# Discrete Dynamical Model ①

## Difference Equations

Discrete because of the way data was collected at discrete time points.

system is always continuous.

Example. A person takes a drug (20mg of the drug). The Kidneys clear 20% of the drug from the blood every 4 hrs.

1) Build a dynamical model for the above
2) How much drug left after 20 hrs?

Time 0 → 20mg.

4hrs → $20 - \left(\frac{20}{100} \times 20\right) = 20(1-0.2) = 20 \times 0.8 = 16$

8 hrs → $16 - \left(\frac{20}{100} \times 16\right) = 16(1-0.2) = 16 \times 0.8 = 12.8$

| Time index $n$ | Real time | |
|---|---|---|
| 0 | 0 | $a(0) = 20$ |
| 1 | 4hrs | $a(1) = 16$ |
| 2 | 8 hrs | $a(2) = 12.8$ |
| 3 | 12 | $\vdots = \vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| | | $a(n) =$ |

$\underline{a(n)}$ is a variable that tracks amount of drug at time index $\underline{n}$

| Real time | |
|---|---|
| 0 | $a(0) = 20$ |
| 4 | $a(1) = 0.8 \times 20 = 0.8\, a(0) = 16$ |
| 8h | $a(2) = 0.8 \times 16 = 0.8\,(a(1)) = 0.8 \times 0.8 \times a(0) = 0.8^2 \times a(0)$ |
| 12 | $a(3) = 0.8^3 a(0) = 0.8\, a(2)$ |
| 16h | $a(4) = 0.8^4 a(0)$ |
| 20h | $a(5) = 0.8^5 a(0) = (0.8)^5 \times 20 = 6.55$ |
| 100h | $a(25) = 0.8^{25} a(0) = (0.8)^{25} \times 20 = 0.076$ |

1 Model $a(n+1) = r\, a(n)$

Solution $a(n) = C r^n \rightarrow$ General solution

Find $C$ using $a(0) \rightarrow$ initial condition

$$a(0) = C r^0 = C$$

$$\boxed{a(n) = a(0)\, r^n}$$

$r = 0.8 \quad a(0) = 20.$

$$\boxed{a(n) = 20\,(0.8)^n} \rightarrow \text{Particular solution}$$

Ex 2 $a(0) = \$1000$ interest is 2%

$r = 1.02$

$a(1) = 1000 + \left(\frac{2}{100} \times 1000\right)$

$= 1000(1 + 0.02)$

$= 1000 \times 1.02$

$= 1020$

$= 1.02\, a(0)$

$a(n+1) = 1.02\, a(n)$ — Model

General Solution $a(n) = c\, r^n$

$a(0) = c\, r^0 = c$

$$\boxed{a(n) = 1000\,(1.02)^n}$$ — Particular Solution

$a(n) = 10000$ Find $n$

$10000 = 1000\,(1.02)^n$

$10 = 1.02^n$

$\ln(10) = n \ln(1.02)$

$n = \dfrac{\ln(10)}{\ln(1.02)} = 116$ years.

Ex3

$a(0) = 90$    90% loss    15 gain per fish.

original

$$a(1) = \underbrace{90}_{a(0)} - \left(\frac{90}{100} \times \underbrace{90}_{a(0)}\right) + \left[\left(\frac{10}{100} \times \underbrace{90}_{a(0)}\right) \times 15\right]$$

$$= 90(1 - 0.9 + (0.1 \times 15))$$

$$= 90(1 - 0.9 + 1.5)$$

$$= 90(1.6)$$

$$a(1) = a(0) \times (1.6) \qquad \boxed{r = 1.6}$$

Model $a(n+1) = 1.6\, a(n)$

Solution $a(n) = C(1.6)^n \rightarrow$ General

$$a(0) = C(1.6)^0 = C = 90$$

$$\boxed{a(n) = 90(1.6)^n} \rightarrow \text{particular solution}$$

Ex $\quad a(n+1) = 2\,a(n) + 1$ $\qquad \begin{cases} b = 1 \\ r = 2 \neq 1 \\ a(0) = 2 \end{cases}$

$a(0) = 2$

Solution : $\quad a(n) = c\,2^n + \left(\frac{1}{1-2}\right)$

$a(n) = c\,2^n - 1 \rightarrow$ General solution.

