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Infinite integrals of Whittaker and Bessel functions with respect to their indices

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We obtain several new closed-form expressions for the evaluation of a family of infinite-domain integrals of the Whittaker functions $W_{\kappa,\mu}(x)$ and $M_{\kappa,\mu}(x)$ and the modified Bessel functions $I_\mu(x)$ and $K_\mu(x)$ with respect to the index μ. The new family of definite integrals is useful in a variety of contexts in mathematical physics. In particular, the integral involving $K_\mu(x)$ represents a new example of the Kontorovich–Lebedev transform. We discuss the relationship between the results derived here and the previously known integrals of Whittaker and Bessel functions. In some cases, we obtain entirely new expressions, and in other cases, we generalize previously known results. An application to time-dependent radiation transport theory is also discussed. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Integrals involving the Whittaker functions $W_{\kappa,\mu}(x)$ and $M_{\kappa,\mu}(x)$ are of fundamental importance in many areas of mathematical physics, including studies of the structure of the hydrogen atom,[1] analysis of the Schrödinger wave equation,[2] studies of the Coulomb Green's function,[3] investigation of fluctuations in financial markets,[4] and modeling of the spectral evolution resulting from the Compton scattering of radiation by hot electrons.[5–7] In a number of these applications, one finds it necessary to evaluate infinite-domain integrals of Whittaker functions with respect to one of the indices κ or μ. While several formulas can be found in the previous literature for the evaluation of Whittaker function integrals with respect to the first index κ, very few relations are available for evaluating integrals with respect to the second index μ. Our main focus here is on the evaluation of the definite integrals

$$\mathfrak{I}_1 \equiv \int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du \tag{1}$$

and

$$\mathfrak{I}_2 \equiv \int_{-\infty}^\infty \frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)} M_{\kappa,iu}(x)du. \tag{2}$$

Apelblat[8] gave a formula for the integral $\mathfrak{I}_1$ that is applicable when $\Re\mathfrak{e}\ \kappa<\frac{1}{2}$ and $\Re\mathfrak{e}\ \lambda>0$, but no closed-form expressions for the integral $\mathfrak{I}_2$ have appeared in the previous literature. We relax the restrictions on Apelblat's result for $\mathfrak{I}_1$ by deriving a new family of expressions that is valid for arbitrary complex values of κ when $\Re\mathfrak{e}\ \lambda>-1$. We also obtain corresponding expressions for the integral $\mathfrak{I}_2$ that are valid throughout the same domain. Special cases of $\mathfrak{I}_1$ and $\mathfrak{I}_2$ include index

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integrals of the modified Bessel functions $I_\mu(x)$ and $K_\mu(x)$. The new integral involving $K_\mu(x)$ represents a previously unknown, closed-form example of the Kontorovich–Lebedev transform.

II. INDEX INTEGRALS OF PRODUCTS OF WHITTAKER FUNCTIONS

The Whittaker functions $W_{\kappa,\mu}(x)$ and $M_{\kappa,\mu}(x)$ are confluent hypergeometric functions that are related to the Kummer functions $\Phi(a,b,z)$ and $\Psi(a,b,z)$ by[9,10]

$$M_{\kappa,\mu}(x) = e^{-x/2}x^{\mu+1/2}\Phi\left(\tfrac{1}{2}+\mu-\kappa, 1+2\mu; x\right), \tag{3}$$

$$W_{\kappa,\mu}(x) = e^{-x/2}x^{\mu+1/2}\Psi\left(\tfrac{1}{2}+\mu-\kappa, 1+2\mu; x\right). \tag{4}$$

The function $W_{\kappa,\mu}(x)$ can be expressed in terms of $M_{\kappa,\mu}(x)$ using[9]

$$W_{\kappa,\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}M_{\kappa,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}M_{\kappa,-\mu}(x), \tag{5}$$

and the function $M_{\kappa,\mu}(x)$ is given by the power series

$$M_{\kappa,\mu}(x) = e^{-x/2}x^{\mu+1/2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}-\kappa+\mu\right)_n}{(1+2\mu)_n}\frac{x^n}{n!}, \tag{6}$$

where $(a)_n$ denotes the Pochhammer symbol, defined by[10]

$$(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{7}$$

The series in (6) converges for all finite values of x.

Infinite integrals with respect to the parameter μ involving the product of two Whittaker $W_{\kappa,\mu}(x)$ functions were first explored by Becker,[11] who focused on the general form

$$\int_0^\infty \frac{u\sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)}{s+u^2}W_{\kappa,iu}(x)W_{\kappa,iu}(x_0)du, \tag{8}$$

where x and x_0 are real and positive, and s and κ are complex. This integral converges for all values of s in the complex plane, with the exception of the negative real semiaxis provided that $\mathfrak{Re}\ \kappa \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ if $\mathfrak{Im}\ \kappa \neq 0$. It also converges in the special case $s=0$ provided that $\mathfrak{Re}\ \kappa \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

By employing complex contour integration along with various symmetry relations satisfied by the Whittaker functions, it can be shown that[11]

$$\begin{aligned}\int_0^\infty &\frac{u\sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)}{s+u^2}W_{\kappa,iu}(x)W_{\kappa,iu}(x_0)du \\ &= \pi^2\frac{\Gamma\left(\frac{1}{2}-\kappa+\sqrt{s}\right)}{\Gamma(1+2\sqrt{s})}W_{\kappa,\sqrt{s}}(x_{\max})M_{\kappa,\sqrt{s}}(x_{\min}) \\ &\quad - 4\pi^2 e^{-(x+x_0)/2}\sum_{n=0}^{[\mathfrak{Re}\ \kappa-1/2]}\frac{\alpha_n n!}{\Gamma(2\kappa-n)}\frac{(xx_0)^{\kappa-n}}{4s-\alpha_n^2}L_n^{(\alpha_n)}(x)L_n^{(\alpha_n)}(x_0),\end{aligned} \tag{9}$$

where $L_n^{(\alpha_n)}(x)$ represents the Laguerre polynomial, $[a]$ is the integer part of a, and we have made the definitions

$$\alpha_n \equiv 2\kappa - 2n - 1, \quad x_{\min} \equiv \min(x, x_0), \quad x_{\max} \equiv \max(x, x_0). \tag{10}$$

Note that the summation in (9) is carried out only if $\mathfrak{Re}\ \kappa \geq 1/2$.

An interesting special case involving modified Bessel functions is obtained by setting $\kappa=0$ in (9). In this case, the summation is not performed at all, and we can use the identities[10]

$$\Gamma\left(\frac{1}{2} + iu\right)\Gamma\left(\frac{1}{2} - iu\right) = \frac{\pi}{\cosh(\pi u)}, \tag{11}$$

$$\frac{\Gamma\left(\frac{1}{2} + \sqrt{s}\right)}{\Gamma(1 + 2\sqrt{s})} = \frac{\sqrt{\pi}4^{-\sqrt{s}}}{\Gamma(1 + \sqrt{s})}, \tag{12}$$

and

$$\sinh(2\pi u) = 2 \sinh(\pi u)\cosh(\pi u) \tag{13}$$

to conclude that

$$\int_0^\infty \frac{u \sinh(\pi u)}{s + u^2} W_{0,iu}(x) W_{0,iu}(x_0) du = \frac{\pi^{3/2} 4^{-\sqrt{s}}}{2\Gamma(1 + \sqrt{s})} W_{0,\sqrt{s}}(x_{\max}) M_{0,\sqrt{s}}(x_{\min}). \tag{14}$$

