Chapter 12

Developing Persistent and Flexible Problem Solvers with a Growth Mindset

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I just kept going like a snow plow stuck in the road. I didn't wait for the spring to come. I kept going.

—Griffin, fifth grade

THIS CHAPTER describes research from a group of mathematics teachers and a university researcher who collaborated through Lesson Study, a form of professional development that focuses on research lessons. At the beginning of our Lesson Study, we developed our research aim and overarching goal: to develop persistent and flexible problem solvers. Through collaborative planning and designing of problem-driven lessons throughout the academic year, the teacher–researchers developed classroom communities of inquiry and specific strategies that promoted students' persistence and flexible thinking in problem solving. Teachers observed marked progression in students' productive dispositions toward mathematics throughout the school year. Students developed a "growth mindset" (Dweck 2006) focused on effort and persistence in learning mathematics. The Lesson Study model of professional development also influenced teachers' instructional practices in developing persistent and flexible problem solvers.

Research Background

According to Dweck (2006), a Stanford University psychologist with three decades of research on achievement and success, two mindsets about learning exist: a *growth mindset* and a *fixed mindset*. A growth mindset holds that your basic qualities are things that you can cultivate through your efforts, whereas a fixed mindset holds that your qualities are "carved in stone" (i.e., your intelligence is something you can't change very much). When facing challenging problems, children who believe that effort drives intelligence tend to do better than children who believe that intelligence is a fixed quality that they cannot change. According to research on competence and motivation (Elliot and Dweck 2005; Weiner 2005), students can attribute their successes and failures to ability (e.g., "I'm just [good/bad] at mathematics"), effort (e.g., "I [worked/did not work] hard enough"), luck, or powerful people (e.g., "the teacher [loves/hates] me"). A student with a fixed mindset avoids challenges, gives up easily, sees effort as fruitless or worse, ignores useful negative feedback, and feels threatened by the success of others. Meanwhile, a student with a growth mindset embraces challenge, persists despite setbacks, sees effort as the path to mastery, learns from mistakes and criticisms, and finds lessons and inspiration in the success of others. People with a growth mindset believe that they can develop their abilities through hard work, persistence, and dedication; brains and talents are merely a starting base (Dweck 2006).

Research also suggests that good problem solvers are qualitatively different from poor problem solvers (National Research Council 2004; Schoenfeld 2007). Good problem solvers are flexible and resourceful. They have many ways to think about problems: "alternative approaches if they get stuck, ways of making progress when they hit roadblocks, of being efficient with (and making use of) what they know. They also have a certain kind of mathematical disposition—a willingness to pit themselves against difficult mathematical challenges under the assumption that they will be able to make progress on them, and the tenacity to keep at the task when others have given up" (Schoenfeld 2007, p. 60). Problem solvers experience a range of emotions associated with different stages in the solution process. Mathematicians who successfully solve problems say that having done so contributes to an appreciation for the "power and beauty of mathematics" (National Council of Teachers of Mathematics [NCTM] 1989, p. 77) and the "joy of banging your head against a mathematical wall, and then discovering that there might be ways of either going around or over that wall" (Olkin and Schoenfeld 1994, p. 43). Good problem solvers also are more willing to engage with a task for a length of time, so that the task ceases to be a "puzzle" and becomes a problem (Schoenfeld 2007).

Creating opportunities for success in mathematics is important, but offering students a series of easy tasks can lead to a false sense of self-efficacy and can limit access to challenging mathematics. Ironically, research indicates that students need to experience periodic challenge and even momentary failure to develop higher levels of self-efficacy and task persistence (Middleton and Spanias 1999). Achieving a balance between opportunities for success and opportunities to solve problems that require considerable individual or group effort requires teachers to design curricular materials and instructional practices carefully (Woodward 1999). In the following sections, we describe our efforts to develop persistent and flexible problem solvers by looking deeply at instructional practices and mathematical tasks.