At $n=0$ $\quad a(0) = c\,2^0 - 1 = 2$

$c - 1 = 2 \qquad c = 3$

Particular solution $\quad a(n) = 3(2^n) - 1$

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what if $\; r = -1 \qquad a(n+1) = -1\,a(n) + 2$

$a(0) = \underline{\underline{5}}$

neutrally stable

$a(1) = -1\,a(0) + 2$

$= (-1 \times 5) + 2 = \underline{\underline{-3}}$

$a(2) = -1\,a(1) + 2$

$= -1 \times (-3) + 2 = \underline{\underline{5}}.$

$a(3) = -1\,a(2) + 2$

$= -1(5) + 2 = \underline{\underline{-3}}$

(7)

$\underline{\underline{r > 1}}$ $\quad a(n+1) = 2\,a(n) \longrightarrow$ Diverging Dynami[c] system.

<u>unstable systems</u>

$\underline{\underline{r < 1}}$ but not $= -1$ and $r > -1$

$-1 < r < 1 \quad r \neq 0$ ✓
converging (stable)

$\underline{\underline{r < -1}}$ $\quad a(n+1) = -2\,a(n) \quad a(0) = 10$

$a(1) = -2 \times 10 = -20$

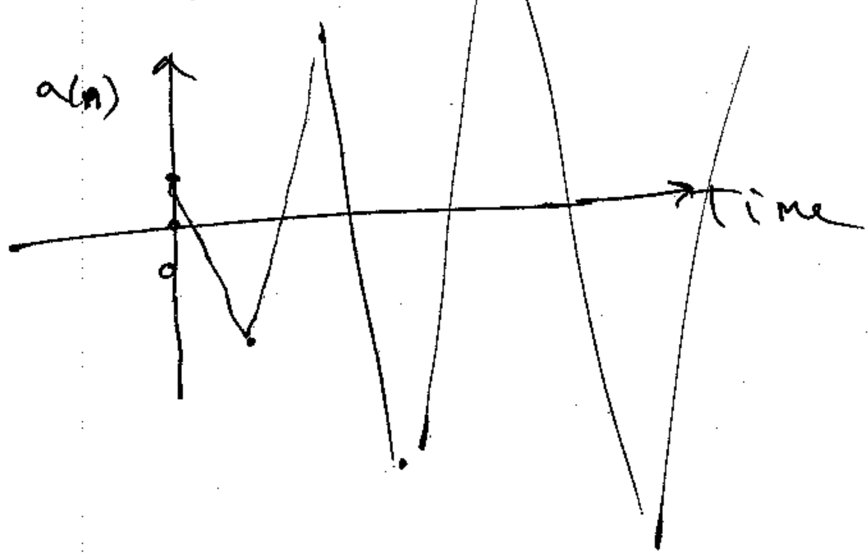
$a(2) = -2 \times (-20) = +40$

$a(3) = -2\,(40) = -80$

$a(4) = -2 \times (-80) = +160$

~~oscillating~~ & also ~~Diverging~~

Unstable.



| | | |
|---|---|---|
| $r > 1$ | Diverging | unstable. |
| $r < -1$ | oscillating & Diverging | unstable. |
| $r = -1$ | ~~neutrally~~ Toggles between Two points. | neutrally stable |
| $r = 1$, $b = 0$ | Stays whereever it is | stable. |
| $r = 1$, $b \neq 0$ | Diverging. b is positive → goes to +infinity. b is negative → goes to -infinity | unstable. |
| $-1 < r < 1$, $r \neq 0$. | Converging | stable. |

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Equilibrium point of the system.

point where once at equilibrium the system. it will always stay at equilibrium.

## Model 2.

$$a(n+1) = r\, a(n) + b$$

constant.

**Solution.**

$$a(n) = C r^n + \frac{b}{1-r} \quad \text{if } r \neq 1$$

$$a(n) = bn + C \quad \text{if } r = 1$$

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**Proof:-** $a(n+1) = C r^{(n+1)} + \frac{b}{1-r} =$

$$r\left(C r^n + \frac{b}{1-r}\right) + b.$$

$$C r^{(n+1)} + \frac{b}{1-r} = C r^{n+1} + r\left(\frac{b}{1-r}\right) + b$$

$$= C r^{(n+1)} + \frac{\cancel{br} + b - \cancel{br}}{1-r}$$

$$= C r^{(n+1)} + \frac{b}{1-r}$$

$a(n+1) = r\,a(n) + b \qquad b \neq 0.$
$\qquad\qquad\qquad\qquad\qquad r \neq 0$