This result is convergent for all complex values of s, excluding the negative real semiaxis. Hence, the point $s=0$ is convergent in this case. The Whittaker functions in (14) are related to the modified Bessel functions I_μ and K_μ via[9,10]

$$M_{0,\mu}(x) = 4^\mu \Gamma(1 + \mu)\sqrt{x} I_\mu\left(\frac{x}{2}\right), \tag{15}$$

$$W_{0,\mu}(x) = \sqrt{\frac{x}{\pi}} K_\mu\left(\frac{x}{2}\right). \tag{16}$$

Combining (14)–(16), we find that

$$\int_0^\infty \frac{u \sinh(\pi u)}{s + u^2} K_{iu}\left(\frac{x}{2}\right) K_{iu}\left(\frac{x_0}{2}\right) du = \frac{\pi^2}{2} K_{\sqrt{s}}\left(\frac{x_{\max}}{2}\right) I_{\sqrt{s}}\left(\frac{x_{\min}}{2}\right) \tag{17}$$

or, equivalently,

$$\int_0^\infty \frac{u \sinh(\pi u)}{\lambda^2 + u^2} K_{iu}(z) K_{iu}(z_0) du = \frac{\pi^2}{2} K_\lambda(z_{\max}) I_\lambda(z_{\min}), \tag{18}$$

where $\lambda=\sqrt{s}$, $z=x/2$, $z_0=x_0/2$, and

$$z_{\min} \equiv \min(z, z_0), \quad z_{\max} \equiv \max(z, z_0). \tag{19}$$

Equation (18) is an example of the Kontorovich–Lebedev transform[12] that applies for $\mathfrak{Re}\ \lambda > 0$ or $\lambda=0$. This generalizes the equivalent result given by Eq. (6.794.10) from Gradshteyn and Ryzhik,[9] which is valid only for positive integer values of λ. A related integral was also considered by Yakubovich.[13]

III. INDEX INTEGRALS OF $W_{\kappa,\mu}(x)$

Our primary goal in this article is to use the general result given by (9) to evaluate the index integrals $\mathfrak{I}_1$ and $\mathfrak{I}_2$ defined in (1) and (2), respectively. We shall focus on the evaluation of the $W_{\kappa,\mu}(x)$ integral $\mathfrak{I}_1$ first. Two distinct results for $\mathfrak{I}_1$ are obtained, depending on whether the real part of λ is positive or negative. The two respective expressions are derived by multiplying (9) by either $e^{-x_0/2}x_0^{\sqrt{s}-1/2}$ or $e^{-x_0/2}x_0^{-\sqrt{s}-1/2}$ and then integrating with respect to x_0 over the domain $0<x_0<\infty$. This process requires the evaluation of definite integrals of the Laguerre polynomial $L_n^{(\alpha_n)}(x_0)$ and the Whittaker function $W_{\kappa,iu}(x_0)$ appearing on the right- and left-hand sides of (9), respectively. Based on Eq. (7.414.11) from Gradshteyn and Ryzhik,[9] we can write the general form for the relevant Laguerre polynomial integral as

$$\int_0^\infty e^{-x_0}x_0^{\kappa-n+\beta-1/2}L_n^{(2\kappa-2n-1)}(x_0)dx_0 = \frac{\Gamma\left(\kappa-n+\beta+\frac{1}{2}\right)\Gamma\left(\kappa-\beta-\frac{1}{2}\right)}{n!\,\Gamma\left(\kappa-n-\beta-\frac{1}{2}\right)}, \tag{20}$$

which converges provided that $\mathfrak{Re}(\kappa-n+\beta+\frac{1}{2})>0$. Likewise, we can employ Eq. (7.621.11) from Gradshteyn and Ryzhik[9] to obtain for the required Whittaker function integral the general result

$$\int_0^\infty e^{-x_0/2}x_0^{\beta-1/2}W_{\kappa,iu}(x_0)dx_0 = \frac{\Gamma(1+\beta+iu)\Gamma(1+\beta-iu)}{\Gamma\left(\frac{3}{2}-\kappa+\beta\right)}, \tag{21}$$

which converges provided that $\mathfrak{Re}(1+\beta\pm iu)>0$. Since the maximum value of n appearing in (9) is $[\mathfrak{Re}\ \kappa-\frac{1}{2}]$, it follows that the criterion $\mathfrak{Re}\ \beta>-1$ is a sufficient condition for the convergence of both (20) and (21).

Because of the switch in parameters that occurs at $x=x_0$, we are also obliged to carry out *indefinite* integrals of the Whittaker functions appearing on the right-hand side of (9). The necessary indefinite integrals can be evaluated by employing the fundamental integral formulas for the confluent hypergeometric functions Φ and Ψ given by Eqs. (3.2.4), (3.2.5), (3.2.10), and (3.2.11) from Slater,[13] which state that

$$\int e^{-x}\Phi(a,b,x)dx = \frac{b-1}{1+a-b}e^{-x}\Phi(a,b-1,x) + C_0, \tag{22}$$

$$\int e^{-x}x^{b-1}\Phi(a,b,x)dx = b^{-1}e^{-x}x^b\Phi(a+1,b+1,x) + C_0, \tag{23}$$

$$\int e^{-x}\Psi(a,b,x)dx = -e^{-x}\Psi(a,b-1,x) + C_0, \tag{24}$$

$$\int e^{-x}x^{b-1}\Psi(a,b,x)dx = -e^{-x}x^b\Psi(a+1,b+1,x) + C_0, \tag{25}$$

where C_0 is an arbitrary constant. We can substitute for Φ and Ψ using (3) and (4) to obtain the equivalent Whittaker function integrals

$$\int e^{-x_0/2}x_0^{\sqrt{s}-1/2}M_{\kappa,\sqrt{s}}(x_0)dx_0 = \frac{1}{1+2\sqrt{s}}e^{-x_0/2}x_0^{\sqrt{s}}M_{\kappa-1/2,\sqrt{s}+1/2}(x_0) + C_0, \tag{26}$$

$$\int e^{-x_0/2}x_0^{\sqrt{s}-1/2}W_{\kappa,\sqrt{s}}(x_0)dx_0 = -e^{-x_0/2}x_0^{\sqrt{s}}W_{\kappa-1/2,\sqrt{s}+1/2}(x_0) + C_0, \tag{27}$$

$$\int e^{-x_0/2}x_0^{-\sqrt{s}-1/2}M_{\kappa,\sqrt{s}}(x_0)dx_0 = \frac{2\sqrt{s}}{\frac{1}{2}-\kappa-\sqrt{s}}e^{-x_0/2}x_0^{-\sqrt{s}}M_{\kappa-1/2,\sqrt{s}-1/2}(x_0) + C_0, \tag{28}$$

$$\int e^{-x_0/2}x_0^{-\sqrt{s}-1/2}W_{\kappa,\sqrt{s}}(x_0)dx_0 = -e^{-x_0/2}x_0^{-\sqrt{s}}W_{\kappa-1/2,\sqrt{s}-1/2}(x_0) + C_0. \tag{29}$$

By combining (26)–(29) with the asymptotic relations

$$M_{\kappa,\mu}(x) \to x^{\mu+1/2}, \quad x \to 0, \tag{30}$$

$$W_{\kappa,\mu}(x) \to x^{\kappa}e^{-x/2}, \quad x \to \infty, \tag{31}$$

we obtain for the required indefinite integrals

$$\int_0^x e^{-x_0/2}x_0^{\sqrt{s}-1/2}M_{\kappa,\sqrt{s}}(x_0)dx_0 = \frac{e^{-x/2}x^{\sqrt{s}}}{1+2\sqrt{s}}M_{\kappa-1/2,\sqrt{s}+1/2}(x), \tag{32}$$

$$\int_0^x e^{-x_0/2}x_0^{-\sqrt{s}-1/2}M_{\kappa,\sqrt{s}}(x_0)dx_0 = \frac{2\sqrt{s}}{\frac{1}{2}-\kappa-\sqrt{s}}[e^{-x/2}x^{-\sqrt{s}}M_{\kappa-1/2,\sqrt{s}-1/2}(x) - 1], \tag{33}$$

$$\int_x^\infty e^{-x_0/2}x_0^{\sqrt{s}-1/2}W_{\kappa,\sqrt{s}}(x_0)dx_0 = e^{-x/2}x^{\sqrt{s}}W_{\kappa-1/2,\sqrt{s}+1/2}(x), \tag{34}$$

$$\int_x^\infty e^{-x_0/2}x_0^{-\sqrt{s}-1/2}W_{\kappa,\sqrt{s}}(x_0)dx_0 = e^{-x/2}x^{-\sqrt{s}}W_{\kappa-1/2,\sqrt{s}-1/2}(x). \tag{35}$$

Based on the convergence properties of (20) and (21) and the fact that $\mathfrak{Re}\sqrt{s}>0$ for the principle branch of the square root function, we shall find that operating on (9) with $\int_0^\infty e^{-x_0/2}x_0^{\sqrt{s}-1/2}dx_0$ yields an expression for the fundamental integral $\mathfrak{I}_1$ [Eq. (1)] that is applicable when $\mathfrak{Re}\,\lambda>0$, and operating with $\int_0^\infty e^{-x_0/2}x_0^{-\sqrt{s}-1/2}dx_0$ yields a result that is applicable when $-1<\mathfrak{Re}\,\lambda<0$. We consider each of these cases separately below.

