

Solving ODE

Solve $\dot{x} + 5x = 3$ $x(0) = 3$

$\frac{dx}{dt} + 5x(t) = 3 \rightarrow$ constant input.

Finding $x(t) = ?$

RHS is the input to the ODE.

Solve using Laplace.

Take Laplace Transform on both sides.

$$sX(s) - x(0) + 5X(s) = \frac{3}{s}$$

$$sX(s) - 3 + 5X(s) = \frac{3}{s}$$

$$X(s)(s+5) = \frac{3}{s} + 3$$

$$(s+5)X(s) = \frac{3+3s}{s}$$

$$X(s) = \frac{3+3s}{s(s+5)} = \frac{c_1}{s} + \frac{c_2}{s+5}$$

$$c_1 = \lim_{s \rightarrow 0} sX(s) = \lim_{s \rightarrow 0} s \left(\frac{3+3s}{s(s+5)} \right) = \frac{3}{5}$$

$$c_2 = \lim_{s \rightarrow -5} (s+5)X(s) = \lim_{s \rightarrow -5} \frac{(3+3s)(s+5)}{s(s+5)} \\ = \frac{-12}{-5} = \frac{12}{5}$$

$$X(s) = \frac{3}{s} + \frac{12}{s+5}$$

Take inverse

$$x(t) = \frac{3}{5} + \frac{12}{5} e^{-5t}$$

$$\dot{x} + 5x = 3 \quad x(0) = 3$$

From Table

$$\dot{x} + ax = b \quad x(t) = \frac{b}{a} + ce^{-at}$$

$$a = 5$$
$$b = 3$$

$$x(t) = \frac{3}{5} + ce^{-5t}$$

$$x(0) = \frac{3}{5} + ce^0 = 3$$

$$c = 3 - \frac{3}{5} = \frac{12}{5}$$

$$x(t) = \frac{3}{5} + \frac{12}{5} e^{-5t}$$

Solve $\ddot{x} + 7\dot{x} + 10x = 20$ $x(0) = 5$
 $\dot{x}(0) = 3$

$\ddot{x} + a\dot{x} + bx = c$
 $a = 7$
 $b = 10$
 $c = 20$ (input)

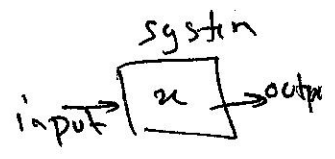
Characteristic equation of an ODE.

Focus only on left side of the equation

$\ddot{x} \Rightarrow s^2$
 $\dot{x} \Rightarrow s$
 $x \Rightarrow 1$

$s^2 + 7s + 10 = 0$

Find the roots.



$(s+5)(s+2) = 0$

Roots $-5, -2$
 $(s_1) (s_2)$

distinct Roots

Solution from table

$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \frac{c}{b}$

$x(t) = c_1 e^{-5t} + c_2 e^{-2t} + \frac{20}{10}$

$x(0) = 5 = c_1 + c_2 + 2$

$c_1 + c_2 = 3$

$\dot{x}(t) = -5c_1 e^{-5t} - 2c_2 e^{-2t}$

$\dot{x}(0) = 3 = -5c_1 - 2c_2$

$c_1 = -3 \quad c_2 = 6$

$x(t) = -3e^{-5t} + 6e^{-2t} + 2$

Ex

$$\text{Solve } \ddot{x} + 4\dot{x} + 53x = 15$$

$$x(0) = 8$$

~~$$\dot{x}(0) = 8$$~~

$$\dot{x}(0) = -19$$

$$\ddot{x} + a\dot{x} + bx = c$$

$$a = 4 \quad b = 53 \quad c = 15$$

RHS \rightarrow constant input (you may use the table)

