

## Spectral formation in x-ray pulsars and associated identities involving the Laguerre polynomials

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The partial differential equation governing the formation of the radiation spectrum in the shocked plasma accreting onto an x-ray pulsar is solved to obtain the associated Green's function describing the scattering of monochromatic radiation injected into the plasma at a fixed altitude above the star. Collisions between the photons and the infalling electrons cause both the ordered and random components of the plasma energy to be transferred to the radiation, which escapes by diffusing through the walls of the accretion column. The analytical solution for the Green's function provides important physical insight into the formation of the observed spectra in x-ray pulsars. Interesting mathematical aspects of the problem include the establishment of a new summation formula involving the Laguerre polynomials, based on the calculation of the photon number density via integration of Green's function. © 2008 American Institute of Physics.  
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### I. INTRODUCTION

The analysis of solutions to partial differential equations satisfying physically prescribed boundary conditions often leads to new insights into the relationships between the various special functions of mathematical physics. In this article, we employ the methods of classical analysis to obtain the closed-form solution for the Green's function describing the scattering of radiation inside the plasma accreting onto the surface of a rotating neutron star in an x-ray pulsar. The method is based on the separation of the two-dimensional partial differential equation into two second-order ordinary differential equations depending on the energy and space variables. By applying suitable boundary conditions in the spatial dimension, we are able to determine the eigenvalues and show that the corresponding spatial eigenfunctions are Laguerre polynomials. The solutions to the associated energy equation are Whittaker functions. The global solution for the two-dimensional Green's function is obtained by forming an expansion based on the energy and space eigenfunctions. The availability of this solution also allows us to calculate the distribution of the radiation escaping from the accreting gas, which forms the spectrum observed at earth. Beyond the direct physical relevance of the Green's function, the method of solution also yields several additional results of mathematical interest, including a new summation identity for the Laguerre polynomials.

We provide a brief overview of the physical problem before proceeding with the main derivation. The fundamental power source for the radiation produced in bright x-ray pulsars is the gravitational accretion (inflow) of fully ionized plasma that is channeled onto the magnetic poles of the rotating neutron star by the strong magnetic field. In these luminous sources, the pressure of the photons governs the dynamical structure of the accretion flow, and therefore the gas must pass through a radiation-dominated shock on its way to the stellar surface. The kinetic energy of the gas is carried away by the high-energy radiation that escapes from the column, which allows the

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material to come to rest on the stellar surface.<sup>1,2</sup> The strong compression of the infalling gas drives its temperatures up to several million Kelvins. The gas therefore radiates x rays, which appear to pulsate due to the star's rotation. The observed x-ray spectrum is often distinctly nonthermal, indicating that nonequilibrium processes are playing an important role in the radiative transfer occurring inside the accretion column.<sup>3</sup>

The nonthermal shape of the spectrum is due to the transfer of energy from the gas to the photons via electron scattering. The plasma possesses both "ordered" kinetic energy associated with the inflow and "random" kinetic energy associated with the thermal motion of the particles. These two types of energy are transferred to the photons via first- and second-order Fermi processes, respectively. Our primary goal in this article is to obtain the exact solution for the Green's function describing the effect of electron scattering on monoenergetic seed photons injected from a source located at an arbitrary altitude inside the accretion column. The Green's function contains a complete representation of the fundamental physics governing the propagation of the photons in the physical and energy spaces. Since the transport equation governing the radiation spectrum is linear, we can compute the solution associated with any seed photon distribution via convolution. In x-ray pulsars, the most important sources of seed photons are bremsstrahlung, cyclotron, and blackbody emission.<sup>4</sup> The Green's function provides a direct means for investigating how photons produced by the various source mechanisms contribute to the observed nonthermal spectra in x-ray pulsars.

## II. FUNDAMENTAL EQUATIONS

The plasma flowing onto the neutron star is assumed to have a cylindrical geometry maintained by the strong dipole magnetic field. We define the spatial coordinate  $z$  as the altitude measured from the stellar surface along the axis of the cylindrical accretion column. The gas flows onto the star with velocity  $v(z) < 0$ , which vanishes at the surface of the star. The Green's function,  $f_G(z_0, z, \epsilon_0, \epsilon)$ , is defined as the radiation distribution measured at location  $z$  and energy  $\epsilon$  resulting from the injection of  $\dot{N}_0$  photons per second with energy  $\epsilon_0$  from a monochromatic source located at altitude  $z_0$  inside the column. In a steady-state situation,  $f_G$  satisfies the Kompaneets transport equation<sup>4,5</sup>

$$v \frac{\partial f_G}{\partial z} = \frac{dv}{dz} \frac{\epsilon}{3} \frac{\partial f_G}{\partial \epsilon} + \frac{\partial}{\partial z} \left( \frac{c}{3n_e \sigma_{\parallel}} \frac{\partial f_G}{\partial z} \right) - \frac{f_G}{t_{\text{esc}}} + \frac{n_e \bar{\sigma} c}{m_e c^2} \frac{1}{\epsilon^2} \frac{\partial}{\partial \epsilon} \left[ \epsilon^4 \left( f_G + kT_e \frac{\partial f_G}{\partial \epsilon} \right) \right] + \frac{\dot{N}_0 \delta(\epsilon - \epsilon_0) \delta(z - z_0)}{\pi r_0^2 \epsilon_0^2}, \quad (1)$$

where  $r_0$  is the radius of the accretion column,  $t_{\text{esc}}$  represents the mean time photons spend in the plasma before escaping through the walls of the column,  $\bar{\sigma}$  denotes the angle-averaged electron scattering cross section,  $c$  is the speed of light,  $k$  is Boltzmann's constant, and  $T_e$ ,  $n_e$ , and  $m_e$  denote the electron temperature, number density, and mass, respectively. The left-hand side of (1) represents the comoving time derivative of the Green's function, and the terms on the right-hand side represent first-order Fermi energization ("bulk Comptonization"), spatial diffusion along the column axis, photon escape, stochastic (thermal) Comptonization, and photon injection, respectively.