A. Application of $\int_0^\infty e^{-x_0/2}x_0^{\sqrt{s}-1/2}dx_0$

Operating on (9) with $\int_0^\infty e^{-x_0/2}x_0^{\sqrt{s}-1/2}dx_0$ and combining the result with (20), (21), (32), and (34), we find that

$$\int_0^\infty \frac{u\,\sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\sqrt{s}+1+iu)\Gamma(\sqrt{s}+1-iu)}{\Gamma\left(\sqrt{s}-\kappa+\frac{3}{2}\right)(s+u^2)}W_{\kappa,iu}(x)du$$

$$= \frac{\pi^2\Gamma\left(\frac{1}{2}-\kappa+\sqrt{s}\right)x^{\sqrt{s}}}{\Gamma(2+2\sqrt{s})e^{x/2}}[W_{\kappa,\sqrt{s}}(x)M_{\kappa-1/2,\sqrt{s}+1/2}(x)+(1+2\sqrt{s})M_{\kappa,\sqrt{s}}(x)W_{\kappa-1/2,\sqrt{s}+1/2}(x)]$$

$$-4\pi^2e^{-x/2}\sum_{n=0}^{[\mathfrak{Re}\ \kappa-1/2]}\frac{\alpha_n\Gamma\left(\kappa-n+\sqrt{s}+\frac{1}{2}\right)\Gamma\left(\kappa-\sqrt{s}-\frac{1}{2}\right)}{\Gamma\left(\kappa-n-\sqrt{s}-\frac{1}{2}\right)\Gamma(2\kappa-n)}\frac{x^{\kappa-n}}{4s-\alpha_n^2}L_n^{(\alpha_n)}(x), \tag{36}$$

where $\alpha_n \equiv 2\kappa-2n-1$. Additional simplification can be achieved by using Eq. (2.4.27) from Slater[14] to write the Wronskian of the Whittaker functions as

$$\mathfrak{W} \equiv M_{\kappa,\mu}(x)\frac{d}{dx}W_{\kappa,\mu}(x)-W_{\kappa,\mu}(x)\frac{d}{dx}M_{\kappa,\mu}(x) = -\frac{\Gamma(1+2\mu)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}. \tag{37}$$

The derivatives in this expression can be evaluated using Eqs. (2.4.8) and (2.4.19) from Ref. 13, which state that

$$M_{\kappa-1/2,\mu+1/2}(x) = \frac{1+2\mu}{\mu-\kappa+\frac{1}{2}}e^{-x/2}x^{\mu+1}\frac{d}{dx}e^{x/2}x^{-\mu-1/2}M_{\kappa,\mu}(x), \tag{38}$$

$$W_{\kappa-1/2,\mu+1/2}(x) = \frac{1}{\kappa-\mu-\frac{1}{2}}e^{-x/2}x^{\mu+1}\frac{d}{dx}e^{x/2}x^{-\mu-1/2}W_{\kappa,\mu}(x). \tag{39}$$

Combining (37)–(39) with the recurrence relation

$$z\Gamma(z) = \Gamma(z+1), \tag{40}$$

we obtain

$$W_{\kappa,\mu}(x)M_{\kappa-1/2,\mu+1/2}(x)+(1+2\mu)M_{\kappa,\mu}(x)W_{\kappa-1/2,\mu+1/2}(x) = \frac{\Gamma(2+2\mu)x^{1/2}}{\Gamma\left(\mu-\kappa+\frac{3}{2}\right)} \tag{41}$$

or, equivalently,

$$W_{\kappa,\sqrt{s}}(x)M_{\kappa-1/2,\sqrt{s}+1/2}(x)+(1+2\sqrt{s})M_{\kappa,\sqrt{s}}(x)W_{\kappa-1/2,\sqrt{s}+1/2}(x) = \frac{\Gamma(2+2\sqrt{s})x^{1/2}}{\Gamma\left(\sqrt{s}-\kappa+\frac{3}{2}\right)}. \tag{42}$$

The final result for the integral $\mathfrak{I}_1$ is achieved by combining (36) and (42) with the identity

$$\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z), \tag{43}$$

which yields, after some algebra,

$$\begin{aligned}\mathfrak{I}_1 &\equiv \int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du \\ &= \pi^2\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)x^{\lambda+1/2}e^{-x/2} \\ &\quad - \frac{\pi^3 e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\ \kappa-1/2]}\frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)},\end{aligned} \tag{44}$$

where we have set $\lambda=\sqrt{s}$. Equation (44) is applicable for $\mathfrak{Re}\ \lambda>0$ provided that $\mathfrak{Re}\ \kappa\neq\frac{1}{2},\frac{3}{2},\ldots$ if $\mathfrak{Im}\ \kappa\neq 0$. It also applies when $\lambda=0$ provided that $\mathfrak{Re}\ \kappa\neq\frac{1}{2},\frac{3}{2},\ldots$ This result generalizes Eq. (17.3.3) from Apelblat,[8] which does not include the summation and is therefore only applicable for $\mathfrak{Re}\ \kappa<\frac{1}{2}$. Equation (44) is one of the main results of the article, and it represents an interesting closed-form example of the Whittaker function index transformation discussed in Ref. 15.

B. Application of $\int_0^\infty e^{-x_0/2}x_0^{-\sqrt{s}-1/2}dx_0$

Next we apply the operator $\int_0^\infty e^{-x_0/2}x_0^{-\sqrt{s}-1/2}dx_0$ to (9) and combine the result with (20), (21), (33), and (35) to obtain

$$\begin{aligned}&\int_0^\infty \frac{u\sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(1-\sqrt{s}+iu)\Gamma(1-\sqrt{s}-iu)}{\Gamma\left(\frac{3}{2}-\kappa-\sqrt{s}\right)(s+u^2)}W_{\kappa,iu}(x)du \\ &= \frac{\pi^2\Gamma\left(\frac{1}{2}-\kappa+\sqrt{s}\right)}{\Gamma(1+2\sqrt{s})e^{x/2}x^{\sqrt{s}}}\left\{\frac{2\sqrt{s}W_{\kappa,\sqrt{s}}(x)}{\frac{1}{2}-\kappa-\sqrt{s}}[M_{\kappa-1/2,\sqrt{s}-1/2}(x)-e^{x/2}x^{\sqrt{s}}]+M_{\kappa,\sqrt{s}}(x)W_{\kappa-1/2,\sqrt{s}-1/2}(x)\right\} \\ &\quad -4\pi^2e^{-x/2}\sum_{n=0}^{[\mathfrak{Re}\kappa-1/2]}\frac{\alpha_n\Gamma\left(\kappa-n-\sqrt{s}+\frac{1}{2}\right)\Gamma\left(\kappa+\sqrt{s}-\frac{1}{2}\right)}{\Gamma\left(\kappa-n+\sqrt{s}-\frac{1}{2}\right)\Gamma(2\kappa-n)}\frac{x^{\kappa-n}}{4s-\alpha_n^2}L_n^{(\alpha_n)}(x),\end{aligned} \tag{45}$$

where $\alpha_n\equiv 2\kappa-2n-1$. In this case, we can make further progress by utilizing Eqs. (2.4.7) and (2.4.20) from Ref. 13 to write

