Tying It All Together

Classroom Practices That Promote Mathematical Proficiency for All Students

What classroom practices will promote mathematical proficiency for all my students? As a fifth-grade mathematics teacher, I taught four classes: an accelerated third-grade group using what we called compacted math (third- and fourth-grade curriculum condensed into one academic year of study); an inclusion classroom that the special education teacher and I co-taught; and two classes composed of students with a wide range of abilities and learning styles. Some of my students loved the challenge I presented in mathematics, while others already showed signs of math avoidance and even math phobia. These students did not feel confident in their ability to reason through problems or apply strategies and even doubted their computational skills. As a result of their school experience, they saw mathematics as confusing and irrelevant to their lives; most disheartening, they associated it with failure.

In my search for answers, I came across a book by the National Research Council called *Adding It Up: Helping Children Learn Mathematics* (2001). *Adding It Up*, a reading requirement for one of my graduate courses, explores how students in grades pre-K–8 learn mathematics. The editors discuss how teaching, curricula, and teacher education should be changed to improve mathematics learning. Most important, for me, was the illustration of the five interdependent components of mathematical proficiency and the description of how students develop this proficiency (see **fig. 1**). The image made it so clear to me that these five intertwined strands—conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition—were the critical strands for developing mathematically proficient students. My own students

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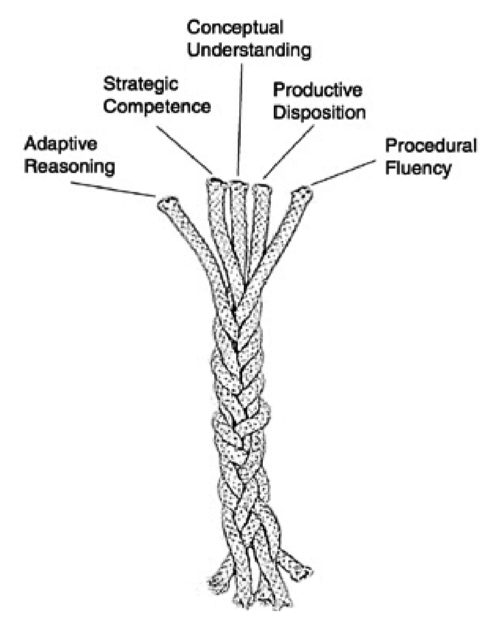
Figure 1

Five strands of mathematics proficiency



Source: National Research Council (2001)

who stood out as "math stars" were the ones who exhibited all these traits. They were the ones who understood the concept, learned the procedures with meaning, solved problems using efficient strategies, defended and justified their reasoning, and found mathematical investigations challenging and engaging. Their mathematical understanding, like the intertwined rope in the illustration, was connected and strong. Through my graduate course, my classmates and I read and relished every chapter of *Adding It Up* and discussed in length how this research translated into classroom practices.

As the new school year began, my goal in mathematics was to integrate the five strands into my daily teaching practices and adopt this conceptual framework. In the process, I developed several learning activities that created classroom structures for promoting mathematical proficiency.

The Five Strands of Mathematical Proficiency

Before I discuss the classroom practices that promoted mathematics proficiency, it is important to establish a common definition for mathematics proficiency. I will use the definitions set forth in *Adding It Up:*

1. *Conceptual understanding* refers to the "integrated and functional grasp of mathematical ideas," which "enables them [students] to learn new ideas by connecting those ideas to what they already know." It is the "comprehension of mathematical concepts, operations, and relations." A few of the benefits of building conceptual understanding are that it supports retention and prevents common errors.
2. *Procedural fluency* is defined as "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately."
3. *Strategic competence* is the "ability to formulate, represent, and solve mathematical problems."
4. *Adaptive reasoning* is the "capacity for logical thought, reflection, explanation, and justification."
5. *Productive disposition* is the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (NRC 2001, p. 116)

These five strands are interconnected and must all work together for students to be mathematically proficient. In many traditional classrooms, procedural fluency plays a dominant role in defining mathematical proficiency. However, on the basis of this broader definition, four other critical strands need to be developed for students to be mathematically proficient. Teaching practices must also reflect these interrelated components. I will share some classroom practices that promoted these five strands of proficiency.