Context of Our Classroom Design Research through Lesson Study

The design research process enabled the teachers and researcher to document the Lesson Study, to develop sequences of instructional strategies and tools, and to analyze student learning and the means by which that learning was supported. During the academic year, the classroom teachers and the researcher collaboratively planned four problem-driven lessons focused on developing persistent and flexible problem solvers. Before each lesson, the teachers and researcher met to discuss the lesson objective, important mathematics, the design of the mathematical task, and the expected flow of the lesson. Also, the group spent considerable time discussing students' anticipated responses and common misconceptions in an effort to develop conceptual supports that would scaffold the tasks for diverse learners. We collected data to document the design process through the pre- and postlesson discussions and through artifacts such as lesson plans, task sheets, and conceptual supports. When teaching the lesson, each teacher had an observer who recorded notes on students' engagement, responses, and questions that elicited rich mathematical discourse. During postlesson meetings, the teachers and researcher met to discuss the outcome of the lesson, evidence of student learning, and how the lesson design contributed to developing persistent and flexible problem solvers.

Through each cycle, the teachers and researcher designed different pedagogical tools to support our research aim. One such tool was a set of prompts that encouraged learners to reflect orally or in writing about their use of communication, flexible thinking, and persistence as they approached mathematical tasks (fig. 12.1). These prompts emerged from collaborative discussions among the Lesson Study teachers about the research goals. Before the school year began, we discussed what characterized persistent and flexible problem solvers who communicated clearly and respectfully. We designed the prompts to share our vision with our students and help them internalize these traits. The chart was organized so that teachers could have students respond to a particular column or row or give students choices about which prompts to use. This reflection not only gave the students insight into their problem-solving process but also gave the teacher–researchers a window into students' motivation, persistence, and flexibility in thinking.

Clear Communication	Respectful Communication	Flexible Thinking	Persistence
What math words could help us share our thinking about this problem? Choose 2 and explain what they mean in your own words.	Did someone else solve the problem in a way you had not thought of? Explain what you learned by listening to a classmate.	What other problems or math topics does this remind you of? Explain your connection.	What did you do if you got stuck or felt frustrated?
What could you use *besides words* to show how to solve the problem? Explain how this representation would help someone understand.	Did you ask for help or offer to help a classmate? Explain how working together helped solve the problem.	Briefly describe at least 2 ways to solve the problem. Which is easier for you?	What helped you try your best? *or* What do you need to change so that you can try your best next time?
If you needed to make your work easier for someone else to understand, what would you change?	What helped you share and listen respectfully when we discussed the problem? *or* What do you need to change so that you can share and listen respectfully next time?	What strategies did you use that you think will be helpful again for future problems?	Do you feel more or less confident about math after trying this problem? Explain why.

Fig. 12.1. Reflection prompts to encourage persistence and flexibility in problem solving

In the following sections, we share classroom accounts that describe students' development of persistent and flexible thinking. The participating teachers taught fourth- and fifth-grade students of diverse ability levels, including students with individualized education plans, students of average ability, and gifted-and-talented students.

Lessons That Elicited Students' Persistence and Flexibility in Problem Solving

Through our research lessons, we discovered essential design features that elicited students' persistence and motivation for solving problems. One design element involved presenting rigorous mathematical tasks that challenged students' thinking and required justification and reasoning. In a lesson called "Possible Solution Set," we posed a task where students found all the possible ways to have a three-digit house number whose digits had a sum of 12. In addition to discovery of number combinations, the underlying problem-solving focus was to encourage students to use a table or an organized list to keep track of number combinations. For an extension, we asked students to find all the three-digit house numbers whose digits had a product of 24. Once students discovered a mathematical strategy, giving them related problems or classes of problems was important so that they could transfer their strategy development to other problem types.

Through this lesson, students developed persistence as they worked and discovered multiple answers that satisfied the criteria (fig. 12.2). Several students began by listing random combinations of numbers, but through collective inquiry, those students soon realized that their method was not efficient for keeping track of all the number combinations. In fact, once they noticed their classmates' using a variety of strategies, such as a table, a tree diagram, or an organized list, these students developed an appreciation for different problem-solving strategies. As the teachers circulated through the room asking for solutions, students could see that several ways to solve the problem existed. Certain students were also better at verbalizing their strategy and thinking processes, whereas others created excellent tables and organized lists. We noticed that students were motivated to find all the possible combinations. When we asked, "How do you know that you have all the number combinations?" students had to prove their thinking. This emphasis on justification reinforced the importance of persisting until students were certain that they had solved the problem. One student commented, "I feel more confident after doing this problem because I really get stuck on knowing when to do an organized list, but now I know when to make one."