The transport equation employed here is similar to the one analyzed by Becker<sup>1</sup> and Becker and Wolff,<sup>2</sup> except that thermal Comptonization has now been included via the appearance of the Kompaneets (1957) operator, which is the fourth term on the right-hand side of (1). The radiation number density  $n_G$  and energy density  $U_G$  associated with the Green's function are given by

$$n_G(z) = \int_0^{\infty} \epsilon^2 f_G(z_0, z, \epsilon_0, \epsilon) d\epsilon, \quad U_G(z) = \int_0^{\infty} \epsilon^3 f_G(z_0, z, \epsilon_0, \epsilon) d\epsilon. \quad (2)$$

The Green's function provides fundamental physical insight into the spectral redistribution process, and it also allows us to calculate the particular solution for the spectrum  $f(z, \epsilon)$  associated with an arbitrary photon source  $Q(z, \epsilon)$  using the integral convolution<sup>6</sup>

$$f(z, \epsilon) = \int_0^\infty \int_0^\infty \frac{f_G(z_0, z, \epsilon_0, \epsilon)}{N_0} \epsilon_0^2 Q(z_0, \epsilon_0) d\epsilon_0 dz_0, \quad (3)$$

where the source function  $Q$  is normalized so that  $\epsilon^2 Q(z, \epsilon) d\epsilon dz$  gives the number of seed photons injected per unit time in the altitude range between  $z$  and  $z+dz$  and in the energy range between  $\epsilon$  and  $\epsilon+d\epsilon$ .

Following Lyubarskii and Sunyaev,<sup>7</sup> we will assume that the electron temperature  $T_e$  has a constant value, which is physically reasonable since most of the Comptonization occurs in a relatively compact region near the base of the accretion column. In this situation, it is convenient to work in terms of the dimensionless energy variable  $\chi$ , defined by

$$\chi(\epsilon) \equiv \frac{\epsilon}{kT_e}, \quad (4)$$

and the dimensionless optical depth variable  $\tau$ , defined by

$$d\tau \equiv n_e(z) \sigma_{\parallel} dz, \quad \tau(z) \equiv \int_0^z n_e(z') \sigma_{\parallel} dz', \quad (5)$$

where  $z$  and  $\tau$  both vanish at the stellar surface. The electron number density  $n_e$  is related to the inflow velocity  $v$  via

$$\dot{M} \equiv \pi r_0^2 m_p n_e |v| = \text{const}, \quad (6)$$

where  $\dot{M}$  is the mass accretion rate onto the magnetic pole of the star and  $m_p$  is the proton mass. Note that we can write the Green's function as either  $f_G(z_0, z, \epsilon_0, \epsilon)$  or  $f_G(\tau_0, \tau, \chi_0, \chi)$  since the variables  $(z, \epsilon)$  and  $(\chi, \tau)$  are interchangeable by utilizing (4) and (5).

Making the change of variable from  $z$  to  $\tau$  in (1), we find after some algebra that the transport equation for the Green's function can be written in the form

$$\frac{v}{c} \frac{\partial f_G}{\partial \tau} = \frac{1}{c} \frac{dv}{d\tau} \frac{\chi}{3} \frac{\partial f_G}{\partial \chi} + \frac{1}{3} \frac{\partial^2 f_G}{\partial \tau^2} - \frac{\xi^2 v^2}{c^2} f_G + \frac{\bar{\sigma}}{\sigma_{\parallel}} \frac{kT_e}{m_e c^2} \frac{1}{\chi^2} \frac{\partial}{\partial \chi} \left[ \chi^4 \left( f_G + \frac{\partial f_G}{\partial \chi} \right) \right] + \frac{\dot{N}_0 \delta(\chi - \chi_0) \delta(\tau - \tau_0)}{\pi r_0^2 c k T_e \epsilon_0^2}, \quad (7)$$

where  $\tau_0 \equiv \tau(z_0)$ ,  $\chi_0 \equiv \chi(\epsilon_0)$ , and we have introduced the dimensionless constant

$$\xi \equiv \frac{\pi r_0 m_p c}{\dot{M} (\sigma_{\parallel} \sigma_{\perp})^{1/2}}, \quad (8)$$

which determines the importance of the escape of photons from the accretion column. It can be shown that  $\xi$  is roughly equal to the ratio of the dynamical (accretion) timescale divided by the timescale for the photons to diffuse through the column walls,<sup>4</sup> and therefore the condition  $\xi \sim 1$  must be satisfied in order to ensure that the radiation pressure decelerates the gas to rest at the stellar surface, with the kinetic energy of the material carried away by the photons escaping through the column walls.<sup>8</sup>

Following Becker and Wolff,<sup>2</sup> we compute the mean escape time using the diffusive prescription

$$t_{\text{esc}}(z) = \frac{r_0 \tau_{\perp}}{c}, \quad \tau_{\perp}(z) = n_e \sigma_{\perp} r_0, \quad (9)$$

where  $\tau_{\perp}$  represents the perpendicular scattering optical thickness of the cylindrical accretion column. Note that  $\tau_{\perp}$  and  $t_{\text{esc}}$  are each functions of  $z$  through their dependence on the electron number density  $n_e(z)$ . Becker<sup>8</sup> confirmed that the diffusion approximation employed in (9) is valid

since  $\tau_{\perp} > 1$  for typical x-ray pulsar parameters. The technical approach used to solve for the Green's function, carried out in Sec. III, involves the derivation of eigenvalues and associated eigenfunctions based on the set of spatial boundary conditions for the problem.<sup>9-12</sup>