Classroom Practices That Promote the Five Strands

"Modeling Math Meaningfully"

As I began my planning for the school year, one of the first questions I grappled with was, what is the best way to teach for and assess conceptual understanding? As I searched for ways to approach this goal, an idea came from my readings about Lesh et al.'s translation model (2003), which states that students make more meaningful connections when representing a mathematical idea in multiple modes: manipulatives, pictures, real-life contexts, verbal symbols, and written symbols. This model emphasizes that translations within and among various modes of representation make concepts meaningful for students (see **fig. 2**).

To give students opportunities to work with multiple modes, I created an activity called “Modeling Math Meaningfully” (see **fig. 3**), in which students represented their mathematical understanding in the five modes. I designed four quadrants where students did the following:

- Wrote the problem with numbers (symbols)
- Drew a picture of the problem
- Wrote a real-life story or situation describing when they would use this concept
- Explained how they solved the problem through the use of manipulatives

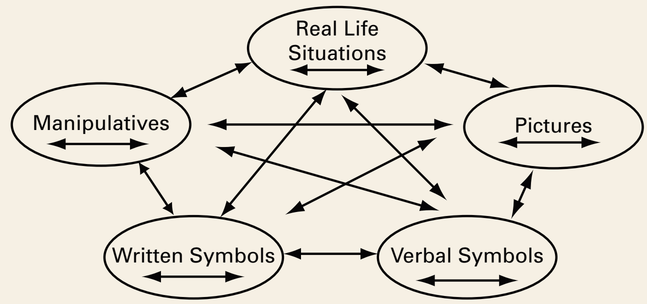
(The verbal mode is implied in the students’ discussion of what is recorded in all quadrants.)

In the beginning, the students had difficulty representing their understanding in a variety of modes, so I went through the process using various types of problems. Further, many students had not used manipulatives to justify their mathematical reasoning or had not been asked to represent their mathematical understanding using multiple modes. One of the most challenging processes when teaching with manipulatives is to facilitate students’ ability to transfer what they do with the manipulatives to their conceptual and procedural understanding. Various studies have identified this problem of the disconnect between one’s actions with the manipulatives and one’s symbolic notations of these actions (Kaput 1992). Representing these different modes on a single activity sheet allowed students to make the connection among them more intentionally. The connection among representations also supports NCTM’s Representation Standard, which stresses that all students should be able to “select, apply, and translate among mathematical representations to solve problems” (NCTM 2000, p. 67).

Through this activity, I was able to gauge my students’ conceptual understanding by assessing their representations in the four quadrants. The quadrant “I can write it with numbers,” in which they showed

