Models

Exponential Model

Model: $a(n+1) = r\,a(n) + b \cdot s^n$

Solution: $a(n) = c\,r^n + \frac{b}{s-r}\,s^n$ if $r \neq s$

(OR)

$a(n) = c\,r^n + \frac{bn}{r}\,r^n$ if $r = s$

<u>Ex</u> $a(n+1) = 2\,a(n) + 3^n \quad a(0) = 1$

Solve.

$r = 2 \quad s = 3 \quad b = 1$

$r \neq s$

$a(n) = c\,2^n + \frac{1}{3-2}\,3^n = c\,2^n + 3^n$ → General solution

$a(0) = 1 = c + 1 \qquad c = 0$

$\boxed{a(n) = 3^n}$ → particular solution

Ex

$I = 0.08$ or 8% interest compounded ~~monthly~~ annually. (yearly)

$a(0) = 0$

Deposit \$100 at the end of each year.

Deposit increases 5% each year over the last year's deposit.

How much in the bank after n years?

$$r = \left(1 + \frac{I}{m}\right) = \left(1 + \frac{0.08}{1}\right) = 1.08$$

Deposit Model

$d(0) = 0$

$d(1) = 100$

$d(2) = 100 + \left(100 \times \frac{5}{100}\right)$

$= 100(1 + 0.05)$

$d(2) = 100(1.05) = 105$

$d(n) = d(0)(1.05)^n$

M $d(n+1) = k\, d(n)$

Solution $d(n) = c\, k^n$

M $d(n+1) = 1.05\, d(n)$

Solution $d(n) = (1.05)^n \times 100$

Account balance model

$a(n+1) = r\, a(n) + \text{Deposit}$

$a(n+1) = r\, a(n) + 100 \times (1.05)^n$

$\boxed{a(n+1) = 1.08\, a(n) + 100(1.05)^n}$

Model Exponential Model

$r = 1.08$

$b = 100$

$s = 1.05$

$r \neq s$

Solution:

$$a(n) = c\, r^n + \frac{b}{s-r}\, s^n \qquad r \neq s.$$

$$a(n) = c\,[1.08)^n + \frac{100}{(1.05-1.08)}\,(1.05)^n$$

$$a(0) = 0 = c + \frac{100}{(1.05-1.08)}$$

$$c = \frac{100}{0.03} = 3333.33$$

$$a(n) = 3333.33\,(1.08)^n - 3333.33\,(1.05)^n$$

$$\boxed{a(n) = 3333.33\,[(1.08)^n - (1.05)^n]}$$

Model 4

$a(n+1) = r\,a(n) + \text{polynomial}$

polynomial $\quad n^2+1 \rightarrow 2^{nd}$ degree

$n^2+n+1 \rightarrow 2^{nd}$ degree.

$n+5 \rightarrow 1^{st}$ degree.

Solution.

		ADD
n^2+1	$\rightarrow$	An^2+Bn+D
n^2+n+1	$\rightarrow$	An^2+Bn+D.
$n+5$	$\rightarrow$	$An+B$
n^2	$\rightarrow$	An^2+Bn+D
n	$\rightarrow$	$An+B$

If $r \neq 1$

Model $a(n+1) = 2\,a(n) + \underbrace{n+1}_{\text{polynomial}}, \quad a(0)=0$

Solve.

