447 Proj.1

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#1 *Solve Algebraically*

$$(x - A_1)^2 + (y - B_1)^2 + (z - C_1)^2 = [c(t_1 - d)]^2 \quad (1)$$
$$(x - A_2)^2 + (y - B_2)^2 + (z - C_2)^2 = [c(t_2 - d)]^2 \quad (2)$$
$$(x - A_3)^3 + (y - B_3)^3 + (z - C_3)^3 = [c(t_3 - d)]^3 \quad (3)$$
$$(x - A_4)^4 + (y - B_4)^4 + (z - C_4)^4 = [c(t_4 - d)]^4 \quad (4)$$

Expand the squares

$$x^2 - 2xA_1 + A_1^2 + y^2 - 2yB_1 + B_1^2 + z^2 - 2zC_1 + C_1^2 = c^2(d^2 - 2dt_1 + t_1^2)$$
$$x^2 - 2xA_2 + A_2^2 + y^2 - 2yB_2 + B_2^2 + z^2 - 2zC_2 + C_2^2 = c^2(d^2 - 2dt_2 + t_2^2)$$
$$x^2 - 2xA_3 + A_3^2 + y^2 - 2yB_3 + B_3^2 + z^2 - 2zC_3 + C_3^2 = c^2(d^2 - 2dt_3 + t_3^2)$$
$$x^2 - 2xA_4 + A_4^2 + y^2 - 2yB_4 + B_4^2 + z^2 - 2zC_4 + C_4^2 = c^2(d^2 - 2dt_4 + t_4^2)$$

Subtract equations 2 from equation 1 to get a linear equation

$$\begin{aligned} & x^2 - 2xA_1 + A_1^2 + y^2 - 2yB_1 + B_1^2 + z^2 - 2zC_1 + C_1^2 = c^2(d^2 - 2dt_1 + t_1^2) \\ - \; & \underline{x^2 - 2xA_2 + A_2^2 + y^2 - 2yB_2 + B_2^2 + z^2 - 2zC_2 + C_2^2 = c^2(d^2 - 2dt_2 + t_2^2)} \\ = \; & -2xA_1 + 2xA_2 + A_1^2 - A_2^2 - 2yB_1 + 2yB_2 + B_1^2 - B_2^2 - 2zC_1 + 2zC_2 + C_1^2 - C_2^2 \\ & \qquad = c^2(-2dt_1 + 2dt_2 + t_1^2 - t_2^2) \end{aligned}$$

Rewrite combining terms.

$$2x(A_2 - A_1) + A_1^2 - A_2^2 + 2y(B_2 - B_1) + B_1^2 - B_2^2 + 2z(C_2 - C_1) + C_1^2 - C_2^2 = 2dc^2(t_2 - t_1) + t_1^2 - t_2^2 \quad (5)$$

Do the same for Equation 3 and 4:

$$2x(A_3 - A_1) + A_1^2 - A_3^2 + 2y(B_3 - B_1) + B_1^2 - B_3^2 + 2z(C_3 - C_1) + C_1^2 - C_3^2 = 2dc^2(t_3 - t_1) + t_1^2 - t_3^2 \quad (6)$$
$$2x(A_4 - A_1) + A_1^2 - A_4^2 + 2y(B_4 - B_1) + B_1^2 - B_4^2 + 2z(C_4 - C_1) + C_1^2 - C_4^2 = 2dc^2(t_4 - t_1) + t_1^2 - t_4^2 \quad (7)$$

Solve for d in Equation 5:

$$\frac{2x(A_2 - A_1) + A_1^2 - A_2^2 + 2y(B_2 - B_1) + B_1^2 - B_2^2 + 2z(C_2 - C_1) + C_1^2 - C_2^2 - t_1^2 + t_2^2}{2c^2(t_2 - t_1)} = d$$

Rewrite to for simplicity and to point out constants.

$$\frac{2x(A_2 - A_1) + 2y(B_2 - B_1) + 2z(C_2 - C_1) + A_1^2 - A_2^2 + B_1^2 - B_2^2 + C_1^2 - C_2^2 + t_2^2 - t_1^2}{2c^2(t_2 - t_1)} = d$$