$$M_{\kappa-1/2,\mu-1/2}(x)=\frac{1}{2\mu}e^{-x/2}x^{1-\mu}\frac{d}{dx}e^{x/2}x^{\mu-1/2}M_{\kappa,\mu}(x), \tag{46}$$

$$W_{\kappa-1/2,\mu-1/2}(x)=\frac{1}{\kappa+\mu-\frac{1}{2}}e^{-x/2}x^{1-\mu}\frac{d}{dx}e^{x/2}x^{\mu-1/2}W_{\kappa,\mu}(x), \tag{47}$$

which can be combined with (37) for the Wronskian to obtain

$$\frac{2\mu}{\frac{1}{2}-\kappa-\mu} W_{\kappa,\mu}(x) M_{\kappa-1/2,\mu-1/2}(x) + M_{\kappa,\mu}(x) W_{\kappa-1/2,\mu-1/2}(x) = \frac{\Gamma(1+2\mu)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)} \frac{x^{1/2}}{\frac{1}{2}-\mu-\kappa} \tag{48}$$

or, equivalently,

$$\frac{2\sqrt{s}}{\frac{1}{2}-\kappa-\sqrt{s}} W_{\kappa,\sqrt{s}}(x) M_{\kappa-1/2,\sqrt{s}-1/2}(x) + M_{\kappa,\sqrt{s}}(x) W_{\kappa-1/2,\sqrt{s}-1/2}(x) = \frac{\Gamma(1+2\sqrt{s})}{\Gamma\left(\sqrt{s}-\kappa+\frac{1}{2}\right)} \frac{x^{1/2}}{\frac{1}{2}-\sqrt{s}-\kappa}. \tag{49}$$

Combining (40), (43), (45), and (49), we obtain after simplification

$$\begin{aligned}\mathfrak{I}_1 &\equiv \int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du \\ &= \pi^2\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2} - \frac{\Gamma\left(\frac{1}{2}-\kappa-\lambda\right)}{\Gamma(-2\lambda)}W_{\kappa,\lambda}(x)\right] \\ &\quad - \frac{\pi^3 e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)},\end{aligned} \tag{50}$$

where we have set $\lambda = -\sqrt{s}$. Equation (50) is applicable for $-1 < \mathfrak{Re}\,\lambda < 0$ provided that $\mathfrak{Re}\,\kappa \neq \frac{1}{2}, \frac{3}{2}, \ldots$ if $\mathfrak{Im}\,\kappa \neq 0$. The formula also applies when $\lambda = 0$ provided that $\mathfrak{Re}\,\kappa \neq \frac{1}{2}, \frac{3}{2}, \ldots$. Equation (50) is the second main result of the article, and it complements (44), which applies for $\mathfrak{Re}\,\lambda > 0$.

We consider several special cases for $\mathfrak{I}_1$ below, corresponding to particular values of the parameters κ and λ. For brevity, we focus here on situations with $\kappa=0$, $\kappa=1$, $\kappa=2$, or $\lambda=0$.

C. $\kappa=0$

The case with $\kappa=0$ with an arbitrary value of λ is of particular significance because it yields a new index integral involving the modified Bessel function $K_\mu(x)$ that is a previously unknown example of the Kontorovich–Lebedev transform.[13] When $\kappa=0$, the summation in (50) is not performed at all, and the formula reduces to

$$\begin{aligned}&\int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}+iu\right)\Gamma\left(\frac{1}{2}-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{0,iu}(x)du \\ &\quad = \pi^2\Gamma\left(\lambda+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2} - \frac{\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(-2\lambda)}W_{0,\lambda}(x)\right].\end{aligned} \tag{51}$$

By combining this relation with (11) and (13), and the identity

$$\frac{\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(-2\lambda)} = \frac{2^{1+2\lambda}\sqrt{\pi}}{\Gamma(-\lambda)}, \tag{52}$$

we can obtain the equivalent result

$$\int_0^\infty u\sinh(\pi u)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{0,iu}(x)du=\frac{\pi}{2}\Gamma\left(\lambda+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{2^{1+2\lambda}\sqrt{\pi}}{\Gamma(-\lambda)}W_{0,\lambda}(x)\right],\tag{53}$$

which is applicable if $-1<\mathfrak{Re}\,\lambda<0$ or $\lambda=0$. It also applies when $\mathfrak{Re}\,\lambda>0$ if the $W_{0,\lambda}(x)$ term on the right-hand side is neglected.

An interesting new integral formula involving the modified Bessel function $K_\mu(x)$ can be obtained by using (16) to substitute for the Whittaker functions appearing in (53), which yields

$$\int_0^\infty u\sinh(\pi u)\Gamma(\lambda+iu)\Gamma(\lambda-iu)K_{iu}(z)du=\pi^{3/2}2^{\lambda-1}\Gamma\left(\lambda+\frac{1}{2}\right)\left[z^\lambda e^{-z}-\frac{2^{1+\lambda}}{\Gamma(-\lambda)}K_\lambda(z)\right],\tag{54}$$

where $z=x/2$. This formula, which is valid for $-1<\mathfrak{Re}\,\lambda<0$ or $\lambda=0$, represents a new closed-form example of the Kontorovich–Lebedev transform that has not appeared in the previous literature. The corresponding result obtained for $\mathfrak{Re}\,\lambda>0$ is

$$\int_0^\infty u\sinh(\pi u)\Gamma(\lambda+iu)\Gamma(\lambda-iu)K_{iu}(z)du=\pi^{3/2}2^{\lambda-1}\Gamma\left(\lambda+\frac{1}{2}\right)z^\lambda e^{-z}.\tag{55}$$

This relation can also be obtained by using Eq. (176.12) from Erdélyi *et al.*[12] or Eq. (6.797.4) from Gradshteyn and Ryzhik[9] to write

$$\int_0^\infty u\sinh(\pi u)\Gamma(\lambda+iu)\Gamma(\lambda-iu)K_{iu}(z)K_{iu}(y)du=\pi^{3/2}2^{\lambda-1}\left(\frac{yz}{y+z}\right)^\lambda\Gamma\left(\lambda+\frac{1}{2}\right)K_\lambda(y+z),\tag{56}$$

which is applicable for $\mathfrak{Re}\,\lambda>0$ or $\lambda=0$. We can easily recover (55) from (56) in the limit $y\to\infty$ by utilizing the asymptotic behavior[10]

$$K_\nu(y)\to\sqrt{\frac{\pi}{2y}}e^{-y},\quad y\to\infty.\tag{57}$$

The new integral formula (54) extends the parameter range within which the Kontorovich–Lebedev transform can be evaluated analytically. Equations (54) and (55) both apply when $\lambda=0$, in which case we can employ the identity

$$\Gamma(iu)\Gamma(-iu)=\frac{\pi}{u\sinh(\pi u)}\tag{58}$$

to show that they both reduce to

$$\int_0^\infty K_{iu}(z)du=\frac{\pi}{2}e^{-z},\tag{59}$$

in agreement with Eq. (6.795.1) from Gradshteyn and Ryzhik.[9]

D. $\kappa=1$

When $\kappa=1$, the summation in (50) terminates after a single term, and we have

$$\int_0^\infty u\,\sinh(2\pi u)\Gamma\left(-\frac{1}{2}+iu\right)\Gamma\left(-\frac{1}{2}-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{1,iu}(x)du$$
$$=\pi^2\Gamma\left(\lambda-\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{\Gamma\left(-\frac{1}{2}-\lambda\right)}{\Gamma(-2\lambda)}W_{1,\lambda}(x)\right]-\frac{\pi^3e^{-x/2}}{\cos[\pi(1-\lambda)]}\frac{\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)}xL_0^{(1)}(x). \tag{60}$$