solution $\quad a(n) = C\,2^n + An + B$

$a(n+1) = C\,2^{n+1} + A(n+1) + B$

$C\,2^{n+1} + An + A + B = 2\left[C\,2^n + An + B\right] + n + 1$

$\cancel{C\,2^{n+1}} + An + A + B = \cancel{C\,2^{n+1}} + 2An + 2B + n + 1$

n terms $\quad A\cancel{n} = 2An + n = (2A+1)\cancel{n}$

$A = 2A+1$

$\underline{\underline{A = -1}}$

Constants $\quad A+B = 2B+1$

$A = B+1 \quad B = A-1$

$\underline{\underline{B = -2}}$

$a(n) = c\,2^n - n - 2$ (plug back into general solution) 5

$a(0) = 0 = c - 2$

$c = 2$

$a(n) = 2 \cdot 2^n - n - 2 = 2^{n+1} - n - 2$

HW 2

$a(n+1) = -2\,a(n) + 3n + 2$ $\qquad a(2) = 2$

Solution $a(n) = c(-2)^n + An + B$

$a(n+1) = c(-2)^{n+1} + A(n+1) + B$

$c(-2)^{n+1} + An + A + B = -2\left[c(-2)^n + An + B\right] + 3n + 2$

$\cancel{c(-2)^{n+1}} + An + A + B = \cancel{c(-2)^{n+1}} - 2An - 2B + 3n + 2$

collect n ~~terms~~ $An = -2An + 3n$

$A\cancel{n} = (-2A + 3)\cancel{n}$

$A = -2A + 3$

$3A = 3 \quad A = 1$

constants $A + B = -2B + 2$

$3B = 2 - 1 = 1 \quad B = 1/3$

$a(n) = c(-2)^n + n + 1/3$

$a(2) = 2 = c(-2)^2 + 2 + 1/3$

$\cancel{2} = 4c + \cancel{2} + 1/3 \qquad c = -1/12$

$\boxed{a(n) = -1/12\,(-2)^n + n + 1/3}$

If $r = 1$ in polynomial Models

2^{nd} order $\rightarrow n(An^2 + Bn + D)$

1^{st} order $\rightarrow n(An + B)$

$a(n+1) = a(n) + n + 1 \quad a(0) = 0$

Solution $\underline{\underline{r=1}}$

$a(n) = Cr^n + n(An+B)$

$a(n+1) = Cr^{n+1} + (n+1)(A(n+1)+B)$

$\cancel{C(1)^{n+1} + An^2 + Bn + An + B = C(1)^n + n(An+B) + n + 1}$

$C(1)^{n+1} + (n+1)(An + A + B) = C(1)^n + An^2 + Bn + n + 1$

$\cancel{C} + \cancel{An^2} + An + \cancel{Bn} + An + A + B = \cancel{C} + \cancel{An^2} + \cancel{Bn} + n + 1$

n terms $2An = n \qquad A = 1/2$

Constants $A + B = 1 \qquad B = 1/2$

$a(n) = C + n(1/2\, n + 1/2)$

$a(0) = 0 = C$

$a(n) = n\left(\frac{1}{2}n + \frac{1}{2}\right) = \frac{1}{2}\left[n^2 + n\right]$

Example for 2^{nd} order. Discrete Dy. Model.

$$a(n) = -3.5\, a(n-1) + 2\, a(n-2) \qquad a(0) = 3,\quad a(1) = -3$$

Solve.

Characteristic eqn: $x^2 = -3.5x + 2$

Solve for roots. $x^2 + 3.5x - 2 = 0$

In general $a x^2 + bx + c = 0$

$$\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Roots} \Rightarrow \frac{-3.5 \pm \sqrt{3.5^2 - (4 \times 1 \times -2)}}{2}$$

Roots $\Rightarrow \frac{1}{2}, -4$

$r = 1/2 \quad s = -4$

$r \neq s$

Solution: $a(n) = c_1 \left(\frac{1}{2}\right)^n + c_2 (-4)^n$

$a(0) = 3 = c_1 + c_2$

$a(1) = -3 = c_1 \times \left(\frac{1}{2}\right)^1 + c_2 (-4)^1$

$-3 = \frac{1}{2} c_1 - 4 c_2$

$c_2 = 3 - c_1$

$-3 = \frac{1}{2} c_1 - 4(3 - c_1)$

$-3 = \frac{1}{2} c_1 - 12 + 4c_1$

$9 = 4.5\, c_1 \qquad \boxed{c_1 = 2}$

$c_2 = 3 - 2 = 1$

$$\boxed{a(n) = 2\left(\frac{1}{2}\right)^n + (-4)^n}$$

Exponential financial Example

$I = 0.1$ 10% compounded monthly.

$a(0) = 10{,}000$

First ~~week~~ month you withdraw \$100

Every month after you withdraw 20% more than previous month.

How long will the money last?

withdraw model.