### III. EXACT SOLUTION FOR GREEN'S FUNCTION

Lyubarskii and Sunyaev<sup>7</sup> demonstrated that the transport equation (7) is separable in energy and space if  $\chi \neq \chi_0$  and the velocity profile has the particular form

$$v(\tau) = -\alpha c \tau, \quad (10)$$

where  $\alpha$  is a positive constant. This expression provides a reasonable approximation to the actual velocity profile in an x-ray pulsar accretion column. By combining (5), (6), (8), and (10), we can express  $\tau$  as an explicit function of the altitude  $z$ , obtaining

$$\tau(z) = \left( \frac{\sigma_{\parallel}}{\sigma_{\perp}} \right)^{1/4} \left( \frac{2z}{\alpha \xi r_0} \right)^{1/2}. \quad (11)$$

Using this result to substitute for  $\tau$  in (10), we note that the velocity profile required for separability is related to  $z$  via

$$v(z) = - \left( \frac{\sigma_{\parallel}}{\sigma_{\perp}} \right)^{1/4} \left( \frac{2\alpha z}{\xi r_0} \right)^{1/2} c. \quad (12)$$

This profile describes a flow that stagnates at the stellar surface ( $\tau=0$ ,  $z=0$ ) as required.

Adopting the velocity profile given by (10), we find that the transport equation (7) for the Green's function can be reorganized to obtain

$$\frac{\alpha \chi}{3} \frac{\partial f_G}{\partial \chi} - \frac{\bar{\sigma}}{\sigma_{\parallel} m_e c^2} \frac{1}{\chi^2} \frac{\partial}{\partial \chi} \left[ \chi^4 \left( f_G + \frac{\partial f_G}{\partial \chi} \right) \right] = \frac{1}{3} \frac{\partial^2 f_G}{\partial \tau^2} + \alpha \tau \frac{\partial f_G}{\partial \tau} - \xi^2 \alpha^2 \tau^2 f_G + \frac{\dot{N}_0 \delta(\chi - \chi_0) \delta(\tau - \tau_0)}{\pi r_0^2 c k T_e \epsilon_0^2}. \quad (13)$$

When  $\chi \neq \chi_0$ , the  $\delta$ -function source term on the right-hand side makes no contribution, and the differential equation is therefore linear and homogeneous. In this case, the transport equation can be separated in energy and space using the functions

$$f_{\lambda}(\tau, \chi) = g(\lambda, \tau) h(\lambda, \chi), \quad (14)$$

where  $\lambda$  is the separation constant. We find that the spatial and energy functions,  $g$  and  $h$ , respectively, satisfy the differential equations

$$\frac{1}{3} \frac{d^2 g}{d\tau^2} + \alpha \tau \frac{dg}{d\tau} + \left( \frac{\alpha \lambda}{3} - \xi^2 \alpha^2 \tau^2 \right) g = 0 \quad (15)$$

and

$$\frac{1}{\chi^2} \frac{d}{d\chi} \left[ \chi^4 \left( h + \frac{dh}{d\chi} \right) \right] - \beta \chi \frac{dh}{d\chi} - \beta \lambda h = 0, \quad (16)$$

where the parameter  $\beta$  is defined by

$$\beta \equiv \frac{\alpha \sigma_{\parallel} m_e c^2}{3 \bar{\sigma} k T_e}. \quad (17)$$

It can be shown that  $\beta$  determines the relative importance of bulk and thermal Comptonization. When  $\beta$  is of order unity, the two processes are comparable, and when  $\beta > 1$ , the bulk process dominates.<sup>4</sup>

### A. Eigenvalues and spatial eigenfunctions

In order to obtain the global solution for the spatial separation function  $g$ , we need to understand the physical boundary conditions that  $g$  must satisfy. In the downstream region, as the gas approaches the stellar surface, we require that the advective and diffusive components of the radiation flux must both vanish due to the divergence of the electron density as the gas settles onto the star. The advective flux is indeed negligible at the stellar surface since  $v \rightarrow 0$  as  $\tau \rightarrow 0$  [see Eq. (10)]. However, in order to ensure that the diffusive flux vanishes, we must require that  $dg/d\tau \rightarrow 0$  as  $\tau \rightarrow 0$ . Conversely, in the upstream region, we expect that  $g \rightarrow 0$  as  $\tau \rightarrow \infty$  since no photons can diffuse to large distances in the direction opposing the plasma flow. With these boundary conditions taken into consideration, we find that the fundamental solution for the spatial separation function  $g$  has the general form

$$g(\lambda, \tau) \propto \begin{cases} e^{-\alpha(3+w)\tau^2/4} U(\rho, 1/2, \alpha w \tau^2/2), & \tau \geq \tau_0 \\ e^{-\alpha(3+w)\tau^2/4} M(\rho, 1/2, \alpha w \tau^2/2), & \tau \leq \tau_0, \end{cases} \quad (18)$$

where  $M$  and  $U$  denote confluent hypergeometric functions,<sup>13</sup> and we have made the definitions

$$\rho \equiv \frac{w+3-2\lambda}{4w}, \quad w \equiv (9+12\xi^2)^{1/2}. \quad (19)$$