Characteristic equation

$$\ddot{x} \rightarrow s^2$$

$$\dot{x} \rightarrow s$$

$$x \rightarrow 1$$

$$s^2 + 4s + 53 = 0$$

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-4 \pm \sqrt{4^2 - (4 \times 1 \times 53)}}{2 \times 1}$$

$$\text{Roots} = \frac{-4 \pm \sqrt{-196}}{2}$$

$$= -2 \pm 7i$$

Case 4! $\sigma \pm j\omega$

$$\sigma = -2$$

$$\omega = 7$$

Solution!
from the table

$$x(t) = e^{\sigma t} (c_1 \sin \omega t + c_2 \cos \omega t) + \frac{c}{b}$$

$$x(t) = e^{-2t} (c_1 \sin 7t + c_2 \cos 7t) + \frac{15}{53}$$

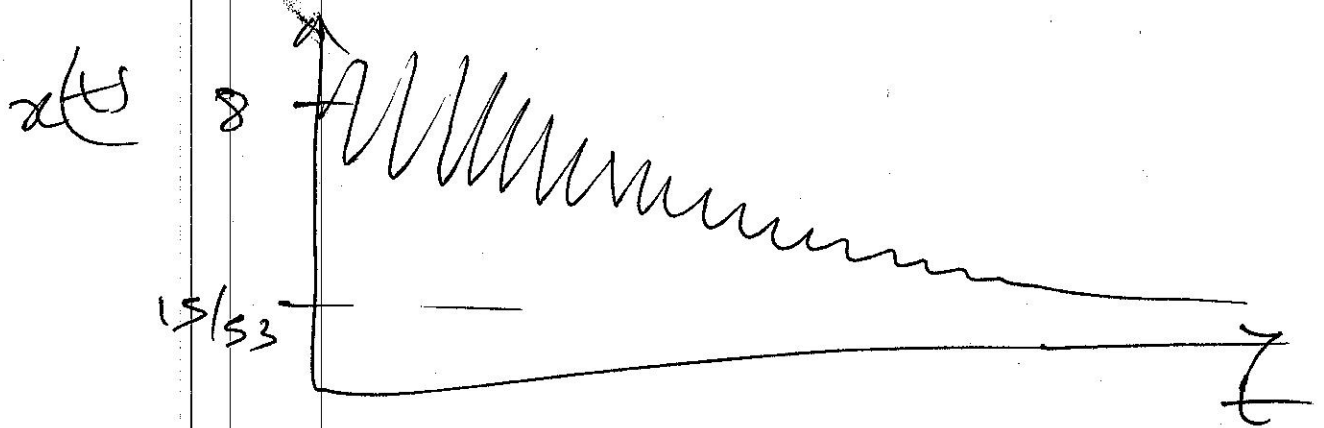
$$\dot{x}(t) = c_1 [e^{-2t} 7 \cos 7t + \sin 7t (-2)e^{-2t}] + c_2 [e^{-2t} -7 \sin 7t + \cos 7t (-2)e^{-2t}]$$

$$x(0) = 8 = c_2 + \frac{15}{53} \quad c_2 = \frac{8 \cdot 53 - 15}{53} = \frac{409}{53} \quad (14)$$

$$x'(0) = -19 = 7c_1 - 2c_2$$

$$c_1 = -27/53$$

Solution $x(t) = e^{-2t} \left(\frac{-27}{53} \sin 7t + \frac{409}{53} \cos 7t \right) + \frac{15}{53}$



Solving ODE with varying input

Solve $\dot{x} + 12x = 5t$ $x(0) = 0$, Find $x(t)$

↓
Not a constant
You cannot use table 2.3.2

Take Laplace Transform on both sides of =

$$sX(s) - x(0) + 12X(s) = \frac{5}{s^2}$$

$$sX(s) + 12X(s) = \frac{5}{s^2}$$

$$X(s) = \frac{5}{s^2(s+12)}$$

Use Partial fractions.

$$X(s) = \frac{5}{s^2(s+12)} = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{s+12}$$

$$c_1 = \lim_{s \rightarrow 0} s^2 X(s) = \lim_{s \rightarrow 0} \frac{5}{s+12} = \frac{5}{12}$$

$$c_2 = \lim_{s \rightarrow 0} \frac{d}{ds} [s^2 X(s)] = \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{5}{s+12} \right]$$

$$\lim_{s \rightarrow 0} \frac{0 - 5}{(s+12)^2} = \frac{-5}{144}$$

$$c_3 = \lim_{s \rightarrow -12} (s+12) \left(\frac{5}{s^2(s+12)} \right) = \frac{5}{144}$$

$$x(t) = \frac{5}{12} t - \frac{5}{144} + \frac{5}{144} e^{-12t}$$

Properties of Laplace Transform (16)

