}{s+2}$$

$$Y = (5s+3)\underbrace{\left[\frac{U}{s+2}\right]}$$

State variable X

$$Y = (5s+3)X$$

$$Y = 5sX + 3X$$

$$y = 5\dot{x} + 3x$$

$$y = 5(-2x+u) + 3x$$

$$\boxed{y = -7x + 5u}$$

$C = -7$

$D = 5$

$$\frac{U}{s+2} = X$$

$$U = sX + 2X$$

$$u(t) = \dot{x} + 2x(t)$$

$$\boxed{\dot{x} = -2x(t) + u(t)}$$

$A = -2$

$B = 1$

Method 1

$A = -2 \quad B = -7$

$C = 1 \quad D = 5$

Method 2

$A = -2 \quad B = 1$

$C = -7 \quad D = 5$

$$\frac{Y}{U} = \frac{\text{output}}{\text{input}} = \frac{4s+7}{5s^2+4s+7}$$

$$\overline{\overline{s\text{-}v}}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

(1.

Find s-v Equations

Method 1 $\frac{Y}{U} = \frac{4s+7}{5s^2+4s+7}$

$\xrightarrow{}$ make this 1 [Also s^2 means 2 state variables]

Divide by $5s^2$

$$\frac{Y}{U} = \frac{\frac{4}{5s} + \frac{7}{5s^2}}{1 + \frac{4}{5s} + \frac{7}{5s^2}}$$

$$Y + \frac{4Y}{5s} + \frac{7Y}{5s^2} = \frac{4U}{5s} + \frac{7U}{5s^2}$$

$$Y = \frac{1}{s^2}\left(\frac{7U-7Y}{5}\right) + \frac{1}{s}\left(\frac{4U-4Y}{5}\right)$$

$$= \underbrace{\frac{1}{s}\left[\frac{4U-4Y}{5} + \underbrace{\frac{1}{s}\left[\frac{7U-7Y}{5}\right]}_{X_1}\right]}_{X_2}$$

$$Y = X_2 \qquad \boxed{y = x_2}$$

$$X_1 = \frac{1}{s}\left(\frac{7U - 7Y}{5}\right)$$

$$sX_1 = \frac{7U - 7Y}{5}$$

$$\dot{x}_1 = \frac{7}{5}u(t) - \frac{7}{5}y(t)$$

$$\boxed{\dot{x}_1 = \frac{7}{5}u(t) - \frac{7}{5}x_2}$$

$$X_2 = \frac{1}{s}\left[\frac{4U - 4Y}{5} + X_1\right]$$

$$sX_2 = \frac{4U - 4Y}{5} + X_1$$

$$\dot{x}_2 = \frac{4}{5}u(t) - \frac{4}{5}y(t) + x_1$$

$$\boxed{\dot{x}_2 = \frac{4}{5}u(t) - \frac{4}{5}x_2 + x_1}$$

In Matrix format

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -7/5 \\ 1 & -4/5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 7/5 \\ 4/5 \end{bmatrix}}_{B} u(t)$$

$$y = [0\ 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,u(t) \qquad y = x_2 \qquad C = [0\ 1] \qquad D = [0]$$