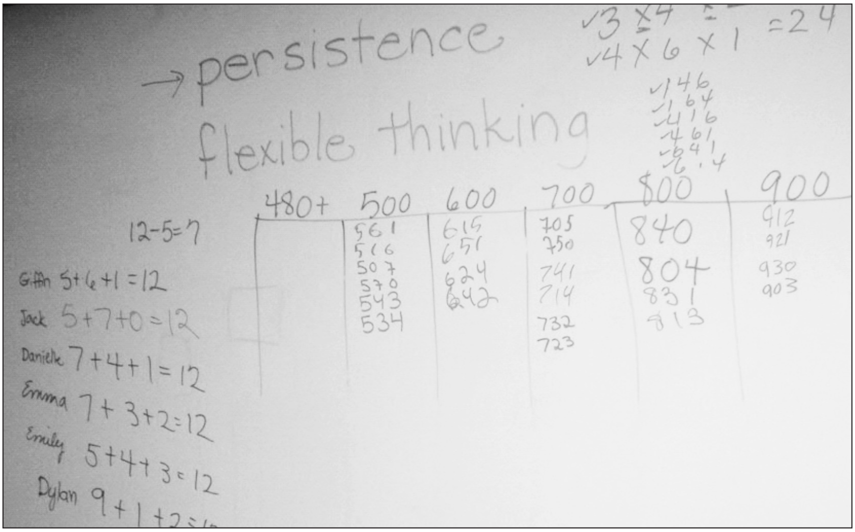

Fig. 12.2. Finding all the combinations by using a variety of strategies

We created another design element that used tiered tasks that allowed for multiple entry points, engagement, and differentiation. We used tiering to adapt an NCTM Illuminations lesson about the properties of triangles (see illuminations .nctm.org/LessonDetail.aspx?ID=U191). For tiered lesson one, “What Can You Build with Triangles?” we explored ways of building different basic shapes from triangles to investigate the properties of a triangle and the relationships among other basic geometric shapes. For tiered lesson two, “What’s Important about Triangles?” students explored relationships among the side lengths of a triangle to determine whether they could construct triangles from these lengths. For an extension and a challenge, in tiered lesson three, “How Many Triangles Can You Construct?” students identified patterns in Sierpinski’s triangle and built a foundation for understanding fractals. The tiered lessons enabled students to work with worthwhile tasks that were neither too easy nor too difficult. Our rationale was based on our understanding that tasks that are too easy may bore students, whereas tasks that are too difficult may frustrate them.

In a fourth-grade classroom, the research lesson focused on tiered lesson two, “What’s Important about Triangles?” which explored relationships among the side lengths of a triangle. Students determined whether they could construct triangles from given lengths. This investigation allowed students to become mathematicians who constructed and tested conjectures by first predicting whether the measurements of the side lengths would make a triangle. Then they constructed the triangle to test their predictions (fig. 12.3). Students could look for commonalities among the measurements that worked compared to the