This relation can be combined with (13), (40), (43), and (52), and the identity

$$\Gamma\left(-\frac{1}{2}+iu\right)\Gamma\left(-\frac{1}{2}-iu\right)=\frac{4\pi}{(1+4u^2)\cosh(\pi u)} \tag{61}$$

to obtain

$$\int_0^\infty \frac{u\,\sinh(\pi u)}{1+4u^2}\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{1,iu}(x)du$$
$$=\frac{\pi}{8}\Gamma\left(\lambda-\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}+\frac{4^{\lambda+1}\sqrt{\pi}}{(1+2\lambda)\Gamma(-\lambda)}W_{1,\lambda}(x)-\Gamma\left(\lambda+\frac{1}{2}\right)xe^{-x/2}\right], \tag{62}$$

which is applicable if $-1<\Re\mathfrak{e}\,\lambda<0$ or $\lambda=0$. It also applies when $\Re\mathfrak{e}\,\lambda>0$ if the $W_{1,\lambda}(x)$ term on the right-hand side is removed. In particular, when $\lambda=0$, we can use (58) to show that (62) reduces to

$$\int_0^\infty \frac{W_{1,iu}(x)}{1+4u^2}du=\frac{e^{-x/2}}{4}(\pi x-\sqrt{\pi x}). \tag{63}$$

E. $\kappa=2$

The summation in (50) yields two terms when $\kappa=2$, and we obtain

$$\int_0^\infty u\,\sinh(2\pi u)\Gamma\left(-\frac{3}{2}+iu\right)\Gamma\left(-\frac{3}{2}-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{2,iu}(x)du$$
$$=\pi^2\Gamma\left(\lambda-\frac{3}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{\Gamma\left(-\frac{3}{2}-\lambda\right)}{\Gamma(-2\lambda)}W_{2,\lambda}(x)\right]$$
$$-\frac{\pi^3e^{-x/2}}{2\cos[\pi(2-\lambda)]}\left[\frac{\Gamma\left(\frac{3}{2}+\lambda\right)}{\Gamma\left(\frac{5}{2}-\lambda\right)}x^2L_0^{(3)}(x)+\frac{\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)}xL_1^{(1)}(x)\right]. \tag{64}$$

Combining this expression with (13), (40), (43), and (52), and the identity

$$\Gamma\left(-\frac{3}{2}+iu\right)\Gamma\left(-\frac{3}{2}-iu\right)=\frac{16\pi}{(1+4u^2)(9+4u^2)\cosh(\pi u)} \tag{65}$$

yields, after simplification,

$$\int_0^\infty \frac{u\sinh(\pi u)}{(9+4u^2)(1+4u^2)}\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{2,iu}(x)du$$
$$=\frac{\pi}{16}\Gamma\left(\lambda-\frac{3}{2}\right)\left[\frac{1}{2}x^{\lambda+1/2}e^{-x/2}-\frac{4^{\lambda+1}\sqrt{\pi}}{(2\lambda+1)(2\lambda+3)\Gamma(-\lambda)}W_{2,\lambda}(x)\right]$$
$$-\frac{\pi}{8}\frac{\Gamma^2\left(\frac{3}{2}+\lambda\right)}{1-4\lambda^2}e^{-x/2}x\left(\frac{x}{3-2\lambda}+\frac{2-x}{1+2\lambda}\right), \tag{66}$$

which is valid for $-1<\mathfrak{Re}\,\lambda<0$ or $\lambda=0$. It also applies when $\mathfrak{Re}\,\lambda>0$ if we neglect the $W_{2,\lambda}(x)$ term on the right-hand side. In the special case $\kappa=2$ and $\lambda=0$, we can combine (58) with (66) to show that (50) reduces to

$$\int_0^\infty \frac{W_{2,iu}(x)}{(9+4u^2)(1+4u^2)}du=\frac{e^{-x/2}}{48}(2\sqrt{\pi x}-3\pi x+\pi x^2). \tag{67}$$

F. $\lambda=0$

It is also interesting to examine the behavior of (44) and (50) for general values of κ when $\lambda=0$. Setting $\lambda=0$ in either equation yields, after simplification,

$$\int_0^\infty u\sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(iu)\Gamma(-iu)W_{\kappa,iu}(x)du$$
$$=\pi^2\Gamma\left(\frac{1}{2}-\kappa\right)\sqrt{x}e^{-x/2}-\frac{2\pi^3e^{-x/2}}{\cos(\pi\kappa)}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)}. \tag{68}$$

This expression can be further reduced by applying (13) and (58) to obtain the final result

$$\int_0^\infty \cosh(\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)W_{\kappa,iu}(x)du$$
$$=\frac{\pi}{2}\Gamma\left(\frac{1}{2}-\kappa\right)\sqrt{x}e^{-x/2}-\frac{\pi^2e^{-x/2}}{\cos(\pi\kappa)}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)}. \tag{69}$$

In the special case $\kappa=0$, the summation is not performed at all, and we can use (11) to show that (69) reduces to

$$\int_0^\infty W_{0,iu}(x)du=\int_0^\infty \sqrt{\frac{x}{\pi}}K_{iu}\left(\frac{x}{2}\right)du=\frac{\sqrt{\pi x}}{2}e^{-x/2}, \tag{70}$$

in agreement with (59). Likewise, one can show that (69) reduces to either (63) or (67) if $\kappa=1$ or $\kappa=2$, respectively.

IV. INDEX INTEGRALS OF $M_{\kappa,\mu}(x)$

In addition to deriving the new closed-form expressions (44) and (50) for the $W_{\kappa,\mu}(x)$ index integral $\mathfrak{I}_1$, defined by

$$\Im_1 \equiv \int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du, \tag{71}$$

our secondary goal in this article is to derive a corresponding set of formulas for evaluating the $M_{\kappa,\mu}(x)$ index integral $\Im_2$, defined by

$$\Im_2 \equiv \int_{-\infty}^\infty \frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)} M_{\kappa,iu}(x)du. \tag{72}$$

We begin by using (5) to express the $W_{\kappa,iu}(x)$ function appearing in (71) as

$$W_{\kappa,iu}(x) = \frac{\Gamma(-2iu)}{\Gamma\left(\frac{1}{2}-\kappa-iu\right)}M_{\kappa,iu}(x) + \frac{\Gamma(2iu)}{\Gamma\left(\frac{1}{2}-\kappa+iu\right)}M_{\kappa,-iu}(x), \tag{73}$$

which implies the obvious symmetry relation

$$W_{\kappa,iu}(x) = W_{\kappa,-iu}(x). \tag{74}$$

Equation (73) can be rearranged to obtain

$$W_{\kappa,iu}(x) = \frac{\Gamma(-2iu)\Gamma(2iu)}{\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}\left[\frac{\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x) + \frac{\Gamma\left(\frac{1}{2}-\kappa-iu\right)}{\Gamma(-2iu)}M_{\kappa,-iu}(x)\right] \tag{75}$$

or, equivalently,

$$W_{\kappa,iu}(x) = \frac{\Gamma(-2iu)\Gamma(1+2iu)}{\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}\left[\frac{\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(1+2iu)}M_{\kappa,iu}(x) - \frac{\Gamma\left(\frac{1}{2}-\kappa-iu\right)}{\Gamma(1-2iu)}M_{\kappa,-iu}(x)\right], \tag{76}$$

where we have utilized the recurrence formula $z\Gamma(z)=\Gamma(z+1)$. Based on the symmetry relation (74), we can rewrite the fundamental integral $\Im_1$ in the alternative form

$$\Im_1 = \frac{1}{2}\int_{-\infty}^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du, \tag{77}$$

which can be combined with (76) to obtain

$$\Im_1 = \frac{1}{2}\int_{-\infty}^\infty u \sinh(2\pi u)\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma(-2iu)\Gamma(1+2iu) \times\left[\frac{\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(1+2iu)}M_{\kappa,iu}(x) - \frac{\Gamma\left(\frac{1}{2}-\kappa-iu\right)}{\Gamma(1-2iu)}M_{\kappa,-iu}(x)\right]du. \tag{78}$$