At equilibrium

$a(n+1) = a(n) = Y$

$Y = r\,Y + b \qquad Y = \dfrac{b}{1-r}$

$a(n+1) = 2\,a(n) + 10 \qquad Y = \dfrac{10}{1-2} = -10$

$a(0) = -10$
$a(1) = 2(-10) + 10 = -10.$
$a(2) = 2(-10) + 10 = -10$

$a(0) = -11$
$a(1) = -22 + 10 = -12$
$a(2) = -24 + 10 = -14$

$a(0) = -9$
$a(1) = -18 + 10 = -8$
$a(2) = -16 + 10 = -6$

$(r > 1)$
Unstable systems also has an equilibrium point.
Disturb it and you can never come back to the equilibrium point.

Real world application for Model - 2.

M: $a(n+1) = r\,a(n) + b$

S: $a(n) = c\,r^n + \dfrac{b}{1-r}$ if $r \neq 1$

(OR)

$a(n) = bn + c$ if $r = 1$

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## Financial Models

R → Rate of interest %
N → years
P – Principal amount

Simple Interest $I = \dfrac{PNR}{100}$

$P = \$1000$

$R = 10\%$ $\quad I = \dfrac{1000 \times 5 \times 10}{100} = 500$

$N = 5$

$A = \text{Amount} = P + I = 1000 + 500 = 1500$

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Compound Interest

$$A = P\left(1 + \left[\frac{R}{100}\right]\right)^N$$

$$= 1000\left(1 + \frac{10}{100}\right)^5 = 1000 \times (1.1)^5 = 1610.51$$

compounded yearly (annually).

$a(0) = 0$

Interest $= 7\%$

First deposit is \$100.

$a(1) = 100$

Yearly compounding

$$a(2) = 100 + \left(100 \times \underbrace{\frac{7}{100}}_{\frac{I}{100}}\right) = 100 + 7 = 107 = 100(1+0.07)$$

monthly compounding.

$$a(2) = 100 + \left(100 \times \frac{7}{12 \times 100}\right)$$

$\uparrow$ M

$$= 100\left(1 + \frac{0.07}{M}\right)$$

$\uparrow$ 12 $\rightarrow$ months

$$\underbrace{\phantom{xx}}_{\frac{I}{M \times 100}}$$

$$\underbrace{1 + \frac{I}{M}}_{r}$$

$$A = P\left(1+\frac{R}{m\times 100}\right)^{N}$$

$m=1$ $N$ in years

$m=2$ $N$ in $1/2$ years

$m=4$ $N$ in quarters.

$m=12$ $N$ in months.

$m=365$ $N$ in days.

Interest compounded monthly

$$A = \$1000\left(1+\frac{10}{12\times 100}\right)^{60}$$

$60 \Rightarrow$ # of months

$$= 1000\,(1.00833)^{60}$$

$$= 1644.98$$

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You are 20 yr old

amount at 65 yr

is \$1,000,000

$I = R = 7\%$

Monthly deposit?

$$a(n+1) = r\,a(n) + b$$

$$a(0) = 0$$

Solution $a(n) = c\,r^{n} + \dfrac{b}{1-r}$

$r = 1 + \dfrac{I}{m}$ if $I$ is already divided by 100.

$$\boxed{\text{Model: } a(n+1) = \underbrace{\left(1+\frac{I}{m}\right)}_{r} a(n) + b}$$

Solution. $a(n) = c r^n + \dfrac{b}{1-r}$

$$= c r^n + \frac{b}{1-\left(1+\frac{I}{m}\right)}$$

Solution ⇒ $a(n) = c r^n - \dfrac{bm}{I}$

$$\boxed{a(n) = c\left(1+\frac{I}{m}\right)^n - \frac{bm}{I}}$$

solution for financial models.

minus sign.

Compounded monthly

$7\% = I = 0.07$

$m = 12$

20 - 65 yrs.