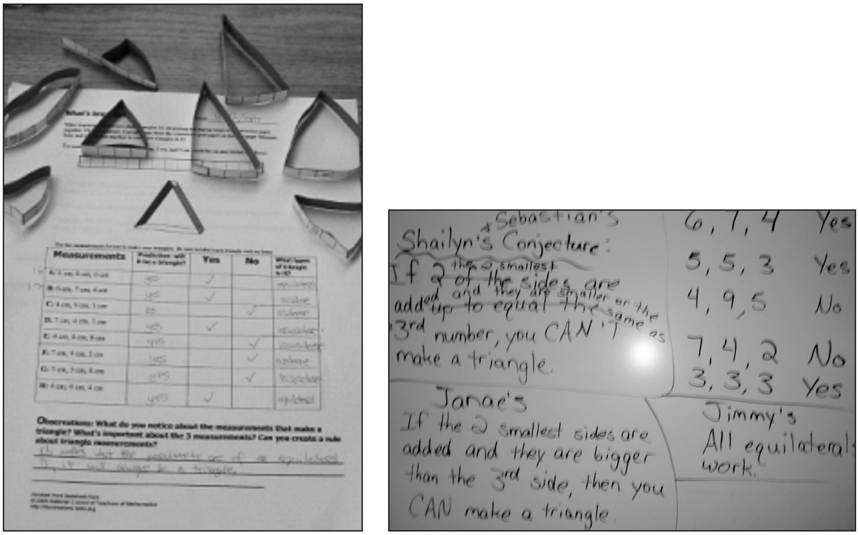

Figure 12.3. Making and testing conjectures about side lengths of a triangle

lengths that did not yield a triangle. This inquiry approach ignited students' curiosity to understand why they obtained their results. As students collectively immersed themselves in this mathematics inquiry, several started to articulate conjectures, and the teacher had students share their conjectures on the board to develop collective knowledge. For example, Shailyn and Sebastian stated that "If two of the [smaller] sides are added and are smaller than the third, you cannot make a triangle." Building on this conjecture, Janae stated, "If the two smaller sides are added and they are bigger than the third, then you can make a triangle." After giving a few examples of measurements and noting "yes" or "no," Jimmy added that "All equilaterals work." The generalizations resulting from individual and shared conjectures mirrored the progression of important mathematical ideas that mathematicians have made throughout history. The shared investigation and community of math inquiry was motivating for students as they discovered properties of triangles.

In a fifth-grade class of advanced students, one teacher focused on tiered lessons two and three. Through the two different levels, all students explored the patterns that occur when triangular pyramids iterate. Through their assigned group discovery activity, students working on lesson two, "What's Important about Triangles?" found exactly how various kinds of triangles are formed and a rule to generate all triangles. Those working on lesson three, "How Many Triangles Can You Construct?" (Kelley 1999), discovered a pattern in the formation of the Sierpinski triangle to determine a rule for its structural iteration. Students drew several iterations of the Sierpinski triangle and then used a computer program to see dynamically how the triangle continues to generate the repetitive pattern. The seemingly buried commonality between both exercises was the need to use emerging patterns to unlock the mysteries behind geometry's strongest and most-studied shape. The rule that the lengths of the two smaller sides of a triangle must have a sum greater than the length of the largest side prodded the others to conclude that the Sierpinski iteration worked in accordance with ascending exponents at every new generational level.

The true test of persistence was building the Sierpinski triangle (fig. 12.4). The specific model that the students built required the prior construction of 256 triangular pyramids from a template. This activity was representational for this class of sixteen students because each stage required precisely sixteen triangular pyramids. Building the model as a class community helped foster the idea

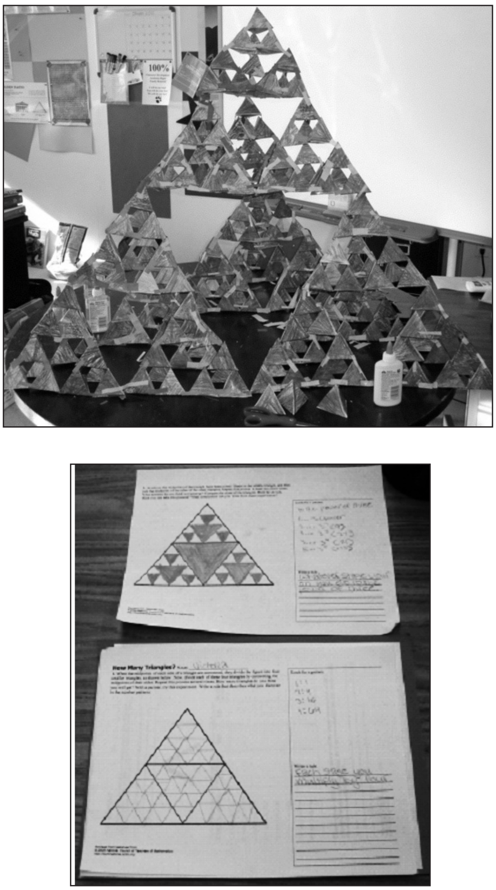

Figure 12.4. Students' development of motivation and persistence through building triangles

that the large-scale triangle was a shared project as well as a mathematical challenge. Mutual respect and collective effort motivated students to complete such a seemingly difficult exercise. The teacher set the tone and offered the nurturing assurance for the community of inquiry to mature. Closing with reflecting on the problem-solving prompts permitted students to share their thinking, but it also gave the teacher valuable clues for how to best facilitate follow-up and extension lessons.