Equation (15) is linear, second order, and homogeneous, and consequently both the function  $g$  and its derivative  $dg/d\tau$  must be continuous at the source location,  $\tau = \tau_0$ . The smooth merger of the  $M$  and  $U$  functions at the source location requires that their Wronskian  $\omega(\tau)$  must vanish at  $\tau = \tau_0$ , where

$$\omega(\tau) \equiv M\left(\rho, \frac{1}{2}, \frac{\alpha w \tau^2}{2}\right) \frac{d}{d\tau} U\left(\rho, \frac{1}{2}, \frac{\alpha w \tau^2}{2}\right) - U\left(\rho, \frac{1}{2}, \frac{\alpha w \tau^2}{2}\right) \frac{d}{d\tau} M\left(\rho, \frac{1}{2}, \frac{\alpha w \tau^2}{2}\right). \quad (20)$$

This condition can be used to solve for the eigenvalues of the separation constant  $\lambda$ . By employing Eq. (13.1.22) from Abramowitz and Stegun<sup>13</sup> to evaluate the Wronskian, we obtain the eigenvalue equation

$$\omega(\tau) = -\frac{\Gamma(\frac{1}{2})(2\alpha w)^{1/2}}{\Gamma(\rho)} e^{\alpha w \tau^2/2} = 0. \quad (21)$$

The left-hand side vanishes when  $\Gamma(\rho) \rightarrow \pm\infty$ , which implies that  $\rho = -n$ , where  $n = 0, 1, 2, \dots$ . By combining this result with (19), we conclude that the eigenvalues  $\lambda_n$  are given by

$$\lambda_n = \frac{4nw + w + 3}{2}, \quad n = 0, 1, 2, \dots \quad (22)$$

When  $\lambda = \lambda_n$ , the spatial separation functions  $g$  reduce to a set of global eigenfunctions  $g_n$  that satisfy the boundary conditions at large and small values of  $\tau$ . In this case, we can use Eqs. (13.6.9) and (13.6.27) from Abramowitz and Stegun<sup>13</sup> to show that the confluent hypergeometric functions  $M$  and  $U$  appearing in (20) are proportional to the generalized Laguerre polynomials  $L_n^{(-1/2)}$ , and consequently the global solutions for the spatial eigenfunctions can be written as

$$g_n(\tau) \equiv g(\lambda_n, \tau) = e^{-\alpha(3+w)\tau^2/4} L_n^{(-1/2)}\left(\frac{\alpha w \tau^2}{2}\right). \quad (23)$$

Based on Eq. (7.414.3) from Gradshteyn and Ryzhik,<sup>14</sup> we note that the spatial eigenfunctions satisfy the orthogonality relation

$$\int_0^\infty e^{3\alpha\tau^2/2} g_n(\tau) g_m(\tau) d\tau = \begin{cases} \Gamma(n+1/2)/n! \sqrt{2\alpha w}, & n = m \\ 0, & n \neq m. \end{cases} \quad (24)$$

## B. Energy eigenfunctions

The solution for the energy separation function  $h$  depends on the boundary conditions imposed in the energy space. As  $\chi \rightarrow 0$ , we require that  $h$  not increase faster than  $\chi^{-3}$  since the Green's function must possess a finite total photon number density [see Eq. (2)]. Conversely, as  $\chi \rightarrow \infty$ , we require that  $h$  decrease more rapidly than  $\chi^{-4}$  in order to ensure that the Green's function contains a finite total photon energy density. Furthermore, in order to avoid an infinite diffusive flux in the energy space at  $\chi = \chi_0$ , the function  $h$  must be continuous there. The fundamental solution for the energy eigenfunction that satisfies the various boundary and continuity conditions can be written as

$$h_n(\chi) \equiv h(\lambda_n, \chi) = \begin{cases} \chi^{\kappa-4} e^{-\chi/2} W_{\kappa, \mu}(\chi_0) M_{\kappa, \mu}(\chi), & \chi \leq \chi_0 \\ \chi^{\kappa-4} e^{-\chi/2} M_{\kappa, \mu}(\chi_0) W_{\kappa, \mu}(\chi), & \chi \geq \chi_0, \end{cases} \quad (25)$$

where  $M_{\kappa, \mu}$  and  $W_{\kappa, \mu}$  denote Whittaker's functions, and we have made the definitions

$$\kappa \equiv \frac{1}{2}(\beta + 4), \quad \mu \equiv \frac{1}{2}[(3 - \beta)^2 + 4\beta\lambda_n]^{1/2}. \quad (26)$$

Note that each of the eigenvalues  $\lambda_n$  results in a different value for  $\mu$ , and the parameter  $\beta$  is defined in (17). Equation (25) can also be written in the more compact form

$$h_n(\chi) = \chi^{\kappa-4} e^{-\chi/2} M_{\kappa, \mu}(\chi_{\min}) W_{\kappa, \mu}(\chi_{\max}), \quad (27)$$

where

$$\chi_{\min} \equiv \min(\chi, \chi_0), \quad \chi_{\max} \equiv \max(\chi, \chi_0). \quad (28)$$