$a(0) = 0 \qquad n = \text{months}$

$= 12 \times 45 = 540$ months.

$$a(0) = c\left(1+\frac{0.07}{12}\right)^0 - \frac{b \times 12}{0.07} = 0$$

$$a(540) = c\left(1+\frac{0.07}{12}\right)^{540} - \frac{b \times 12}{0.07} = 1000000$$

$$\Rightarrow c - \left(\frac{b \times 12}{0.07}\right) = 0 \qquad c = \frac{b \times 12}{0.07}$$

$$a(540) = \underbrace{\frac{b \times 12}{0.07}}_{c}\left(1+\frac{0.07}{12}\right)^{540} - \frac{b \times 12}{0.07} = 1000000$$

$$\frac{b \times 12}{0.07}\left[\left(1+\frac{0.07}{12}\right)^{540} - 1\right] = 1000000$$

$$\frac{b \times 12}{0.07}\left[22.12\right] = 1000000$$

$$b = \frac{1000000 \times 0.07}{12 \times 22.12}$$

$b = \$263.71$ per month deposit.

Model 1 $\quad a(n+1) = r\, a(n)$

$\nearrow$ dynamic quantity $\quad \searrow$ time index

Solution $\quad a(n) = c\, r^n \quad$ Find c using initial condition

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Model 2 $\quad a(n+1) = r\, a(n) + b$

solution: $\quad a(n) = c r^n + \dfrac{b}{1-r} \quad r \neq 1$

(OR) $\quad a(n) = bn + c, \quad r = 1$

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Financial Model

$$r = \left(1 + \frac{I}{m}\right)$$

$I \to \dfrac{\%\text{ interest}}{100}$ Interest

$m = 1 \to$ compounded yearly

$n =$ years

$m = 4 \to$ quarterly.

$n =$ # of quarters

$m = 12 \to$ monthly.

$n =$ # of months

Model:

$$a(n+1) = \left(1 + \frac{I}{m}\right) a(n) + b$$

Solution: $\quad a(n) = c\left(1 + \dfrac{I}{m}\right)^n - \dfrac{bm}{I}$

Ex 2

$I = 8/100 = 0.08$ $\quad$ $n = 60$ months.

$a(0) = 1000$ $\quad$ $m = 12$

$b = 100$ $\quad$ Compounded monthly.

$r = 1 + \frac{I}{m}$

$= 1 + \left(\frac{0.08}{12}\right)$

$= 1.0067$

$\neq 1$

$a(n+1) = 1.0067\, a(n) + 100$

Solution:

$$a(n) = C\,(1.0067)^n + \frac{100}{(1 - 1.0067)}$$

$$a(0) = 1000 = C\,(1.0067)^0 + \frac{100}{-(0.0067)}$$

$$C = 1000 + \frac{100}{0.0067} = 15925.37$$

Find

$$a(60) = 15925.37\,(1.0067)^{60} + \frac{100}{-(0.0067)}$$

$a(60) = \$8426.79$ (approx)

($8400) (approx).

Ex3 Loan Repayment.

$I = 0.06$ compounded monthly.

$m = 12$

30 year loan

$n = \#$ of months.

$n = 360$ months.

$a(0) = 500{,}000$

Find $b$ = monthly payment.

$a(360) = 0$

Model $a(n+1) = \left(1 + \frac{I}{m}\right) a(n) + b$

Solution $a(n) = C\left(1 + \frac{I}{m}\right)^n - \frac{bm}{I}$

minus

$a(0) = 500000 = C - \frac{b12}{0.06}$

$a(360) = 0 = C\left(1 + \frac{0.06}{12}\right)^{360} - \frac{b \times 12}{0.06}$

Subtract $a(0) - a(360)$

$(500000 - 0) = C\left[1 - \left(1 + \frac{0.06}{12}\right)^{360}\right] \cancel{- \frac{b12}{0.06}} \cancel{+ \frac{b \times 12}{0.06}}$

$C = -99550.63$

$\frac{12b}{0.06} = C - 500000$

$b = -2997.73$

How much paid over the life of the loan?

$$2997.73 \times 360 = 1{,}079{,}182.8$$

## AMORTIZATION

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1/2 life of a Radioactive material

$a(0) = X$ $\quad r \to$ rate of decay

$a(n) = X/2 \Rightarrow$ n is the half life.

$r = 0.1$

$$a(n+1) = r\, a(n)$$

$$a(n+1) = 0.1\, a(n)$$

Solution: $a(n) = C r^n$.

$$a(0) = X = C(0.1)^0$$

$$a(n) = \frac{X}{2} = C(0.1)^n$$

$C = X$

$C = X$

$$\frac{X}{2} = X(0.1)^n$$

$$\frac{1}{2} = (0.1)^n$$

$n \log(0.1) = \ln\left(\frac{1}{2}\right)$

$$n = \frac{\ln(0.5)}{\ln(0.1)}$$