Splitting the right-hand side of (78) into two integrals and applying the identity

$$\Gamma(-2iu)\Gamma(1+2iu)=\frac{i\pi}{\sinh(2\pi u)},\tag{79}$$

we find that

$$\begin{aligned}\mathfrak{I}_1&=\frac{i\pi}{2}\int_{-\infty}^{\infty}\frac{u\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(1+2iu)}M_{\kappa,iu}(x)du\\&\quad+\frac{i\pi}{2}\int_{-\infty}^{\infty}\frac{-u\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa-iu\right)}{\Gamma(1-2iu)}M_{\kappa,-iu}(x)du.\end{aligned}\tag{80}$$

The second integral is identical to the first upon replacement of u with $-u$, and therefore we obtain, after simplification,

$$\mathfrak{I}_1=\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x)du.\tag{81}$$

Comparing this result with the definition of $\mathfrak{I}_2$ given by (72), we conclude that

$$\mathfrak{I}_2=\frac{2}{\pi}\mathfrak{I}_1,\tag{82}$$

which can be combined with (44) and (50) to obtain the integral formulas

$$\begin{aligned}\mathfrak{I}_2&\equiv\int_{-\infty}^{\infty}\frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x)du\\&=2\pi\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)x^{\lambda+1/2}e^{-x/2}\\&\quad-\frac{2\pi^2e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)}\end{aligned}\tag{83}$$

for $\mathfrak{Re}\,\lambda>0$, and

$$\begin{aligned}\mathfrak{I}_2&\equiv\int_{-\infty}^{\infty}\frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x)du\\&=2\pi\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{\Gamma\left(\frac{1}{2}-\kappa-\lambda\right)}{\Gamma(-2\lambda)}W_{\kappa,\lambda}(x)\right]\\&\quad-\frac{2\pi^2e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)}\end{aligned}\tag{84}$$

for $-1<\Re\mathfrak{e}\,\lambda<0$. Equations (83) and (84) are valid provided $\Re\mathfrak{e}\,\kappa\neq\frac{1}{2},\frac{3}{2},\ldots$ if $\Im\mathfrak{m}\,\kappa\neq 0$. They also apply when $\lambda=0$ provided that $\Re\mathfrak{e}\,\kappa\neq\frac{1}{2},\frac{3}{2},\ldots$. The convergence of the integral $\mathfrak{J}_2$ when $\lambda=0$ is an interesting phenomenon since it is apparent that the integrand possesses a pole at $u=0$ in this case. However, close inspection reveals that the divergent part of the integrand is an odd function of u and, consequently, the pole makes no net contribution to the integral. Hence we conclude that the integral is technically improper, although it possesses a finite Cauchy principal value. We can treat this case by redefining $\mathfrak{J}_2$ as the Cauchy principal value, i.e.,

$$\mathfrak{J}_2 \equiv \lim_{\epsilon\to 0}\left[\int_{-\infty}^{\epsilon}\frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x)du + \int_{\epsilon}^{\infty}\frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x)du\right], \tag{85}$$

which is finite even when $\lambda=0$. We consider below several special cases for $\mathfrak{J}_2$ obtained when $\kappa=0$, $\kappa=1$, $\kappa=2$, or $\lambda=0$.

A. $\kappa=0$

Equation (82) can be combined with our previous results for $\mathfrak{J}_2$ to evaluate $\mathfrak{J}_1$ for several values of κ and λ. A result of particular significance is obtained by setting $\kappa=0$, in which case (51) and (82) imply that

$$\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{2}+iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)}{\Gamma(2iu)}M_{0,iu}(x)du = 2\pi\Gamma\left(\lambda+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(-2\lambda)}W_{0,\lambda}(x)\right] \tag{86}$$

or, equivalently,

$$\int_{-\infty}^{\infty}\frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)}{4^{iu}\Gamma(iu)}M_{0,iu}(x)du = \sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{2^{1+2\lambda}\sqrt{\pi}}{\Gamma(-\lambda)}W_{0,\lambda}(x)\right], \tag{87}$$

where we have utilized (52) to obtain the final expression, which is valid for $-1<\Re\mathfrak{e}\,\lambda<0$. This result also applies for $\Re\mathfrak{e}\,\lambda>0$ if the $W_{0,\lambda}(x)$ term on the right-hand side is removed.

We can derive an interesting new integral formula for the modified Bessel function I_μ by using (15) and (16) to substitute for the Whittaker functions in (87). After simplification, the result obtained is

$$\int_{-\infty}^{\infty}u\Gamma(\lambda+iu)\Gamma(\lambda-iu)I_{iu}(z)du = i2^{\lambda}\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)\left[\frac{2^{1+\lambda}}{\Gamma(-\lambda)}K_{\lambda}(z)-z^{\lambda}e^{-z}\right], \tag{88}$$

where $z=x/2$. Equation (88) represents a fundamental new index integral involving the function I_μ, valid for $-1<\Re\mathfrak{e}\,\lambda<0$ or $\lambda=0$. Dropping the $K_\lambda(z)$ term on the right-hand side yields an expression applicable when $\Re\mathfrak{e}\,\lambda>0$. In the limit $\lambda\to 0$, we can use (58) to show that (88) reduces to

$$\int_{-\infty}^{\infty}\frac{I_{iu}(z)}{\sinh(\pi u)}du = -ie^{-z}. \tag{89}$$

We can also apply the Bessel function identity

$$I_\nu(z) = e^{-\pi i\nu/2} J_\nu(iz) \tag{90}$$

to obtain the equivalent result

$$\int_{-\infty}^{\infty} \frac{e^{\pi u/2}}{\sinh(\pi u)} J_{iu}(iz) du = -ie^{-z}, \tag{91}$$

which agrees with Eq. (6.796.1) from Gradshteyn and Ryzhik.[9]

B. $\kappa=1$

Setting $\kappa=1$ and combining (60) and (82) yields

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(-\frac{1}{2}+iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)}{\Gamma(2iu)} M_{1,iu}(x) du$$
$$= 2\pi\Gamma\left(\lambda-\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2} - \frac{\Gamma\left(-\frac{1}{2}-\lambda\right)}{\Gamma(-2\lambda)} W_{1,\lambda}(x)\right] - \frac{2\pi^2 e^{-x/2}}{\cos[\pi(1-\lambda)]} \frac{\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)} xL_0^{(1)}(x), \tag{92}$$

which can be simplified using (43) and (52) to obtain

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(-\frac{1}{2}+iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)}{\Gamma(2iu)} M_{1,iu}(x) du$$
$$= 2\pi\Gamma\left(\lambda-\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2} + \frac{4^{\lambda+1}\sqrt{\pi}}{(1+2\lambda)\Gamma(-\lambda)} W_{1,\lambda}(x) - \Gamma\left(\lambda+\frac{1}{2}\right)xe^{-x/2}\right]. \tag{93}$$

This result is applicable if $-1 < \mathfrak{Re}\,\lambda < 0$ or $\lambda=0$. It also applies when $\mathfrak{Re}\,\lambda > 0$ if the $W_{1,\lambda}(x)$ term on the right-hand side is removed. In particular, when $\lambda=0$, we can use (58) to show that (93) reduces to

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(-\frac{1}{2}+iu\right)}{\Gamma(2iu)u\sinh(\pi u)} M_{1,iu}(x) du = 4e^{-x/2}(\pi x - \sqrt{\pi x}). \tag{94}$$

C. $\kappa=2$

Setting $\kappa=2$ and combining (64) and (82) yields

$$\int_{-\infty}^{\infty}\frac{\Gamma\left(-\frac{3}{2}+iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)}{\Gamma(2iu)}M_{2,iu}(x)du$$
$$=2\pi\Gamma\left(\lambda-\frac{3}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{\Gamma\left(-\frac{3}{2}-\lambda\right)}{\Gamma(-2\lambda)}W_{2,\lambda}(x)\right]$$
$$-\frac{\pi^2 e^{-x/2}}{\cos[\pi(2-\lambda)]}\left[\frac{\Gamma\left(\frac{3}{2}+\lambda\right)}{\Gamma\left(\frac{5}{2}-\lambda\right)}x^2L_0^{(3)}(x)+\frac{\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma\left(\frac{3}{2}-\lambda\right)}xL_1^{(1)}(x)\right]. \quad (95)$$