$w(0) = -100$

$w(1) = -120 = 1.2\,w(0)$

$w(2) = -144 = 1.2\,w(1) = 1.2 \times 1.2\,w(0)$

$\vdots \qquad \vdots$

Model $w(n+1) = 1.2\,w(n)$

Solution $w(n) = D(1.2)^n$

$w(0) = -100 = D(1.2)^0$

$D = -100$

$w(n) = -100\,(1.2)^n$

Account balance Model

$a(0) = 10{,}000$

$a(n+1) = 1.0083\,a(n) + (\text{withdraw Model})$

$r = 1 + \frac{I}{m}$

$r = 1 + \frac{0.1}{12}$

$= 1.0083$

$a(n+1) = 1.0083\, a(n) - 100\, (1.2)^n$

$r = 1.0083$

$b = 100$ $\qquad r \neq s$

$s = 1.2$

$a(n) = c\,(1.0083)^n - \dfrac{100}{1.2 - 1.0083}(1.2)^n$

$$\boxed{a(n) = c\,(r)^n + \frac{b}{s-r}\, s^n}$$

$a(0) = 10000 = c - \dfrac{100}{0.1917}$

$c = 10000 + \dfrac{100}{0.1917} = 10000 + 521.65$

10521.65

$= \cancel{9478.35}$

$$\boxed{a(n) = \underbrace{\cancel{9478.35}\ 10521.65\,(1.0083)^n}_{\text{growth model}} \underbrace{-\ 521.65\,(1.2)^n}_{\text{withdraw model}}.}$$

$n =$ 1 year	11618.79	$-$	469.	$= 11149$
$n =$ 10 years	28369	$-$	1.67×10^{11}	$=$ negative

$n =$ 20 years.

How long it lasts. ?

$a(n) = 0 = 10521.65\,(1.0083)^n - 521.65\,(1.2)^n$

Solve for n: $\ln(10521.65) + n\ln(1.0083) = \ln(521.65) + n\ln(1.2)$

$n = \dfrac{\ln(10521.65) - \ln(521.65)}{\ln(1.2) - \ln(1.0083)}$ $\quad n = 17$ months

SUMMARY

Model 1	$a(n+1) = r\,a(n)$
Solution	$a(n) = C r^n$
Model 2	$a(n+1) = r\,a(n) + b$
Solution:	$a(n) = C r^n + \dfrac{b}{1-r} \quad r \neq 1$
	(OR) $a(n) = bn + C$ if $r = 1$

Model 2 - Financial Models.

$$r = \left(1 + \frac{I}{m}\right)$$

Model $a(n+1) = \left(1 + \dfrac{I}{m}\right) a(n) + b$

Solution: $a(n) = C\left(1 + \dfrac{I}{m}\right)^n - \dfrac{bM}{I}$

$M \to 1$, $n \to$ years	compound yearly
$M \to 4$, $n \to$ quarters	quartely
$M \to 2$, $n \to \frac{1}{2}$ years	half yearly
$M \to 12$, $n \to$ months	monthly

Model 3 Exponential

Model: $a(n+1) = r\,a(n) + b \cdot s^n$

Solution: $a(n) = C r^n + \dfrac{b}{s-r}, \quad r \neq s$

(OR) $a(n) = C r^n + \dfrac{bn}{r} r^n, \quad r = s$

Model 4 polynomial

Model : $a(n+1) = r\,a(n) + \text{polynomial}$

Solution. $a(n) = C r^n + [\quad]$

$r \neq 1$	1st order	$An+B$
$r \neq 1$	2nd order	An^2+Bn+D
$r = 1$	1st order	$n(An+B)$
$r = 1$	2nd order	$n(An^2+Bn+D)$

Model 5: 2nd order Discrete Dynamical systems.

<u>Model</u> $a(n+2) = b_1 a(n+1) + b_2 a(n)$

$a(n+1) = b_1 a(n) + b_2 a(n-1)$

Solution: Solve ~~the~~ characteristic equation

$$x^2 = b_1 x + b_2$$

Solve for the Roots of the quadratic equation.

r, s are the roots.

$$a(n) = C_1 r^n + C_2 s^n \quad \text{if } r \neq s$$

(OR) $a(n) = C_1 r^n + n C_2 r^n$ if $r = s$