Figure 2

Lesh’s translation model

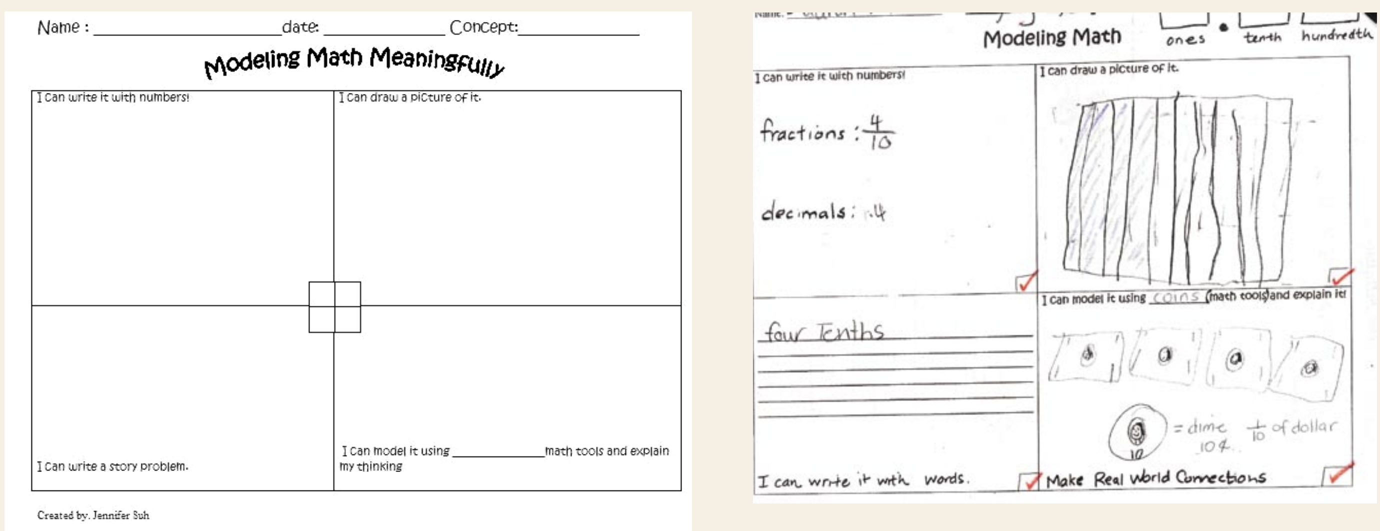


Source: Lesh et al. (2003)

Figure 3

Students’ work on “Modeling Math Meaningfully”: adding fractions with unlike denominators and decimals

a. b.



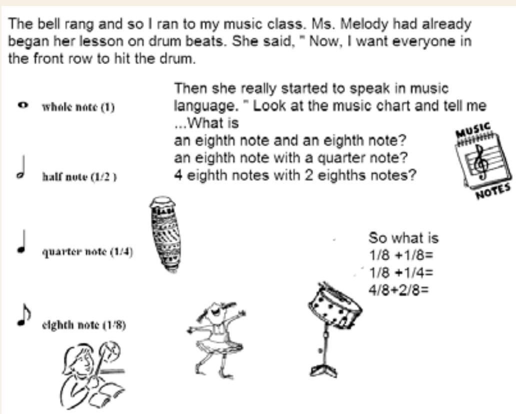
work with numbers, symbols, and algorithms, gave me a glimpse of their procedural fluency. The quadrants "I can write a story problem" and "I can model it using math tools and explain my thinking" revealed their strategic competence and adaptive reasoning skills. The "I Can Draw A Picture Of It" quadrant allowed students to demonstrate their understanding using pictures (see **fig. 3**). These four connected ways served as conceptual support for the students as they made meaning and deepened their relational understanding. Further, teaching and learning in multiple modes revealed the students' personal learning styles and mathematical thinking and allowed me to adapt my instruction accordingly.

"Math Curse"

Research on students' attitude toward mathematics, their belief in their own ability, and the nature of mathematics have been linked to student achievement (McLeod 1992). Thus, one of the most important strands to develop is a productive disposition toward mathematics. Students always want to know: Is mathematics sensible, usable, and relevant to my life? To bring relevance to problem solving, I needed to explore and connect mathematics in my students' world. I began the school year by reading Jon Scieszka's *Math Curse* (1995), a humorous mathematics tale about a boy who wakes up one morning to find that every situation in his life is a mathematics problem. After reading the story, I told my students that they were under the "math curse" and would see mathematics problems everywhere throughout the day. The following class period, they were to bring in a mathematics problem that they had encountered and share it with their classmates. We typed each problem and illustrated it by drawing pictures using the computer or using clip art (see the example in **fig. 4**). The class created a book that included each student's problem related to her day at school. The finished product became a very attractive book for the back-to-school night display. Through this initial activity, I wanted to heighten my students' awareness of mathematics and its functional use so that they could learn to appreciate problem solving. It also was an engaging way to link to the next activity, "Math Happenings."

Figure 4

Sample of student work from the class's own "Math Curse" book



"Math Happenings"

The second strategy that I used throughout the year was "Math Happenings," created by Shirley Maggard, a mathematics specialist at my school. A "math happening" is a real-life problem that can be solved mathematically. On each Monday during the first semester, I came to school and told my students that I had had a math happening. My students were genuinely interested in my "math happening" and wanted to help me solve the problem. One math happening occurred when my husband and I were building a playground set in our backyard, a project that led us into exploring measurement of area and perimeter, budgeting money, and comparing unit prices. Some other math happenings dealt with shopping, planning for my Thanksgiving party, and painting my son's nursery. As I continued to share my real-life problems, my students started to notice mathematical moments in their own lives and brought in problems to share. To assess their mathematical proficiency, I used a rubric that examined students' understanding in the five areas of mathematics proficiency (see **fig. 5**).