Combining this expression with (40), (43), and (52), we obtain, after some algebra,

$$\int_{-\infty}^{\infty}\frac{\Gamma\left(-\frac{3}{2}+iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)}{\Gamma(2iu)}M_{2,iu}(x)du$$
$$=4\pi\Gamma\left(\lambda-\frac{3}{2}\right)\left[\frac{1}{2}x^{\lambda+1/2}e^{-x/2}-\frac{4^{\lambda+1}\sqrt{\pi}}{(2\lambda+1)(2\lambda+3)\Gamma(-\lambda)}W_{2,\lambda}(x)\right]$$
$$-\frac{8\pi\Gamma^2\left(\frac{3}{2}+\lambda\right)}{1-4\lambda^2}e^{-x/2}x\left(\frac{x}{3-2\lambda}+\frac{2-x}{1+2\lambda}\right), \quad (96)$$

which is valid for $-1<\Re\mathfrak{e}\,\lambda<0$ or $\lambda=0$. It also applies when $\Re\mathfrak{e}\,\lambda>0$ if we neglect the $W_{2,\lambda}(x)$ term on the right-hand side. In the special case $\kappa=2$ and $\lambda=0$, we can use (58) to show that (96) reduces to

$$\int_{-\infty}^{\infty}\frac{\Gamma\left(-\frac{3}{2}+iu\right)}{\Gamma(2iu)u\sinh(\pi u)}M_{2,iu}(x)du=\frac{4}{3}e^{-x/2}(2\sqrt{\pi x}-3\pi x+\pi x^2). \quad (97)$$

D. $\lambda=0$

Setting $\lambda=0$ and combining (68) and (82) yields, for general values of κ,

$$\int_{-\infty}^{\infty}\frac{\Gamma(iu)\Gamma(-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)}M_{\kappa,iu}(x)du$$
$$=2\pi\Gamma\left(\frac{1}{2}-\kappa\right)\sqrt{x}e^{-x/2}-\frac{4\pi^2e^{-x/2}}{\cos(\pi\kappa)}\sum_{n=0}^{[\Re\mathfrak{e}\,\kappa-1/2]}\frac{x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)}. \quad (98)$$

By applying (58), this can be rewritten as

$$\int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)u\,\sinh(\pi u)} M_{\kappa,iu}(x)du$$
$$= 2\Gamma\left(\frac{1}{2}-\kappa\right)\sqrt{x}e^{-x/2} - \frac{4\pi e^{-x/2}}{\cos(\pi\kappa)} \sum_{n=0}^{[\Re\mathrm{e}\;\kappa-1/2]} \frac{x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)}. \tag{99}$$

In the special case $\lambda=0$ and $\kappa=0$, we can combine (99) with (15) and (52) to obtain the Bessel function integral

$$\int_{-\infty}^{\infty} \frac{I_{iu}(x/2)}{\sinh(\pi u)} du = -ie^{-x/2}, \tag{100}$$

which agrees with (89).

V. PHYSICAL APPLICATION

The results developed in this article have an important application in the field of high-energy radiation transport, involving the repeated Compton scattering of photons by hot electrons in a plasma with temperature T_e and number density n_e. This process, called "thermal Comptonization," plays a major role in the development of the x-ray continuum spectra observed from active galaxies, black holes, and neutron stars.[6] The Green's function, f_{G}, which describes the time evolution of an initially monoenergetic photon spectrum, satisfies the Kompaneets equation,[5]

$$\frac{\partial f_{\mathrm{G}}}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(f_{\mathrm{G}} + \frac{\partial f_{\mathrm{G}}}{\partial x}\right)\right], \tag{101}$$

where the dimensionless photon energy x and the dimensionless time y are defined by

$$x(\epsilon) \equiv \frac{\epsilon}{kT_e}, \quad y(t) \equiv n_e \sigma_{\mathrm{T}} c \frac{kT_e}{m_e c^2} t, \tag{102}$$

and the quantities t, ϵ, σ_{T}, m_e, c, and k denote time, photon energy, the Thomson cross section, the electron mass, the speed of light, and Boltzmann's constant, respectively. The terms proportional to f_{G} and $\partial f_{\mathrm{G}}/\partial x$ on the right-hand side of (101) represent the effects of electron recoil and stochastic (second-order Fermi) photon energization, respectively. At time $t=0$, the Green's function satisfies the monoenergetic initial condition

$$f_{\mathrm{G}}(x,x_0,y)\big|_{y=0} = x_0^{-2}\delta(x-x_0), \tag{103}$$

where the dimensionless initial energy is defined by

$$x_0 \equiv \frac{\epsilon_0}{kT_e}. \tag{104}$$

The exact solution for the time-dependent Green's function is given by[7]

$$f_{\mathrm{G}}(x,x_0,y) = \frac{32}{\pi} e^{-9y/4} x_0^{-2} x^{-2} e^{(x_0-x)/2} \int_0^{\infty} \frac{u\,\sinh(\pi u)}{(1+4u^2)(9+4u^2)}$$
$$\times W_{2,iu}(x_0) W_{2,iu}(x) e^{-u^2 y} du + \frac{e^{-x}}{2} + \frac{e^{-x-2y}}{2}\frac{(2-x)(2-x_0)}{x_0 x}. \tag{105}$$

The fundamental partial differential equation (101) is linear, and therefore the particular solution for the radiation spectrum resulting from an arbitrary initial distribution can be obtained by convolving the initial spectrum with the Green's function.

The power moments $I_n^{\mathrm{G}}(y)$ associated with the Green's function are defined for $n \geq 2$ by

$$I_n^{\mathrm{G}}(y) \equiv \int_0^\infty x^n f_{\mathrm{G}}(x,x_0,y)dx. \tag{106}$$

Important physical examples include the second and third moments, $I_2^{\mathrm{G}}(y)$ and $I_3^{\mathrm{G}}(y)$, which are related to the radiation number density n_r and the radiation energy density U_r via

$$n_r = (kT_e)^3 I_2^{\mathrm{G}}(y), \quad U_r = (kT_e)^4 I_3^{\mathrm{G}}(y). \tag{107}$$

By combining (103) and (106), we find that the power moments satisfy the initial condition

$$I_n^{\mathrm{G}}(0) = x_0^{n-2}. \tag{108}$$

The general solution for the power moments is obtained by operating on (105) with $\int_0^\infty x^n dx$, which yields

$$I_n^{\mathrm{G}}(y) = \frac{32}{\pi} e^{-9y/4} x_0^{-2} e^{x_0/2} \int_0^\infty \frac{u \sinh(\pi u)}{(1+4u^2)(9+4u^2)} W_{2,iu}(x_0) e^{-u^2 y} \times \int_0^\infty e^{-x/2} x^{n-2} W_{2,iu}(x) dx du + \frac{\Gamma(n+1)}{2} + (2-n)\Gamma(n) e^{-2y}\left(\frac{1}{x_0} - \frac{1}{2}\right). \tag{109}$$

The integration with respect to x can be carried out using Eq. (7.621.11) from Gradshteyn and Ryzhik[9] to obtain

$$I_n^{\mathrm{G}}(y) = \frac{32}{\pi} \frac{e^{-9y/4} e^{x_0/2}}{x_0^2 \Gamma(n-2)} \int_0^\infty \frac{u \sinh(\pi u)\Gamma\left(n+iu-\frac{1}{2}\right)\Gamma\left(n-iu-\frac{1}{2}\right)}{(1+4u^2)(9+4u^2)} W_{2,iu}(x_0) e^{-u^2 y} du + \frac{\Gamma(n+1)}{2} + (2-n)\Gamma(n) e^{-2y}\left(\frac{1}{x_0} - \frac{1}{2}\right). \tag{110}$$

In order to confirm the validity of this solution, it is necessary to demonstrate that it satisfies the initial condition given by (108), which is not obvious at first glance. We can establish this fact by employing one of the integral formulas derived in this article. Specifically, by setting $\lambda = n - \frac{1}{2}$ in (66), we find that

$$\int_0^\infty \frac{u \sinh(\pi u)\Gamma\left(n+iu-\frac{1}{2}\right)\Gamma\left(n-iu-\frac{1}{2}\right)}{(1+4u^2)(9+4u^2)} W_{2,iu}(x_0) du = \frac{\pi}{32}\Gamma(n-2) x_0^n e^{-x_0/2} - \frac{\pi}{8} \frac{\Gamma^2(n+1) e^{-x_0/2} x_0}{1-4\left(n-\frac{1}{2}\right)^2}\left(\frac{x_0}{4-2n} + \frac{2-x_0}{2n}\right), \tag{111}$$

where the $W_{2,\lambda}(x)$ term in (66) has been neglected since $n \geq 2$, and therefore $\lambda > 0$. Setting $y=0$ in (110) and using (111) to evaluate the integral with respect to u, we obtain after simplification

$$I_n^{\mathrm{G}}(0) = x_0^{n-2}, \tag{112}$$

which confirms that the solution for the power moments given by (110) satisfies the initial condition (108) as required.

