Initial Value Theorem (IVT)

Given $x(0)$ find $x(0^+)$

IVT does not apply if the (highest) power of s in the numerator & denominator are equal for $X(s)$

$$\boxed{x(0+) = \lim_{s \to \infty} s\,X(s)}$$

$x(0) = 0$

$$X(s) = \frac{7s+2}{s(s+6)} = \frac{7s+2}{s^2+6s}$$

$$x(0+) = \lim_{s \to \infty} s\left(\frac{7s+2}{s^2+6s}\right) = \lim_{s \to \infty} \frac{7s+2}{s+6}$$

$$= \lim_{s \to \infty} \frac{(7s+2)/s}{(s+6)/s} = \lim_{s \to \infty} \frac{7+\frac{2}{s}}{1+\frac{6}{s}} = 7$$

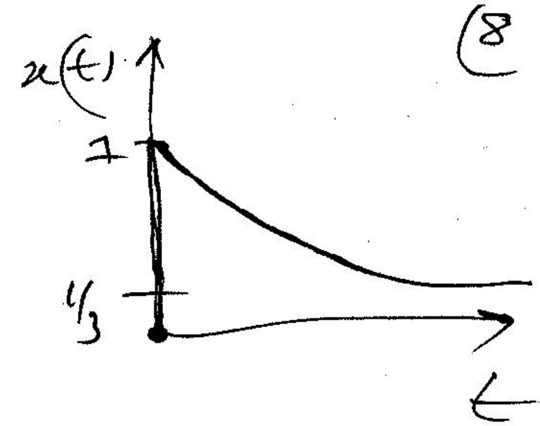
$$x(0+) = 7$$

$$X(s) = \frac{7s+2}{s(s+6)} = \frac{c_1}{s} + \frac{c_2}{s+6}$$

$$c_1 = \lim_{s \to 0} \cancel{s}\left(\frac{7s+2}{\cancel{s}(s+6)}\right) = 2/6 = 1/3$$

$$c_2 = \lim_{s \to -6} \cancel{(s+6)}\left(\frac{7s+2}{s\cancel{(s+6)}}\right) = \frac{-40}{-6} = \frac{20}{3}$$

$$\boxed{x(t) = \frac{1}{3} + \frac{20}{3}e^{-6t}}$$



$$X(s) = \frac{s+4}{s+3}$$ Find $x(0+)$

IVT does not apply

because highest power of s is the same in the num & den of $X(s)$

$x(0) = 0$

If you apply IVT

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} \frac{s(s+4)}{s+3} = \lim_{s\to\infty} \frac{s+4}{1+\frac{3}{s}}$$

(Divide by s)

$$= \infty$$

IVT does not apply

$$X(s) = \frac{s+4}{s+3}$$ Take inverse

$$= \frac{s}{s+3} + \frac{4}{s+3} = C_1 + \frac{C_2}{(s+3)}$$

$$s+4 = c_1(s+3)+c_2$$

$s \rightarrow \quad 1 = c_1$

constants $\rightarrow \quad 4 = 3c_1 + c_2$

$$c_2 = 1$$

$$X(s) = 1 + \frac{1}{s+3}$$

$$\boxed{x(t) = \underbrace{\delta(t)}_{\text{impulse}} + e^{-3t}}$$

Final value theorem (FVT)

Given $x(0)$ find $x(\infty)$

$$\boxed{x(\infty) = \lim_{s \to 0} sX(s)}$$

FVT does not apply if the real roots of the ~~steps~~ denominator of $sX(s)$ is positive, zero or the function $x(t)$ is periodic.

$$X(s) = \frac{9s+2}{s(s+2)}$$

Roots of $sX(s) = \cancel{s}\left[\frac{9s+2}{\cancel{s}(s+2)}\right]$

$$= \frac{9s+2}{(s+2)}$$

Root here is -2 for $sX(s)$

(FVT applies)

$$x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{9s+2}{s+2} = 2/2 = 1$$

$$X(s) = \frac{9s+2}{s^2(s+2)} \qquad sX(s) = \cancel{s}\left(\frac{9s+2}{\cancel{s^2}(s+2)}\right)$$

$$sX(s) = \frac{9s+2}{s(s+2)}$$

Roots $0, -2$

Cannot apply FVT

If you apply

$$x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{9s+2}{s(s+2)}$$

$=$ Divide by 0.