Make this equation more simple by defining new constants:

$$K_1 = A_1^2 - A_2^2 + B_1^2 - B_2^2 + C_1^2 - C_2^2 + t_2^2 - t_1^2$$
$$K_2 = A_1^2 - A_3^2 + B_1^2 - B_3^2 + C_1^2 - C_3^2 + t_3^2 - t_1^2$$
$$K_3 = A_1^2 - A_4^2 + B_1^2 - B_4^2 + C_1^2 - C_4^2 + t_4^2 - t_1^2$$

$$\frac{2x(A_2 - A_1) + 2y(B_2 - B_1) + 2z(C_2 - C_1) + K_1}{2c^2(t_2 - t_1)} = d$$

Plug d into Equation into Equations 6 and 7.

$$\begin{aligned} & 2x(A_3 - A_1) + 2y(B_3 - B_1) + 2z(C_3 - C_1) + K_2 \\ = \; & 2\left(\frac{2x(A_2 - A_1) + 2y(B_2 - B_1) + 2z(C_2 - C_1) + K_1}{2c^2(t_2 - t_1)}\right) c^2(t_3 - t_1) \end{aligned} \tag{8}$$

$$\begin{aligned} & 2x(A_4 - A_1) + 2y(B_4 - B_1) + 2z(C_4 - C_1) + K_3 \\ = \; & 2\left(\frac{2x(A_2 - A_1) + 2y(B_2 - B_1) + 2z(C_2 - C_1) + K_1}{2c^2(t_2 - t_1)}\right) c^2(t_4 - t_1) \end{aligned} \tag{9}$$

Cancel out $2c^2$ and movie the t's over

$$\frac{2x(A_3 - A_1) + 2y(B_3 - B_1) + 2z(C_3 - C_1) + K_2}{t_3 - t_1} = \frac{2x(A_2 - A_1) + 2y(B_2 - B_1) + 2z(C_2 - C_1) + K_1}{t_2 - t_1}$$
$$\frac{2x(A_4 - A_1) + 2y(B_4 - B_1) + 2z(C_4 - C_1) + K_3}{t_4 - t_1} = \frac{2x(A_2 - A_1) + 2y(B_2 - B_1) + 2z(C_2 - C_1) + K_1}{t_2 - t_1}$$

Solve these equations for z:

$$\frac{(t_2 - t_1)\left(2x(A_3 - A_1) + 2y(B_3 - B_1) + K_2\right) - (t_3 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right)}{2[(t_3 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_3 - C_1)]} = z \tag{10}$$

$$\frac{(t_2 - t_1)\left(2x(A_4 - A_1) + 2y(B_4 - B_1) + K_3\right) - (t_4 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right)}{2[(t_4 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_4 - C_1)]} = z \tag{11}$$

Plug 10 into 11:

$$\begin{aligned} & \frac{(t_2 - t_1)\left(2x(A_3 - A_1) + 2y(B_3 - B_1) + K_2\right) - (t_3 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right)}{2[(t_3 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_3 - C_1)]} \\ = & \frac{(t_2 - t_1)\left(2x(A_4 - A_1) + 2y(B_4 - B_1) + K_3\right) - (t_4 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right)}{2[(t_4 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_4 - C_1)]} \end{aligned}$$

This is now an equation of just x and y. Solve this for y?:

$$\begin{aligned} & 2[(t_4 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_4 - C_1)] * \\ & (t_2 - t_1)\left(2x(A_3 - A_1) + 2y(B_3 - B_1) + K_2\right) - (t_3 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right) \\ & = 2[(t_3 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_3 - C_1)] * \\ & (t_2 - t_1)\left(2x(A_4 - A_1) + 2y(B_4 - B_1) + K_3\right) - (t_4 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right) \end{aligned} \tag{12}$$

Simplify with new constants:

$$Q_3 = (t_3 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_3 - C_1)$$
$$Q_4 = (t_4 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_4 - C_1)$$