## C. Eigenfunction expansion

The spatial eigenfunctions  $g_n(y)$  form an orthogonal set, as expected since this is a standard Sturm–Liouville problem. The solution for the Green's function can therefore be expressed as the infinite series

$$f_G(\tau_0, \tau, \chi_0, \chi) = \sum_{n=0}^{\infty} C_n g_n(\tau) h_n(\chi), \quad (29)$$

where the expansion coefficients  $C_n$  are computed by employing the orthogonality of the eigenfunctions, along with the derivative jump condition

$$\lim_{\varepsilon \rightarrow 0} \left. \frac{\partial f_G}{\partial \chi} \right|_{\chi=\chi_0+\varepsilon} - \left. \frac{\partial f_G}{\partial \chi} \right|_{\chi=\chi_0-\varepsilon} = - \frac{3\dot{N}_0 \beta k T_e \delta(\tau - \tau_0)}{\alpha \pi r_0^2 c \epsilon_0^4}, \quad (30)$$

which is obtained by integrating the transport equation (13) with respect to  $\chi$  in a small region surrounding the injection energy  $\chi_0$ . Substituting using the expansion for  $f_G$  yields

$$\lim_{\varepsilon \rightarrow 0} \sum_{n=0}^{\infty} C_n g_n(\tau) [h'_n(\chi_0 + \varepsilon) - h'_n(\chi_0 - \varepsilon)] = - \frac{3\dot{N}_0 \beta k T_e \delta(\tau - \tau_0)}{\alpha \pi r_0^2 c \epsilon_0^4}, \quad (31)$$

where primes denote differentiation with respect to  $\chi$ . By employing (27) for  $h_n$ , we find that

$$\sum_{n=0}^{\infty} C_n g_n(\tau) \chi_0^{\kappa-4} e^{-\chi_0/2} \mathfrak{W}(\chi_0) = - \frac{3\dot{N}_0 \beta k T_e \delta(\tau - \tau_0)}{\alpha \pi r_0^2 c \epsilon_0^4}, \quad (32)$$

where the Wronskian  $\mathfrak{W}$  is defined by

$$\mathfrak{W}(\chi_0) \equiv M_{\kappa,\mu}(\chi_0)W'_{\kappa,\mu}(\chi_0) - W_{\kappa,\mu}(\chi_0)M'_{\kappa,\mu}(\chi_0). \quad (33)$$

We can evaluate  $\mathfrak{W}(\chi_0)$  analytically using Eq. (2.4.27) from Slater,<sup>15</sup> which yields

$$\mathfrak{W}(\chi_0) = -\frac{\Gamma(1+2\mu)}{\Gamma(\mu-\kappa+\frac{1}{2})}. \quad (34)$$

Using this result to substitute for  $\mathfrak{W}(\chi_0)$  in (32) and reorganizing the terms, we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(1+2\mu)C_n g_n(\tau)}{\Gamma(\mu-\kappa+\frac{1}{2})} = \frac{3\dot{N}_0\beta e^{\chi_0/2} \delta(\tau-\tau_0)}{\alpha\pi r_0^2 c \chi_0^\kappa (kT_e)^3}, \quad (35)$$

where  $\mu$  is a function of  $\lambda_n$  via (26). We can now calculate the expansion coefficients  $C_n$  by utilizing the orthogonality of the spatial eigenfunctions  $g_n$  represented by (24). Multiplying both sides of (35) by  $e^{3\alpha\tau^2/2}g_m(\tau)$  and integrating with respect to  $\tau$  from zero to infinity yields, after some algebra,

$$C_n = \frac{3\dot{N}_0\beta\sqrt{2w}e^{\chi_0/2}e^{3\alpha\tau_0^2/2}\Gamma(\mu-\kappa+\frac{1}{2})n!g_n(\tau_0)}{\pi r_0^2 c (kT_e)^3 \chi_0^\kappa \sqrt{\alpha} \Gamma(1+2\mu)\Gamma(n+\frac{1}{2})}. \quad (36)$$

The final closed-form solution for the Green's function, obtained by combining (27), (29), and (36), is given by

$$f_G(\tau_0, \tau, \chi_0, \chi) = \frac{3\dot{N}_0\beta e^{3\alpha\tau_0^2/2}\sqrt{2w}\chi^{\kappa-4}e^{(\chi_0-\chi)/2}}{\pi r_0^2 c (kT_e)^3 \chi_0^\kappa \sqrt{\alpha}} \sum_{n=0}^{\infty} \frac{\Gamma(\mu-\kappa+\frac{1}{2})n!}{\Gamma(1+2\mu)\Gamma(n+\frac{1}{2})} \\ \times g_n(\tau_0)g_n(\tau)M_{\kappa,\mu}(\chi_{\min})W_{\kappa,\mu}(\chi_{\max}), \quad (37)$$

where the spatial eigenfunctions  $g_n$  are computed using (23) and the parameters  $\kappa$  and  $\mu$  are given by (26). This exact, analytical solution for  $f_G$  provides a very efficient means for computing the steady-state Green's function resulting from the continual injection of monochromatic seed photons from a source at an arbitrary location inside the accretion column. The eigenfunction expansion converges rapidly and therefore we can generally obtain an accuracy of at least four significant figures in our calculations of  $f_G$  by terminating the series after the first five to ten terms.

#### D. Numerical examples

In this section, we illustrate the computational method by examining the dependence of the Green's function  $f_G(\tau_0, \tau, \chi_0, \chi)$  on the optical depth  $\tau$  and the dimensionless energy  $\chi$ . We remind the reader that the solution for the Green's function represents the photon spectrum inside the accretion column at the specified position and energy, resulting from the injection of monochromatic photons with dimensionless energy  $\chi_0$  from a source located at optical depth  $\tau_0$ . Analysis of  $f_G$  therefore allows us to explore the effects of bulk and thermal Comptonization as the photons diffuse throughout the plasma, eventually escaping through the walls of the accretion column. In terms of  $\chi$  and  $\tau$ , the radiation number density associated with Green's function is given by [cf. Eq. (2)]

$$n_G(\tau) = (kT_e)^3 \int_0^\infty \chi^2 f_G(\tau_0, \tau, \chi_0, \chi) d\chi. \quad (38)$$

It follows that  $(kT_e)^3 \chi^2 f_G d\chi$  equals the number of photons per unit volume at optical depth  $\tau$  with dimensionless energy between  $\chi$  and  $\chi+d\chi$ .