Cannot apply FVT

11

$$X(s) = \frac{5}{s^2+25} \qquad x(\infty) = ?$$

$$s\,X(s) = \frac{5s}{s^2+25}$$

Roots $\Rightarrow s^2 = -25$

$s = 0 \pm 5i$

real root is 0

Cannot apply FVT

$$x(t) = L^{-1}\left[\frac{5}{s^2+5^2}\right]$$

$$\boxed{x(t) = \sin 5t}$$

periodic function.

Impulse Function $\delta(t)$

delta function.

$$L(\delta(t)) = 1$$

$x(0) = 0 \qquad X(s) = \dfrac{F(s)}{(s+5)}$

$$\boxed{\begin{aligned} f(t) &= \delta(t) \\ F(s) &= 1 \end{aligned}}$$

$(s+5)X(s) = F(s)$

$sX(s) + 5X(s) = F(s)$

$\dot{x} + 5x = f(t) = \delta(t)$

$$X(s) = \frac{1}{s+5}$$

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\frac{s}{s+5} = \lim_{s\to\infty}\frac{s/s}{1+\frac{5}{s}}$$

$$x(0^+) = \lim_{s\to\infty} = 1/1 = 1$$

$\ddot{x} = \delta(t) \qquad x(0) = 5.$

$\qquad\qquad\qquad \dot{x}(0) = 10$

Find

IVT $\leftarrow$ $x(0+) = \lim_{s \to \infty} s X(s)$

$\leftarrow$ $\dot{x}(0+) = \lim_{s \to \infty} s\left[s X(s) - x(0) \right]$

IVT for 1st derrivative of x

$X(s) = ?$

$s^2 X(s) - s x(0) - \dot{x}(0) = 1$

$s^2 X(s) - 5s - 10 = 1$

$X(s) = \dfrac{11 + 5s}{s^2}$

$x(0^+) = \lim_{s \to \infty} s X(s) = \lim_{s \to \infty} \cancel{s}\left(\dfrac{11+5s}{s^{\cancel{2}}} \right)$

$= \lim_{s \to \infty} \dfrac{11 + 5s}{s} = \lim_{s \to \infty} \dfrac{\frac{11}{s} + 5}{1}$

No impact of the impulse on $x(t)$ $\leftarrow$ $\begin{cases} x(0^+) = 5. \\ x(0) = 5 \end{cases}$

$$\dot{x}(0+) = \lim_{s \to \infty} s\left[sX(s) - x(0)\right]$$

$$= \lim_{s \to \infty} s\left[\left(\frac{11+5s}{s}\right) - 5\right]$$

$$= \lim_{s \to \infty} \cancel{s}\left[\frac{11 + \cancel{5s} - \cancel{5s}}{\cancel{s}}\right]$$

$$= \lim_{s \to \infty} (11) = 11$$

$$\begin{cases} \dot{x}(0+) = 11 \\ \dot{x}(0) = 10 \end{cases}$$

Transfer Functions (T.F)

$$T.F. = \frac{X(s)}{F(s)}$$

Expressing ODE in this format is called a T.F.

$$\boxed{F(s) = L(f(t)) = \text{input}}$$

$X(s) = L(x(t))$ = system variable.

$\boxed{\text{T.F. is \underline{only} in the } s\text{-domain}}$

By default $x(0) = 0$ $\dot{x}(0) = 0$

(14

$$\cancel{F}(s) \xrightarrow{\text{input}} \boxed{T \cdot F} \longrightarrow \cancel{X}(s) \text{ Statevariable.}$$

$$\xrightarrow{\text{input}} \boxed{\text{state}} \xrightarrow{\text{output}}$$

Very different

Doesn't apply

$$F(s) \longrightarrow \boxed{\frac{X(s)}{F(s)}} \longrightarrow X(s)$$

T.F.