Through students' responses from their reflection logs, we observed their development of a productive disposition toward mathematics. When asked what they did if they got stuck or felt frustrated, students responded, "We asked for help, and we tried to look at things in a different way" and "I asked for help and offered help. I think working in groups is easier because two people can do more than one." When asked what they could use other than words to show how to solve the problem, students responded, "I believe diagrams trigger people's minds so they understand and visualize the problem better" and "If you find the rule and the pattern, you can better see how a problem works." The research team collected and categorized students' comments according to our research aim of evidence of persistence and flexibility in thinking (table 12.1).

Table 12.1
Evidence of students' development of persistence and flexibility in problem solving

Persistent problem solver	**Flexible problem solver**
"I feel much more confident in math, because this problem showed me different problems, strategies, and persistence. The persistence helped me because I put my mind to it." —Alex	"Using the formula to predict if the sides would make a triangle helped me a lot. It is a good strategy for the future." —Sam
"What helped me try my best was when Michael didn't understand something and made me know I had to try harder to explain it better." —Liam	"This problem reminded me of the shapes that we made with the straws and twist ties." —Danielle
"I felt more confident about math after trying this problem because I proved to myself that if I am persistent, then I can accomplish things in math that I set my mind to." —Lauren	"I like trial and error because you start with a big guess and narrow it down." —Griffin
	"A strategy that will help me in the future would be the rule that we found out today." —Emma

Table 12.1—*Continued*

Persistent problem solver	**Flexible problem solver**
"I feel a lot more confident about math after those problems because I know what it feels like to be persistent, and I like it! So I'm going to keep going for that feeling." —Emily	"A strategy that I would use again after this problem would be guessing. I think this because many problems involve estimating. I'm guessing more and doing it better." —Alex
"What helped me to do my best was the hard questions. The more confusing it was, the more I liked it to try my best." —Liam	"I think that doing the number sentences will help me in the future." —Molly
"I just kept going like a snow plow stuck in the road. I didn't wait for the spring to come. I kept going." —Griffin	"This reminds me of when we tried to find perimeter in the beginning of the year. When we first did this, we could barely multiply and divide." —Liam

Core Instructional Practices for Developing Persistent and Flexible Problem Solvers

As we used research lessons to analyze our teaching, we identified four core instructional practices that were instrumental in establishing classroom norms that had a positive effect on students' disposition toward mathematics (fig. 12.5). These four core instructional practices complemented each other and converged to build a safe and stimulating classroom environment that nurtured a community of mathematics inquiry.

First and most important was establishing a community of mathematics inquiry that embraced challenges. Students were more motivated when they felt that they were part of a vibrant and rigorous learning community. Classroom norms that encouraged persistent and flexible problem solvers took time to build. These norms included attributing value to struggle, respecting diverse strategies, communicating mathematical ideas, seeking and giving help when students got stuck, evaluating different strategies for their advantages and disadvantages, self-correcting, being flexible enough to change one's ideas to garner further mathematical insight, and placing value on being a good problem solver.

Second, the emphasis on respectful and clear mathematics communication allowed students to engage in rich, in-depth mathematics argumentation with