In Fig. 1, we use (37) to plot  $\chi^2 f_G$  as a function of  $\chi$  and  $\tau$  for the parameter values  $\dot{N}_0=1$ ,  $T_e=10^7$  K,  $r_0=10^4$  cm,  $\alpha=0.1$ ,  $\xi=1.5$ ,  $\chi_0=0.1$ ,  $\tau_0=0.5$ , and either  $\beta=0.3$  or  $\beta=1.5$ . We remind the reader that the value of  $\beta$  determines the relative importance of bulk and thermal Compton-

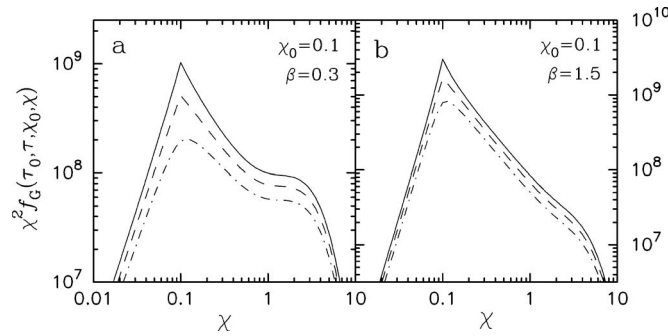


FIG. 1. The Green's function  $\chi^2 f_G$  describing the photon spectrum inside an x-ray pulsar accretion column [Eq. (37)] plotted as a function of the dimensionless photon energy  $\chi$  for  $\dot{N}_0=1$ ,  $T_e=10^7$  K,  $r_0=10^4$  cm,  $\alpha=0.1$ ,  $\xi=1.5$ ,  $\chi_0=0.1$ ,  $\tau_0=0.5$ , and (a)  $\beta=0.3$  and (b)  $\beta=1.5$ . The values of the optical depth are  $\tau=0.01$  (solid line),  $\tau=1.0$  (dashed line), and  $\tau=1.5$  (dot-dashed line).

ization [see Eq. (17)]. Both spectra in Fig. 1 display a peak at the energy of the injected photons,  $\chi=0.1$ . For the case with  $\beta=0.3$ , there is also a noticeable Wien hump in the spectrum around  $\chi\sim 2$  due to the effect of thermal Comptonization.<sup>3</sup> However, when  $\beta=1.5$ , the Wien feature is completely hidden by the power-law shape associated with the dominant bulk Comptonization process. The Green's function is plotted in Fig. 2 for the same parameters used in Fig. 1, except that we now set  $\chi_0=0.5$ , so that the energy of the injected photons,  $\epsilon_0$ , is comparable to the thermal energy of the electrons,  $kT_e$ . In this case, the Wien hump is still visible when  $\beta=0.3$ , although it has begun to merge into the peak associated with the injected photons because of the larger value of  $\chi_0$ . Note that all of the spectra in Figs. 1 and 2 display a normalization that decreases with increasing  $\tau$ , due to the fact that the downward advection of the electrons tends to “trap” the photons near the bottom of the accretion column, close to the stellar surface.<sup>8</sup>

Equation (37) can also be used to calculate the spectrum of the radiation escaping through the walls of the accretion column, which comprises the radiation spectrum observed at earth. Because the x-ray pulsars are distant cosmic sources, the observed spectra are the result of emission escaping over the *entire* vertical extent of the accretion column, and therefore we must perform a vertical integration with respect to  $\tau$  in order to compare the model spectra with actual x-ray observations. This procedure is carried out in Ref. 4, where the resulting spectra are compared with the x-ray data for several astrophysical sources, confirming excellent agreement.

#### IV. IDENTITIES INVOLVING THE LAGUERRE POLYNOMIALS

We are primarily interested here in the mathematical properties of the solution for the Green's function given by (37). In the present application, it is possible to compute the photon number

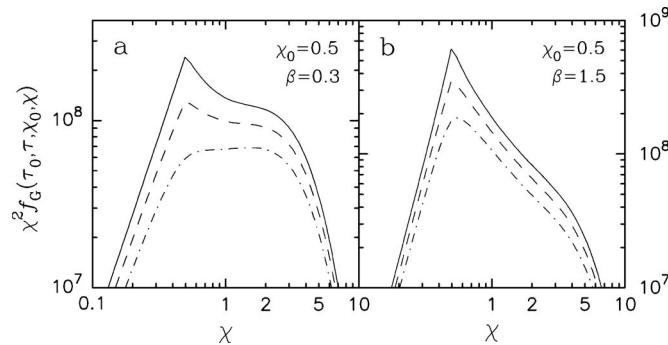


FIG. 2. Same as Fig. 1, except  $\chi_0=0.5$ . In this case, the energy of the injected photons is comparable to the thermal energy of the electrons.

density  $n_G$  either by integrating the series expansion for the Green's function term by term or by solving directly the ordinary differential equation satisfied by  $n_G$ . By equating the results obtained for the photon number density using the two methods, we can derive an interesting new summation identity for the Laguerre polynomials  $L_n^{(-1/2)}(x)$  appearing in the expression for the spatial eigenfunctions  $g_n(\tau)$ . We carry out this procedure below.