The benefit of "Math Happenings" was that students became familiar with the concept of problem posing as well as problem solving. Although these two terms may sound similar, problem posing and problem solving involve different mental processes: formulating a problem and solving a problem. Naturally, my mathematics class became a place where problems surrounded us, and every day we learned strategies that made us better problem solvers. In addition, this classroom practice promoted NCTM's Connections Standard, which emphasizes the importance of applying mathematics in contexts outside the classroom (NCTM 2000). Further, having personal mathematics stories to tell the class made problem solving in mathematics relevant and useful in their lives.

“Convince Me” and “Poster Proofs”

It is important for students to have a chance to discuss their mathematical ideas, argue, and justify their reasoning. Creating a classroom that values students’ thinking is a critical feature for a successful learning environment. To develop strategic competence and adaptive reasoning, students need opportunities to share and compare their solution strategies and explore alternative solution paths. One of the most important values in creating this atmosphere is respect for one another’s ideas.

Through justifying and reasoning, students learn that mathematics makes sense, knowledge that in turn enhances their productive disposition toward mathematics. Students need many opportunities to exercise reasoning and proof through verbal and written activities. To achieve these goals, I created two classroom activities called “Convince Me” and “Poster Proofs.”

In “Convince Me,” students are given time to make sense of the mathematics they are learning by making effective deductive arguments. Just as different types of writing are modeled in language arts, writing in mathematics needs a lot of modeling by the teacher. Instead of telling my students, “Now it is time to write about the math we learned today,” I guided them through a step-by-step process that gave them access to the mathematics vocabulary and methods in making a logical argument to justify their thinking (see **fig. 6**).

The mathematical writing process empowered my students and taught them to defend their answers. It also gave them time to unpack and organize their thinking and an opportunity to correct their misconceptions. Having a chance to make sense of their understanding individually helped them feel confident about their understanding when the time came for them to share their ideas with a partner or with the class. As NCTM’s Communication Standard suggests, the students were able to consolidate their thinking and communicate their mathematical ideas coherently and clearly to their peers (NCTM 2000).

When I wanted my students to collectively work on sense making, I had them work collaboratively through “Poster Proofs.” First, I gave students a nonroutine mathematics problem to solve. Each student worked individually for a couple of minutes. Then the students put their heads together, shared their solution paths, and displayed them on a poster. They discussed their solution strategies, compared one another’s work, and shared their poster with the class. This approach helped the students appreciate the multiple strategies, and, as recommended by NCTM’s *Principles and Standards for School Mathematics* (2000), it allowed students to develop a range of strategies for solving problems, such as using diagrams, and looking for patterns. I presented the class with a problem from NCTM’s *Navigating through Algebra in Grades 3–5* (Cuevas and Yeatts 2001, p. 18) called “Tiling a Patio” (see **fig. 7**).

Figure 5

Assessing Mathematical Proficiency

Activity__________________ Date: __________ Group_____

Students	1	2	3	4
Conceptual Understanding				
• Understands problems or task; • Make connections to similar problems; • Uses models and multiple representations.				
Procedural Understanding				
• Accurate computation; • Proper use of algorithm				
Strategic Competence				
• Formulates and carries out a plan; • Can create similar problems; • Can solve using appropriate strategies.				
Adaptive Reasoning				
• Justifies responses logically; • Reflects on and explains procedures; • Explains concepts clearly.				
Productive Disposition				
• Tackles difficult tasks; • Persevere; • Shows confidence in own ability; • Collaborates/Shares ideas.				

Scoring Rubric
3 Fully Accomplishes the task
2 Partially accomplishes the task
1 Does not accomplish the task

a. Rubric for assessing group mathematical proficiency

Assessing Mathematical Proficiency

Activity__________________ Date: __________

Student Name	Rating and Comments
Conceptual Understanding	
• Understands problems or task • Make connections to similar problems • Uses models and multiple representations	
Procedural Understanding	
• Proper use of algorithm • Accurate computation	
Strategic Competence	
• Formulates and carries out a plan • Can create similar problems • Can solve using appropriate strategies	
Adaptive Reasoning	
• Justifies responses logically • Reflects on and explains procedures • Explains concepts clearly	
Productive Disposition	
• Tackles difficult tasks • Persevere • Shows confidence in own ability • Collaborates/Shares ideas	