Then:

$$\begin{aligned}
&2x[2Q_4(t_2 - t_1)(A_3 - A_1) - (t_3 - t_1)(A_2 - A_1)] + \\
&2y[2Q_4(t_2 - t_1)(B_3 - B_1) + (t_3 - t_1)(B_2 - B_1)] + \\
&\qquad K_2 2Q_4(t_2 - t_1) + K_1(t_3 - t_1)
\end{aligned}$$

$$\begin{aligned}
= &2x[2Q_3(t_2 - t_1)(A_4 - A_1) - (t_4 - t_1)(A_2 - A_1)] + \\
&2y[2Q_3(t_2 - t_1)(B_4 - B_1) + (t_4 - t_1)(B_2 - B_1)] + \\
&\qquad K_3 2Q_3(t_2 - t_1) + K_1(t_4 - t_1)
\end{aligned}$$

more constants:

$$X_1 = [2Q_4(t_2 - t_1)(A_3 - A_1) - (t_3 - t_1)(A_2 - A_1)]$$
$$X_2 = [2Q_3(t_2 - t_1)(A_4 - A_1) - (t_4 - t_1)(A_2 - A_1)]$$
$$Y_1 = [2Q_4(t_2 - t_1)(B_3 - B_1) + (t_3 - t_1)(B_2 - B_1)]$$
$$Y_2 = [2Q_3(t_2 - t_1)(B_4 - B_1) + (t_4 - t_1)(B_2 - B_1)]$$

$$2xX_1 + 2yY_1 + K_2 2Q_4(t_2 - t_1) + K_1(t_3 - t_1)$$

$$= 2xX_2 + 2yY_2 + K_3 2Q_3(t_2 - t_1) + K_1(t_4 - t_1)$$

$$y = \frac{2x(X_2 - X_1) + K_3 2Q_3(t_2 - t_1) + K_1(t_4 - t_1) - K_2 2Q_4(t_2 - t_1) - K_1(t_3 - t_1)}{(Y_1 - Y_2)} \quad (13)$$

$$y = \left(\frac{2(X_2 - X_1)}{(Y_1 - Y_2)}\right) x + \frac{K_3 2Q_3(t_2 - t_1) + K_1(t_4 - t_1) - K_2 2Q_4(t_2 - t_1) - K_1(t_3 - t_1)}{(Y_1 - Y_2)} \quad (14)$$

$$\alpha_1 = \frac{2(X_2 - X_1)}{(Y_1 - Y_2)}$$
$$\beta_1 = \frac{K_3 2Q_3(t_2 - t_1) + K_1(t_4 - t_1) - K_2 2Q_4(t_2 - t_1) - K_1(t_3 - t_1)}{(Y_1 - Y_2)}$$

then,

$$y = \alpha_1 x + \beta_1$$

Now simplify z(x,y) and plug this in:

$$z = \frac{(t_2 - t_1)\left(2x(A_4 - A_1) + 2y(B_4 - B_1) + K_2\right) - (t_4 - t_1)\left(2x(A_2 - A_1) + 2y(B_2 - B_1) + K_1\right)}{2[(t_4 - t_1)(C_2 - C_1) - (t_2 - t_1)(C_4 - C_1)]}$$

$$z = x\frac{2(t_2-t_1)(A_4-A_1)-2(t_4-t_1)(A_2-A_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + y\frac{2(t_2-t_1)(B_4-B_1)-2(t_4-t_1)(B_2-B_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + \frac{(t_2-t_1)K_2-(t_4-t_1)K_1}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]}$$

$$z = x\frac{2(t_2-t_1)(A_4-A_1)-2(t_4-t_1)(A_2-A_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + (\alpha_1 x+\beta_1)\frac{2(t_2-t_1)(B_4-B_1)-2(t_4-t_1)(B_2-B_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + \frac{(t_2-t_1)K_2-(t_4-t_1)K_1}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]}$$

$$z = x\left(\frac{2(t_2-t_1)(A_4-A_1)-2(t_4-t_1)(A_2-A_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + \alpha_1\frac{2(t_2-t_1)(B_4-B_1)-2(t_4-t_1)(B_2-B_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]}\right) + (\beta_1)\frac{2(t_2-t_1)(B_4-B_1)-2(t_4-t_1)(B_2-B_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + \frac{(t_2-t_1)K_2-(t_4-t_1)K_1}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]}$$

$$\alpha_2 = \frac{2(t_2-t_1)(A_4-A_1)-2(t_4-t_1)(A_2-A_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + \alpha_1\frac{2(t_2-t_1)(B_4-B_1)-2(t_4-t_1)(B_2-B_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]}$$

$$\beta_2 = \beta_1\frac{2(t_2-t_1)(B_4-B_1)-2(t_4-t_1)(B_2-B_1)}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]} + \frac{(t_2-t_1)K_2-(t_4-t_1)K_1}{2[(t_4-t_1)(C_2-C_1)-(t_2-t_1)(C_4-C_1)]}$$