Table 2.2-2 page 42 3rd Edition of book

Prop 7 $\mathcal{L} [e^{-at} x(t)] = X(s+a)$

Example Suppose $x(t) = t$

Find $\mathcal{L} [(e^{-at}) x(t)]$

Product of two functions of time t

Use prop 7 to solve

Find $\mathcal{L} (x(t)) = X(s)$

Then replace s with $s+a$

for $\mathcal{L} [e^{-at} x(t)]$

$$\mathcal{L} (x(t)) = \mathcal{L} [t] = \frac{1}{s^2}$$

$$\mathcal{L} [e^{-at} x(t)] = \mathcal{L} [e^{-at} t] = \frac{1}{(s+a)^2}$$

Ex $x(t) = \sin bt$

Find $\mathcal{L} [e^{-at} x(t)]$

$$\mathcal{L} (\sin bt) = \frac{b}{s^2 + b^2} \quad / \quad \mathcal{L} (e^{-at} \sin bt) = \frac{b}{(s+a)^2 + b^2}$$

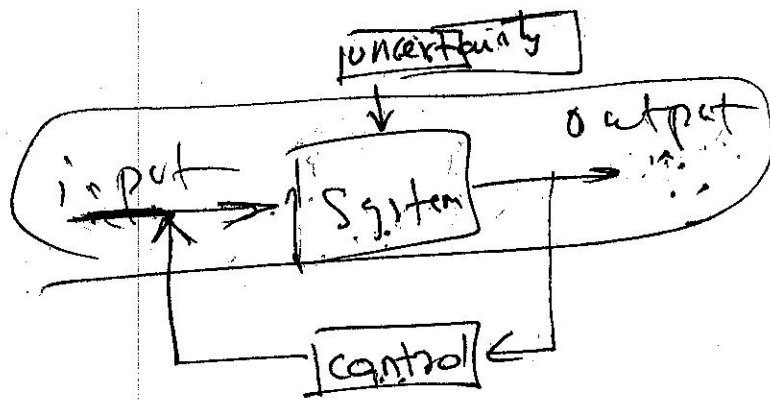
(17)

Property of $L[t x(t)] = -\frac{dX(s)}{ds}$

$$x(t) = e^{-at} \quad X(s) = \frac{1}{s+a}$$

$$L[t e^{-at}] = -\frac{d}{ds} \left[\frac{1}{s+a} \right]$$

$$= -\left(\frac{0-1}{(s+a)^2} \right) = \frac{1}{(s+a)^2}$$



Solving ODE with variable inputs

$$\dot{x} + 5x = 7te^{-3t} \quad x(0) = 0$$

Solve for $x(t)$

Take Laplace Transforms on both sides.

$$sX(s) - \overset{0}{x(0)} + 5X(s) = 7\left(\frac{1}{(s+3)^2}\right)$$

Property 7
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Find $L(t)$ and replace s with

$$(s+a) \quad L(t) = \frac{1}{s^2}$$

$$L(te^{-3t}) = \frac{1}{(s+3)^2} \quad \text{by}$$

property 7

$$(s+5)X(s) = \frac{7}{(s+3)^2}$$

$$X(s) = \frac{7}{(s+5)(s+3)^2}$$

$$X(s) = \frac{c_1}{(s+5)} + \frac{c_2}{(s+3)} + \frac{c_3}{(s+3)^2} \quad (2)$$

$$c_1 = \lim_{s \rightarrow -5} \frac{(s+5) \cdot 7}{(s+5)(s+3)^2} = \frac{7}{4}$$

$$c_3 = \lim_{s \rightarrow -3} \frac{(s+3)^2 \cdot 7}{(s+5)(s+3)^2} = \frac{7}{2}$$

$$c_2 = \lim_{s \rightarrow -3} \frac{d}{ds} \left(\frac{(s+3)^2 \cdot 7}{(s+3)(s+5)} \right)$$

$$= \lim_{s \rightarrow -3} \frac{0 - 7}{(s+5)^2} = \frac{-7}{4}$$

Take L^{-1}

$$x(t) = \frac{7}{4} e^{-5t} + \left(\frac{-7}{4} \right) e^{-3t} + \frac{7}{2} t e^{-3t}$$