Scoring Rubric
3 Fully Accomplishes the task
2 Partially accomplishes the task
1 Does not accomplish the task

b. Rubric for assessing individual mathematical proficiency

I had my students work on this problem individually and then work on "Poster Proofs." This approach allowed for students with different learning styles to be exposed to varied levels of solution strategies that ranged from the basic building blocks of algebraic thinking to very sophisticated thinking (see **fig. 8**). For example, one student used manipulatives to build each patio and thus determine the number of tiles for the next patio and the tenth patio (strategy A), while another drew a picture of the patios to determine the numbers of tiles (strategy B). One student discerned a number pattern in the table that she created (strategy C). One group of students shared their insight with the number patterns for the inside square design being "the patio numbers times the number" and the outside tiles being "the patio number times four plus four." They essentially verbalized the algebraic expression N^2 and the linear expression of plus four ($n + 4$). In addition, when the students explained their solution, they used the overhead tiles to show how they saw that the number of the outside tiles was the same as the number of the interior tiles plus four (the four corner tiles) (strategy D). Allowing the students to use manipulatives, drawings, words, and numeric symbols supported students with varied learning styles and facilitated differentiation with one problem-solving task.

Figure 6

A student's work on a "Convince Me" problem: "Which is greater 3/4 or 5/8? How do you know?"

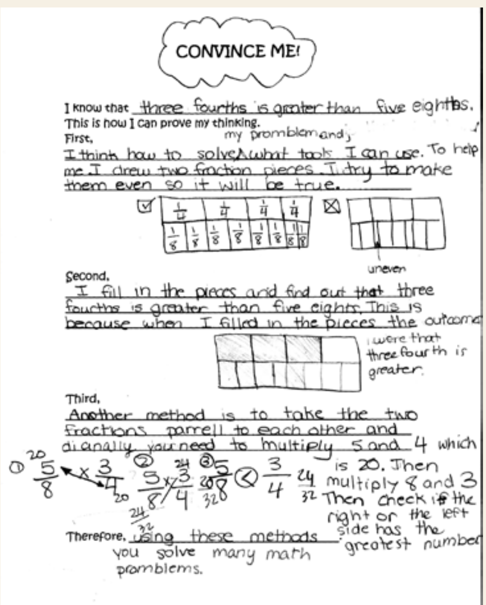
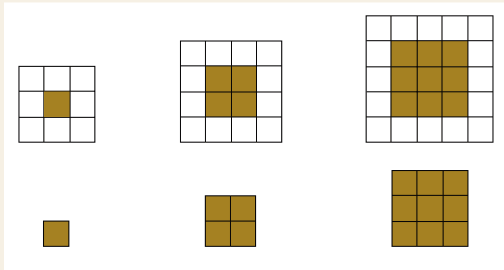


Figure 7

The Tiling a Patio problem

Alfredo Gomez is designing square patios. Each patio has a square garden area in the center. Alfredo uses brown tiles to represent the soil of the garden. Around each garden, he designs a border of white tiles. The pictures show the three smallest square patios that he can design with brown tiles for the garden and white tiles for the border.



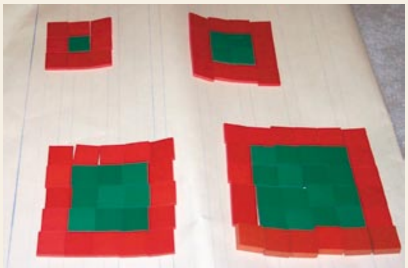
Students Transformed into Mathematicians

Once I focused my teaching on the five strands of mathematics proficiency, the most noticeable change in my class was the change in my students' disposition toward mathematics. Their enthusiasm for "Math Happenings," for example, was contagious, as students begged to share their personal mathematics stories. They started to see mathematics outside the classroom and shared with their classmates how mathematics can help them in the real world. After working with "Modeling Math Meaningfully" for a couple of months, the students began to think and represent their mathematics with multiple representations without prompts. Thus, this activity promoted their representational fluency, as recommended by NCTM (2000), allowing my students to communicate their mathematics approaches, arguments, and understanding to their classmates. *Mathematics* morphed from a noun into a verb. It was not just something to study in the middle of the school day; it became an important activity. My students became mathematicians "mathematizing," solving real-life problems, justifying, and explaining interesting patterns and relationships.