### A. Integration of the Green's function

By using (37) to substitute for  $f_G$  in (38) and integrating term by term, we find that the associated photon number density  $n_G$  can be written as

$$n_G(\tau) = \frac{3\dot{N}_0\beta e^{3\alpha\tau_0^2/2}\sqrt{2w}e^{\chi_0/2}}{\pi r_0^2 c \chi_0^\kappa \sqrt{\alpha}} \sum_{n=0}^{\infty} \frac{\Gamma(\mu - \kappa + \frac{1}{2})n!}{\Gamma(1 + 2\mu)\Gamma(n + \frac{1}{2})} g_n(\tau_0)g_n(\tau)K_G(\chi_0), \quad (39)$$

where

$$K_G(\chi_0) \equiv W_{\kappa,\mu}(\chi_0) \int_0^{\chi_0} \chi^{\kappa-2} e^{-\chi/2} M_{\kappa,\mu}(\chi) d\chi + M_{\kappa,\mu}(\chi_0) \int_{\chi_0}^{\infty} \chi^{\kappa-2} e^{-\chi/2} W_{\kappa,\mu}(\chi) d\chi. \quad (40)$$

The two indefinite integrals on the right-hand side of (40) can be evaluated using Eqs. (3.2.6) and (3.2.12) from Slater.<sup>15</sup> After some algebra, we find that

$$\int_0^{\chi_0} \chi^{\kappa-2} e^{-\chi/2} M_{\kappa,\mu}(\chi) d\chi = \frac{e^{-\chi_0/2} \chi_0^{\kappa-1}}{\kappa + \mu - \frac{1}{2}} M_{\kappa-1,\mu}(\chi_0) \quad (41)$$

and

$$\int_{\chi_0}^{\infty} \chi^{\kappa-2} e^{-\chi/2} W_{\kappa,\mu}(\chi) d\chi = e^{-\chi_0/2} \chi_0^{\kappa-1} W_{\kappa-1,\mu}(\chi_0). \quad (42)$$

Combining relations yields

$$K_G(\chi_0) = \frac{e^{-\chi_0/2} \chi_0^{\kappa-1}}{\kappa + \mu - \frac{1}{2}} W_{\kappa,\mu}(\chi_0) M_{\kappa-1,\mu}(\chi_0) + e^{-\chi_0/2} \chi_0^{\kappa-1} M_{\kappa,\mu}(\chi_0) W_{\kappa-1,\mu}(\chi_0) \quad (43)$$

or, equivalently,

$$K_G(\chi_0) = \frac{\chi_0^\kappa e^{-\chi_0/2}}{\kappa + \mu - \frac{1}{2}} [M'_{\kappa-1,\mu}(\chi_0) W_{\kappa-1,\mu}(\chi_0) - W'_{\kappa-1,\mu}(\chi_0) M_{\kappa-1,\mu}(\chi_0)], \quad (44)$$

where we have used Eqs. (13.4.32) and (13.4.33) from Abramowitz and Stegun,<sup>13</sup> and primes denote differentiation with respect to  $\chi_0$ . The Wronskian on the right-hand side of (44) can be evaluated using (34) to obtain

$$K_G(\chi_0) = \frac{\chi_0^\kappa e^{-\chi_0/2}}{\kappa + \mu - \frac{1}{2}} \frac{\Gamma(1 + 2\mu)}{\Gamma(\mu - \kappa + \frac{3}{2})}, \quad (45)$$

which can be further simplified by applying (26), yielding

$$K_G(\chi_0) = \frac{\chi_0^\kappa e^{-\chi_0/2}}{\beta(\lambda_n - 3)} \frac{\Gamma(1 + 2\mu)}{\Gamma(\mu - \kappa + \frac{1}{2})}, \quad (46)$$

where the eigenvalues  $\lambda_n$  are given by (22). By using (46) to substitute for  $K_G(\chi_0)$  in (39) and simplifying, we find that the photon number density can be evaluated using the expansion

$$n_G(\tau) = \frac{3\dot{N}_0 e^{3\alpha\tau_0^2/2} \sqrt{2w}}{\pi r_0^2 c \sqrt{\alpha}} \sum_{n=0}^{\infty} \frac{n! g_n(\tau_0) g_n(\tau)}{\Gamma(n + \frac{1}{2})(\lambda_n - 3)}. \quad (47)$$

This expression allows the computation of the photon number density at any optical depth  $\tau$  inside the accretion column for given values of the photon injection rate  $\dot{N}_0$  and the source location  $\tau_0$ .

## B. Solution of the differential equation

The availability of the transport equation (13) provides us with an alternative means for computing the photon number density  $n_G$  by directly solving the differential equation

$$\frac{1}{3} \frac{d^2 n_G}{d\tau^2} + \alpha\tau \frac{dn_G}{d\tau} + \alpha n_G - \xi^2 \alpha^2 \tau^2 n_G = - \frac{\dot{N}_0 \delta(\tau - \tau_0)}{\pi r_0^2 c}, \quad (48)$$

which is derived by operating on (13) with  $(kT_e)^3 \int_0^\infty \chi^2 d\chi$ . The homogeneous version of (48) obtained when  $\tau \neq \tau_0$  admits the fundamental solutions

$$n_G(\tau) = \begin{cases} A e^{-\alpha(3+w)\tau^2/4} U(a, 1/2, \alpha w \tau^2/2), & \tau > \tau_0 \\ B e^{-\alpha(3+w)\tau^2/4} M(a, 1/2, \alpha w \tau^2/2), & \tau < \tau_0, \end{cases} \quad (49)$$

where  $A$  and  $B$  are constants and

$$a \equiv \frac{w-3}{4w}, \quad w \equiv (9 + 12\xi^2)^{1/2}. \quad (50)$$

The solutions in (49) are consistent with the physical boundary conditions at large and small  $\tau$  discussed in Sec. III A [see Eq. (18)]. The constants  $A$  and  $B$  are determined by requiring that  $n_G(\tau)$  be continuous at the source location  $\tau_0$  and that it satisfy the derivative jump condition