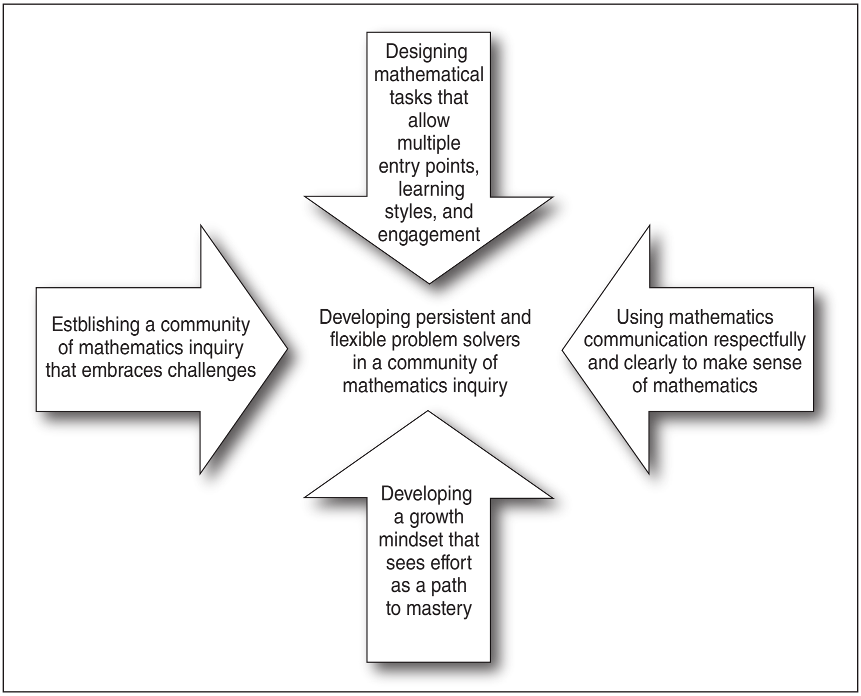

Figure 12.5. Four practices to establish classroom norms to develop persistent and flexible problem solvers

reasoning and proof, just as mathematicians do. Students learned to express their ideas by using multiple tools such as drawings, models, words, and numbers to convince one another, elaborate on each other's ideas, and translate among representations. Mathematical communication included in-depth discussions when students shared and compared strategies, verbalizing the metacognitive process that explained why one pursued a worthwhile strategy while abandoning inefficient ones, determining what questions to ask when one got stuck, and permitting students to defend their answer to build confidence in their reasoning. At times, giving students the space and time to respectfully argue mathematical ideas and convince one another gave reluctant learners and ones who needed more scaffolding the opportunity to make sense of the mathematics. Viewing wrong answers as partially correct and reflecting on finding the part that is wrong and understanding why it is wrong can be a powerful aid to understanding and promoting metacognitive competencies.

Third, designing meaningful mathematics tasks that accommodated multiple entries, learning styles, and engagement was the instructional backdrop for the described activities to happen in the mathematics classroom. As teachers, we learned to use questions to guide and coach our students and to know when to intervene and when to let students grapple with the problem. We modeled the use of Polya's (1957) steps to problem solving by verbalizing the self-monitoring process that is vital during problem solving. We discussed persistence and flexibility and how those ideas apply to problem solving in mathematics. We illustrated what persistent students and their work looked like and then reinforced these behaviors when students demonstrated those dispositions. We likewise modeled and acknowledged examples of flexible thinking. Research calls this modeling "cognitive modeling," in which one verbalizes one's metacognitive processes when solving problems (National Research Council 2004, p. 241). Modeling, acknowledging, and highlighting student behaviors that we wanted to see was important. Those behaviors were visible not only in action but also in students' work.

Finally, developing a growth mindset that views effort as a path to mastery was integral for not only the students but also the classroom teachers. To make this an explicit expectation, the researcher developed an assessment rubric to evaluate students' progress in developing mathematical proficiency through the school year on the basis of demonstrated effort in these areas (fig. 12.6). Using this rubric, teachers noted students' development of productive dispositions toward mathematics along with the other important strands of mathematics proficiency, such as conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. Giving frequent feedback helped students recognize their progress in learning and gave them chances to do even better, which was motivating.

The teachers also supplied an exit pass so that students could self-assess their effort and reflect on their learning (fig. 12.7).

Helping students learn to appreciate multiple approaches to problem solving gave students an appreciation for the flexibility in thinking required to solve complex problems. Modeling reflection on the problem and discussions of various means and methods to solve the problem (manipulatives, charts, diagrams on centimeter paper, looking for patterns, and finally searching for a rule) created a foundation for the students' "problem attack." We recorded these strategies and structures on the board, as well as the term *persistence* and what it looks like in student behaviors. Students could refer to these visual clues for plans of attack if they thought their plan might need to be rejected and another plan substituted. This repertoire of strategies gave students alternative pathways to find solutions (kinesthetic, visual, and auditory). In later discussions, most students felt that diagrams and charts helped them find patterns and that patterns led to full solutions. They supported the theory that building the problem with manipulatives, and then recording their work as diagrams, led to confidence in their solutions.