VI. CONCLUSION

In this article we have derived several new closed-form expressions that are useful for evaluating infinite-domain integrals of the Whittaker functions $W_{\kappa,\mu}(x)$ and $M_{\kappa,\mu}(x)$ and the modified Bessel functions $I_\mu(x)$ and $K_\mu(x)$ with respect to the index μ. The definite integrals developed here are of significance in a wide variety of applications in mathematical physics, including radiation transport, financial analysis, and quantum mechanics. The main conclusions of the article are summarized below.

Our primary results, obtained in Sec. III, are the new integral formulas given by (44) and (50), which state that

$$\mathfrak{I}_1 \equiv \int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du$$
$$= \pi^2\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)x^{\lambda+1/2}e^{-x/2}$$
$$-\frac{\pi^3 e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)} \tag{113}$$

for $\mathfrak{Re}\,\lambda>0$, and

$$\mathfrak{I}_1 \equiv \int_0^\infty u \sinh(2\pi u)\Gamma\left(\frac{1}{2}-\kappa+iu\right)\Gamma\left(\frac{1}{2}-\kappa-iu\right)\Gamma(\lambda+iu)\Gamma(\lambda-iu)W_{\kappa,iu}(x)du$$
$$= \pi^2\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2}-\frac{\Gamma\left(\frac{1}{2}-\kappa-\lambda\right)}{\Gamma(-2\lambda)}W_{\kappa,\lambda}(x)\right]$$
$$-\frac{\pi^3 e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]}\frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)} \tag{114}$$

for $-1<\mathfrak{Re}\,\lambda<0$. Equations (113) and (114) are applicable provided that $\mathfrak{Re}\,\kappa\neq\frac{1}{2},\frac{3}{2},\ldots$ if $\mathfrak{Im}\,\kappa\neq 0$. They can also be utilized in the special case $\lambda=0$, provided that $\mathfrak{Re}\,\kappa\neq\frac{1}{2},\frac{3}{2},\ldots$. These fundamental relations allow the exact evaluation of all of the convergent cases of the integral $\mathfrak{I}_1$ without the need to resort to numerical integration and, consequently, they significantly generalize Apelblat's[8] Eq. (17.3.3), which is restricted to the domain $\mathfrak{Re}\,\kappa<\frac{1}{2}$ and $\mathfrak{Re}\,\lambda>0$.

Equations (113) and (114) are closed-form examples of the Whittaker function index transformation discussed by Srivastava *et al.*[14] A particularly interesting result involving the modified Bessel function K_μ is obtained by setting $\kappa=0$, which yields [see Eq. (54)]

$$\int_0^\infty u \sinh(\pi u)\Gamma(\lambda+iu)\Gamma(\lambda-iu)K_{iu}(z)du = \pi^{3/2}2^{\lambda-1}\Gamma\left(\lambda+\frac{1}{2}\right)\left[z^\lambda e^{-z}-\frac{2^{1+\lambda}}{\Gamma(-\lambda)}K_\lambda(z)\right]. \tag{115}$$

This expression represents a new example of the Kontorovich–Lebedev transform valid if $-1<\mathfrak{Re}\,\lambda<0$ or $\lambda=0$. Equation (115) can also be applied when $\mathfrak{Re}\,\lambda>0$ if we neglect the $K_\lambda(z)$ term on the right-hand side.

In Sec. IV, we used our results for the fundamental $W_{\kappa,\mu}(x)$ index integral $\mathfrak{I}_1$ given by (113) and (114) to develop corresponding expressions for the $M_{\kappa,\mu}(x)$ index integral $\mathfrak{I}_2$ defined in (2). The final results obtained are [see Eqs. (83) and (84)]

$$\begin{aligned}\mathfrak{I}_2 &\equiv \int_{-\infty}^{\infty} \frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)} M_{\kappa,iu}(x)du \\ &= 2\pi\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)x^{\lambda+1/2}e^{-x/2} \\ &\quad - \frac{2\pi^2 e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]} \frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)}\end{aligned} \tag{116}$$

for $\mathfrak{Re}\,\lambda>0$, and

$$\begin{aligned}\mathfrak{I}_2 &\equiv \int_{-\infty}^{\infty} \frac{\Gamma(\lambda+iu)\Gamma(\lambda-iu)\Gamma\left(\frac{1}{2}-\kappa+iu\right)}{\Gamma(2iu)} M_{\kappa,iu}(x)du \\ &= 2\pi\Gamma\left(\lambda-\kappa+\frac{1}{2}\right)\left[x^{\lambda+1/2}e^{-x/2} - \frac{\Gamma\left(\frac{1}{2}-\kappa-\lambda\right)}{\Gamma(-2\lambda)}W_{\kappa,\lambda}(x)\right] \\ &\quad - \frac{2\pi^2 e^{-x/2}}{\cos[\pi(\kappa-\lambda)]}\sum_{n=0}^{[\mathfrak{Re}\,\kappa-1/2]} \frac{(2\kappa-2n-1)\Gamma\left(\kappa+\lambda-n-\frac{1}{2}\right)x^{\kappa-n}L_n^{(2\kappa-2n-1)}(x)}{\Gamma(2\kappa-n)\Gamma\left(\kappa-\lambda-n+\frac{1}{2}\right)}\end{aligned} \tag{117}$$

for $-1<\mathfrak{Re}\,\lambda<0$. Equations (116) and (117) are valid provided that $\mathfrak{Re}\,\kappa\neq\frac{1}{2},\frac{3}{2},\ldots$ if $\mathfrak{Im}\,\kappa\neq0$. They also apply when $\lambda=0$ provided that $\mathfrak{Re}\,\kappa\neq\frac{1}{2},\frac{3}{2}$, etc.

Setting $\kappa=0$ in (117) yields the interesting new expression [see Eq. (88)]

$$\int_{-\infty}^{\infty} u\Gamma(\lambda+iu)\Gamma(\lambda-iu)I_{iu}(z)du = i2^{\lambda}\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)\left[\frac{2^{1+\lambda}}{\Gamma(-\lambda)}K_\lambda(z)-z^\lambda e^{-z}\right], \tag{118}$$

which is a previously unknown integral formula for the modified Bessel function $I_{iu}(z)$ valid for $-1<\mathfrak{Re}\,\lambda<0$ or $\lambda=0$. It also applies when $\mathfrak{Re}\,\lambda>0$ if the $K_\lambda(z)$ term is neglected.

In addition to our general results for the integrals $\mathfrak{I}_1$ and $\mathfrak{I}_2$, we also presented a variety of specialized expressions associated with particular values of κ and λ, and in Sec. V we applied the result obtained for $\mathfrak{I}_2$ when $\kappa=2$ to the solution of a physical problem involving radiation transport. Finally, we note that by utilizing (3) and (4), one can easily transform the fundamental results given by (44), (50), (83), and (84) to obtain an analogous set of definite integrals involving the Kummer confluent hypergeometric functions $\Phi(a,b,z)$ and $\Psi(a,b,z)$.

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