$\boxed{\text{Input multiplies the T.F to give } X(s)}$

$$F(s) \times (T \cdot F) = X(s)$$

$$F(s) \longrightarrow \boxed{\frac{1}{s+2}} \longrightarrow X(s)$$

$$F(s) \times \frac{1}{(s+2)} = X(s)$$

$$X(s) = \frac{F(s)}{s+2}$$

$$(s+2)\, X(s) = F(s)$$

$$s\,X(s) + 2X(s) = F(s)$$

$$\underbrace{s\,X(s) - x(0)} + 2X(s) = F(s)$$

$$\boxed{\dot{x} + 2x = f(t)}$$

15

Find T.F

$$5\ddot{x} + 30\dot{x} + 40x = 6f(t) - 20g(t)$$

$$\frac{X(s)}{F(s)} \qquad \frac{X(s)}{G(s)}$$

$$5s^2X(s) + 30sX(s) + 40X(s) = 6F(s) - 20G(s)$$

$$(5s^2 + 30s + 40)X(s) = 6F(s) - 20G(s)$$

$$\frac{X(s)}{F(s)} = \frac{6}{5s^2+30s+40}$$

$$\frac{X(s)}{G(s)} = \frac{-20}{5s^2+30s+40}$$

Stability of ODE

Find the characteristic roots.

If the real part of the roots are ALL negative then system is stable

If any one of the real part is ~~pos~~ positive the system is unstable

If anyone of the real roots is zero then it is neutrally stable (oscillates).

↓

sine or cosine in the system.

$$\underbrace{\ddot{x} - 25x}\; = 10$$

$$s^2 - 0s - 25 = 0$$

$\ddot{x} \to s^2$

$\dot{x} \to s$

$x \to 1$

$$s^2 = 25$$

$$s = \pm 5$$

unstable system (one root is positive).

$$\ddot{x} + 25x = 10$$

$$s^2 + 25 = 0$$

$$s = \sqrt{-25} = 0 \pm 5i$$

neutrally stable because of 0 in the real part.

Free, forced, steady & transient parts of the solution to ODE

Do not plug the values of $x(0)$ and $\dot{x}(0)$.

Free → depends on initial condition.

forced → depend on input

Steady → stays over time.

transient → vanishes over time.

$\ddot{x} + 25x = 10$ Find the part of the solution to the ODE

$a = 0$
$b = 25$
$c = 10$.

Roots

$s^2 + 25 = 0$

$s = 0 \pm 5i$ $\omega = 5$
$\sigma = 0$

$x(t) = c_1 \sin\omega t + c_2 \cos\omega t + \frac{c}{b}$

$x(0) = c_2 + \frac{c}{b} \Rightarrow \boxed{c_2 = x(0) - \frac{c}{b}}$

$\dot{x}(t) = c_1 \omega \cos\omega t - c_2 \omega \sin\omega t$

$\dot{x}(0) = c_1 \omega$ $c_1 = \frac{\dot{x}(0)}{\omega}$

$$x(t) = \frac{\dot{x}(0)}{\omega}\sin\omega t + \left(x(0) - \frac{c}{b}\right)\cos\omega t + \frac{c}{b}$$

$$x(t) = \frac{\dot{x}(0)}{\omega}\sin\omega t + x(0)\cos\omega t - \frac{c}{b}\cos\omega t + \frac{c}{b}$$

free :- $\frac{\dot{x}(0)}{\omega}\sin\omega t + x(0)\cos\omega t$

forced :- $-\frac{c}{b}\cos\omega t + \frac{c}{b}$

Steady :- All parts.

Transient :- None.

$$X(s) = \frac{6s+15}{s^2+20s+125} \rightarrow (s+10)^2 + 5^2$$

Find the parts.

$$(s^2+20s+125)X(s) = 6s+15$$

$$s^2X(s) + 20sX(s) + 125X(s) = 6s+15$$

$$s^2X(s) - s\cancel{x(0)}^{\,0} - \cancel{\dot{x}(0)}^{\,0} + 20sX(s) - \cancel{x(0)}^{\,0} + 125X(s) = 6s+15$$

free does not exist because intial conditions are zero.

$$\frac{6s+15}{s^2+20s+125} = \frac{c_1(s+10)}{(s+10)^2+5^2} + \frac{c_2(5)}{(s+10)^2+5^2}$$

$$6s+15 = c_1 s + 10c_1 + 5c_2$$

s-terms → $6 = c_1$

Constats → $15 = 10c_1 + 5c_2$

$$15 = 60 + 5c_2$$

$$c_2 = \frac{15-60}{5}$$

$\mathcal{L}^{-1}$

$$x(t) = 6e^{-10t}\cos 5t + \left(\frac{15-60}{5}\right)e^{-10t}\sin 5t$$

Forced ⇒ Both terms because of 6 & 15 are coming from RHS of the ODE (input coefficients)

Free ⇒ None → initial conditions are zero

steady ⇒ None → Terms vanish over time

Transiat → Both → Terms vanish over time.