Assessing Mathematical Proficiency
Activity____________________________ Date: ____________

Student Name	Effort
Productive Disposition	
Tackles difficult tasks	
Perseveres	
Shows confidence in own ability	
Collaborates/shares ideas respectfully	
Strategic Competence	
Uses strategies flexibly	
Formulates and carries out a plan	
Creates similar problems	
Uses appropriate strategies	
Communication of Reasoning and Proof	
Justifies responses logically	
Reflects on and explains procedures	
Explains concepts clearly using the language of mathematics	
Conceptual Understanding	
Understands the problems or tasks	
Makes connections to similar problems	
Uses models and multiple representations flexibly	
Procedural Understanding	
Uses algorithm properly	
Computes accurately	
Scoring Rubric 3: Secure (Student demonstrates effort consistently.) 2: Developing (Student demonstrates effort most of the time.) 1: Beginning (Student demonstrates effort some of the time.) 0: Not demonstrated (Effort not demonstrated.)	

Fig. 12.6. Assessing mathematical proficiency

EXIT PASS

Today, I put effort in my math thinking and math work. ☺ 😐 ☹
I learned that . . .
I still need help on . . .

Fig. 12.7. Exit pass

Conclusion

Before teachers can create classroom norms that foster persistent and flexible problem solvers, the teacher and students must agree on some beliefs and behaviors about teaching and learning. All members of the mathematics community need to commit to developing a growth mindset (table 12.2). One of the biggest commitments that teachers must make when building a community of mathematics inquiry is to give students the time and space to grapple with meaningful mathematics investigations. Students will at first feel frustrated and not know where to begin if they are accustomed to teachers' spoon-feeding them through problems. However, teachers need to become comfortable with this feeling and realize that this is the first stage of problem solving. As students persevere through problems, they will gain an appreciation for solving problems and learn how to make sense of mathematics.

Table 12.2
Commitment to a growth mindset and building a community of mathematics inquiry

Students' commitment to a growth mindset	**Teachers' commitment to growth mindsets**
I will persevere through problems and be productive. "Stick with it!" attitude.	I will give students time and space to grapple with problems and validate their efforts and persistence. I will distribute practice over time and give challenging problems.
I will make sense of mathematics through my written work and my participation in discussion as we work together.	I will choose meaningful and productive tasks and guide students' effort in learning important mathematics and building collective knowledge.
I will consider multiple strategies to learn the most efficient ways to approach a problem.	I will anticipate students' responses and elicit, support, and extend students' thinking.

Through the Lesson Study professional development, teachers also grappled with mathematics problems and learned the value of persistence and flexibility in thinking when solving problems. This experience was essential for teachers to understand the importance of having a productive disposition toward mathematics. Taking this experience and translating it into classroom practice, teachers recognized that changes in beliefs and behaviors needed to start with them. Through Lesson Study, we continued to focus on our overarching goal, which helped us be intentional in developing classroom norms that would facilitate this change. Enabling children to develop persistence and flexibility boosted their self-confidence and helped them embrace the importance of mathematical thinking.

REFERENCES

Dweck, Carol. *Mindset: The New Psychology of Success*. New York: Ballantine Books, 2006.

Elliot, Andrew, and Carol Dweck, eds. *Handbook of Competence and Motivation*. New York: Guilford Press, 2005.

Kelley, Paul. "Build a Sierpinski Pyramid." *Mathematics Teacher* 92 (May 1999): 384–86.

Middleton, James, and Photini Spanias. "Motivation for Achievement in Mathematics: Findings, Generalizations, and Criticisms of the Research." *Journal for Research in Mathematics Education* 30 (January 1999): 65–88.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: NCTM, 1989.

National Research Council. *How Students Learn Mathematics in the Classroom.* Washington, D.C.: National Academies Press, 2004.

Olkin, Ingram, and Alan Schoenfeld. "A Discussion of Bruce Reznick's Chapter." In *Mathematical Thinking and Problem Solving,* edited by Alan Schoenfeld, pp. 39–51. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1994.

Polya, George. *How to Solve It*. Princeton, N.J.: Princeton University Press, 1957.

Schoenfeld, Alan. *Assessing Mathematical Proficiency.* New York: Cambridge University Press, 2007.

Weiner, Bernard. "Motivation from an Attributional Perspective and the Social Psychology." In *Handbook of Competence and Motivation*, edited by Andrew Elliot and Carol Dweck, pp. 73–84. New York: Guilford Press, 2005.

Woodward, John. *Self-Concept, Self-Esteem, and Attributional Issues in Secondary Math Classes.* Presented at the Pacific Coast Research Conference, San Diego, 1999.

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