Teaching with these five strands as a guide allowed for differentiation for my diverse learners with respect to curriculum and instruction. All the activities were open-ended and could be modified to adjust for different learning styles. For example, for the "Convince Me" activity, my higher-ability

Figure 8

Three students' strategies for solving the Tiling a Patio problem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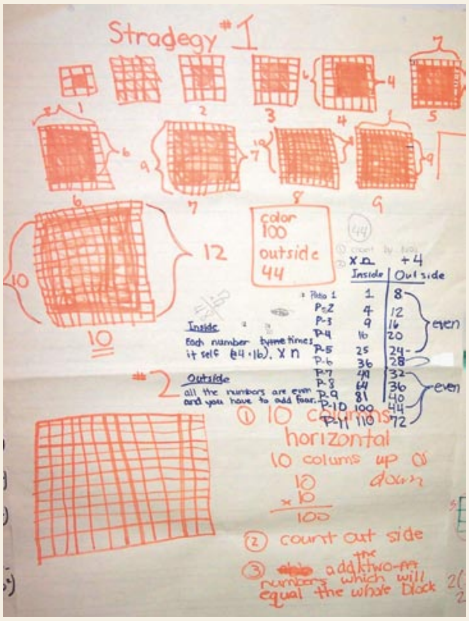

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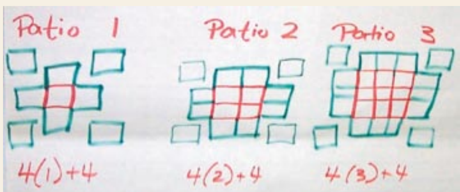

c.

students compared the fractions 3/4 and 5/8 while my special-needs students worked on comparing and justifying their reasoning with the fractions 3/4 and 1/2. For many of my ESOL students, to prove their thinking in English was a challenge, so I allowed them to work with students who spoke their native language. In this way, these students were able to go through the same processes and hear their bilingual classmates translate their proofs into English. Allowing students to use pictures, graphs, and manipulatives helped students who had limited English explain their thinking through images.

Finally, one of the most important lessons learned from my action research was discovering that teaching for mathematical proficiency aligns closely with NCTM's Standards. As mentioned, Problem Solving, Reasoning and Proof, Communication, Connections, and Representation are processes that promoted the development of the five strands of mathematical proficiency. The suggested classroom activities are not secret ingredients for building mathematically proficient students. Rather, these activities provide students with opportunities for building conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions toward mathematics. Teaching and learning for mathematical proficiency is a complex task, but teachers who implement sound practice and effective strategies help our students tie the five strands together.

Bibliography

Cuevas, Gilbert J., and Karol Yeatts. *Navigating through Algebra: Grades 3–5. Principles and Standards for School Mathematics* Navigations Series. Reston, VA: National Council of Teachers of Mathematics, 2001.

Kaput, J. "Technology and Mathematics Education." In *Handbook of Research on Mathematics Teaching and Learning*, edited by D. A. Grouws, pp. 515–56. Reston, VA: National Council of Teachers of Mathematics, 1992.

Lesh, R., K. Cramer, H. Doerr, T. Post, and J. Zawojewski. "Using a Translation Model for Curriculum Development and Classroom Instruction." In *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching,* edited by R. Lesh and H. Doerr. Mahwah, NJ: Lawrence Erlbaum Associates, 2003.

McLeod, D. B. "Research on Affect in Mathematics Education: A Reconceptualization." In *Handbook of Research on Mathematics Teaching and Learning*, edited by D. A. Grouws, pp. 575–96. Reston, VA: National Council of Teachers of Mathematics, 1992.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

National Research Council. *Adding It Up: Helping Children Learn Mathematics*, edited by J Kilpatrick, J. Swafford, and B. Findell. Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press, 2001.

Children's Literature

Scieszka, Jon. Illustrated by Lane Smith. *Math Curse.* New York: Viking Books, 1995. ▲