$$\lim_{\varepsilon \rightarrow 0} \left. \frac{dn_G}{d\tau} \right|_{\tau=\tau_0+\varepsilon} - \left. \frac{dn_G}{d\tau} \right|_{\tau=\tau_0-\varepsilon} = - \frac{3\dot{N}_0}{\pi r_0^2 c}, \quad (51)$$

obtained by integrating (48) with respect to  $\tau$  in a small region around  $\tau_0$ . Combining (49) and (51) with the continuity condition and the expression for the Wronskian given by (21), we find after some algebra that

$$A = \frac{3\dot{N}_0 \Gamma(a)}{\pi r_0^2 c \sqrt{2\pi\alpha w}} e^{\alpha(3-w)\tau_0^2/4} M\left(a, \frac{1}{2}, \frac{\alpha w \tau_0^2}{2}\right), \quad (52)$$

$$B = \frac{3\dot{N}_0 \Gamma(a)}{\pi r_0^2 c \sqrt{2\pi\alpha w}} e^{\alpha(3-w)\tau_0^2/4} U\left(a, \frac{1}{2}, \frac{\alpha w \tau_0^2}{2}\right), \quad (53)$$

and consequently the global solution for the photon number density is given by

$$n_G(\tau) = \frac{3\dot{N}_0 \Gamma(a)}{\pi r_0^2 c \sqrt{2\pi\alpha w}} e^{\alpha(3-w)\tau_0^2/4} e^{-\alpha(3+w)\tau^2/4} M\left(a, \frac{1}{2}, \frac{\alpha w \tau_{\min}^2}{2}\right) U\left(a, \frac{1}{2}, \frac{\alpha w \tau_{\max}^2}{2}\right), \quad (54)$$

where

$$\tau_{\min} \equiv \min(\tau, \tau_0), \quad \tau_{\max} \equiv \max(\tau, \tau_0). \quad (55)$$

### C. Summation identity for the Laguerre polynomials

Equations (47) and (54) provide two independent expressions that can each be used to calculate the photon number density  $n_G$ . This fact allows us to derive an interesting new summation identity involving the Laguerre polynomials  $L_n^{(-1/2)}(x)$  appearing in Eq. (23) for the spatial eigenfunctions  $g_n$ . By setting (47) and (54) equal and substituting for  $\lambda_n$  using (22), we obtain after some simplification

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(-1/2)}\left(\frac{\alpha w \tau_0^2}{2}\right) L_n^{(-1/2)}\left(\frac{\alpha w \tau^2}{2}\right)}{(n+a)\Gamma\left(n+\frac{1}{2}\right)} = \frac{\Gamma(a)}{\sqrt{\pi}} M\left(a, \frac{1}{2}, \frac{\alpha w \tau_{\min}^2}{2}\right) U\left(a, \frac{1}{2}, \frac{\alpha w \tau_{\max}^2}{2}\right). \quad (56)$$

Setting  $x = \alpha w \tau^2/2$  and  $x_0 = \alpha w \tau_0^2/2$  yields the more compact form

$$\sum_{n=0}^{\infty} \frac{n! L_n^{(-1/2)}(x_0) L_n^{(-1/2)}(x)}{(n+a)\Gamma\left(n+\frac{1}{2}\right)} = \frac{\Gamma(a)}{\sqrt{\pi}} M\left(a, \frac{1}{2}, x_{\min}\right) U\left(a, \frac{1}{2}, x_{\max}\right), \quad (57)$$

where

$$x_{\min} \equiv \min(x, x_0), \quad x_{\max} \equiv \max(x, x_0). \quad (58)$$

Equation (57) has not appeared in the previous literature, although Chatterjea,<sup>16</sup> Exton,<sup>17</sup> Srivastava,<sup>18</sup> and Manocha<sup>19</sup> have derived several related identities.

### V. CONCLUSION

In this article, we have applied the methods of classical analysis to derive the closed-form solution for the Green's function  $f_G$  given by (37), describing the bulk and thermal Comptonization of monochromatic seed photons scattered by hot, infalling electrons inside an x-ray pulsar accretion column. X-ray pulsars are rotating neutron stars, which are the most dense and compact solid objects known to exist in the universe. These enigmatic sources are characterized by super-strong magnetic, gravitational, and radiation fields, making them the most extreme physical "laboratories" that we can observe. It is therefore of great theoretical interest to obtain the best possible understanding of the spectral formation process in x-ray pulsars because the observed radiation provides us with the only available window into their physical nature. As demonstrated in Figs. 1 and 2, the Green's function is characterized by a power-law shape at moderate photon energies, with an exponential turnover at higher energies, in agreement with the spectra of many sources.<sup>4</sup> We conclude that bulk and thermal Comptonization in the shocked pulsar accretion column provides a natural explanation for the typical high-energy spectra produced by x-ray pulsars.

Our primary interest here is in exploring the mathematical properties of the Green's function. In particular, we established that by performing two independent calculations of the photon number density  $n_G$  associated with  $f_G$ , one can obtain an interesting new summation identity involving the Laguerre polynomials,  $L_n^{(-1/2)}(x)$ , expressed by (57). The derivation is based on the simultaneous calculation of  $n_G$  using either term-by-term integration of the Green's function expansion (37), which yields (47), or instead via the direct solution of the differential equation for  $n_G$  given by (48), which leads to (54). The new identity for the Laguerre polynomials obtained here is related to various similar expressions presented by Chatterjea,<sup>16</sup> Exton,<sup>17</sup> Srivastava,<sup>18</sup> Manocha,<sup>19</sup> Varma,<sup>20</sup> and Gradshteyn and Ryzhik,<sup>14</sup> although our results are distinct from theirs. We expect that our new expression may be of potential benefit in various problems of mathematical physics.

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