$$z = \alpha_2 x + \beta_2$$

Now d:

$$d = \frac{2x(A_2-A_1)+2y(B_2-B_1)+2z(C_2-C_1)+K_1}{2c^2(t_2-t_1)}$$

$$d = \frac{2x(A_2-A_1)+2(\alpha_1 x+\beta_1)(B_2-B_1)+2(\alpha_2 x+\beta_2)(C_2-C_1)+K_1}{2c^2(t_2-t_1)}$$

$$d = x\frac{2(A_2-A_1)+2\alpha_1(B_2-B_1)+2\alpha_2(C_2-C_1)}{2c^2(t_2-t_1)} + \frac{2\beta_1(B_2-B_1)+2\beta_2(C_2-C_1)+K_1}{2c^2(t_2-t_1)}$$

$$\alpha_3 = \frac{2(A_2-A_1)+2\alpha_1(B_2-B_1)+2\alpha_2(C_2-C_1)}{2c^2(t_2-t_1)}$$

$$\beta_3 = \frac{2\beta_1(B_2-B_1)+2\beta_2(C_2-C_1)+K_1}{2c^2(t_2-t_1)}$$

$$d = \alpha_3 x + \beta_3$$

$$
\begin{aligned}
x &= x \\
y &= \alpha_1 x + \beta_1 \\
z &= \alpha_2 x + \beta_2 \\
d &= \alpha_3 x + \beta_3
\end{aligned}
$$

$$(x - A_1)^2 + (y - B_1)^2 + (z - C_1)^2 = [c(t_1 - d)]^2$$

$$(x - A_1)^2 + ((\alpha_1 x + \beta_1) - B_1)^2 + ((\alpha_2 x + \beta_2) - C_1)^2 = [c(t_1 - (\alpha_3 x + \beta_3))]^2$$

$$(x - A_1)^2 + (\alpha_1 x + \beta_1 - B_1)^2 + (\alpha_2 x + \beta_2 - C_1)^2 = [c(t_1 - \alpha_3 x + \beta_3)]^2$$

$$(x - A_1)^2 + (\alpha_1 x + \beta_1 - B_1)^2 + (\alpha_2 x + \beta_2 - C_1)^2 = [c(-\alpha_3 x + t_1 + \beta_3)]^2$$

$$
\begin{aligned}
x^2 - 2A_1 x + A_1^2 + \alpha_1^2 x^2 + 2(\beta_1 - B_1)x + (\beta_1 - B_1)^2 + \alpha_2^2 x^2 + 2(\beta_2 - C_1)x + (\beta_2 - C_1)^2 = \\
c^2[\alpha_3^2 x^2 - 2\alpha_3 x(t_1 + \beta_3) + (t_1 + \beta_3)^2]
\end{aligned}
$$

$$
\begin{aligned}
\left(1 + \alpha_1^2 + \alpha_2^2 - c^2\alpha_3^2\right) x^2 + \left(-2A_1 + 2(\beta_1 - B_1) + 2(\beta_2 - C_1) + 2\alpha_3 c^2 (t_1 + \beta_3)\right) x \\
+ \left(A_1^2 + (\beta_1 - B_1)^2 + (\beta_2 - C_1)^2 - c^2(t_1 + \beta_3)^2\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= 1 + \alpha_1^2 + \alpha_2^2 - c^2\alpha_3^2 \\
\Psi &= -2A_1 + 2(\beta_1 - B_1) + 2(\beta_2 - C_1) + 2\alpha_3 c^2 (t_1 + \beta_3) \\
\Theta &= A_1^2 + (\beta_1 - B_1)^2 + (\beta_2 - C_1)^2 - c^2(t_1 + \beta_3)^2
\end{aligned}
$$

then the simplified equation for x is:

$$\gamma x^2 + \Psi x + \Theta = 0$$

solve for x via the quadratic equation:

$$x = \frac{-\Psi \pm \sqrt{\Psi^2 - 4\gamma\Theta}}{2\gamma}$$

$$
\begin{aligned}
y &= \alpha_1 x + \beta_1 \\
z &= \alpha_2 x + \beta_2 \\
d &= \alpha_3 x + \beta_3
\end{aligned}
$$