Modeling and Analysis of non-linear large-deformations of plates with application to Micro-Air Vehicles: An Analytical and Computational approach

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Abstract

In this work we consider the development of a computational methodology to study stability and nonlinear dynamics of large deformation plate models. The main application will be the computational modeling of flexible wing designs for Micro Air Vehicles. We develop the model by using a geometrically nonlinear Green strain-displacement formulation, a materially linear constitutive stress-strain formulation, and a Hamiltonian energy approach to develop a governing system of coupled partial differential equations for the axial and transverse displacements. We develop an appropriate energy norm for a class of boundary conditions where we prove a stability estimate for a simplified version of the model. The model developed will be numerically validated for benchmark applications.

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1 Introduction

A Micro Air Vehicle (MAV) is a type of Unmanned Aerial Vehicle (UAV) that has a wingspan of roughly 15 centimeters, much smaller than the current UAVs in the field today. Recent advances in sensor and camera miniaturization have made MAVs popular choices as platforms for conducting surveillance in hostile environments where maneuverability is at a premium. In addition to the obvious defense applications, MAVs are also used in search-and-resuce, ecological surveys, disaster relief, and climate monitoring. Advances in artificial intelligence research has also made the organization of MAV swarms, which alter their flight plan and sensor coverage based on received input from other networked drones, much more practical for the near future. As this limits the number of needed human operators, MAVs are sure to increase in utility and popularity as time goes on.

In the past, MAVs have used fixed wing designs, which suffer from a phenomenon called *flutter*. Flutter is caused by vibrations from aerodynamic forces interacting with vibrations of the material of the wing, causing dangerous oscillations that can become strong enough to snap the wing in two. Flexible wings, inspired by biology, solve this problem and improve on the design in terms of maneuverability and wind tolerance [4].

Flexible wing design is relatively new, so newer models must be developed to simulate the aerodynamics of a membrane wing. We use plate theory to understand the structural mechanics of the flexible wing. Older models used classical Newtonian mechanics to derive and analyze such models [1], however, we use a Hamiltonian approach to derive the model. The Hamiltonian approach works by describing the energy of the entire system. This makes the model momentum -invariant, while classical mechanics depend on momentum. In addition to the new approach, we also incorporate nonlinearities into our model. For example, in traditional models, the axial displacement was assumed to be zero and the average axial force is assumed constant over the plate [5]. In nonlinear models, the axial displacement are not trivial [2]. Since physical systems are for the most part nonlinear, an accurate model of the wing structure must be nonlinear. The first nonlinearity is called geometric, because it is a result of the nonlinear strain-displacement relations, when the deformations are large. The material nonlinearity is a result of a nonlinear stress-strain relation, however, in this paper we use a linear relationship.

The outline of this paper is as follows. First, we present the development of the mathematical model for the dynamic behavior of a nonlinear plate undergoing deformation both in transverse and axial directions using a Hamiltonian approach. Section 3 presents a new stability result for the associated non-linear boundary value problem for axial and transverse displacements by obtaining an energy-estimate for the excited non-linear plate. Note that, we have only considered this model for simplicity and the analysis presented can be extended to more complex problems as well.

2 Derivation of Mathematical Model

The primary goal of this section is to derive a system of partial differential equations that describe the total energy of the plate. To do this, we must establish three interrelated quantities: displacement, stress, and strain. These quantities are then used to define the kinetic and potential energies of the system. After the formulas for energy are defined, we use Hamilton's principle to

derive the final model.

2.1 Displacement

We need functions to represent the total displacement of the plate at any point in our (x_1, x_2, x_3) plane. We use the Kirchoff-Love theory of plates to define the displacement, which assumed a three dimensional plate, if thin enough, can be represented by a two-dimensional mid-plane. The following three kinematic assumptions hold: straight lines normal to the mid-surface remain straight after deformation, straight lines normal to the mid-surface remain normal to the mid-surface after deformation, and the thickness of the plate does not change during a deformation. We use u to denote the in-plane axial deformation of the x_1-axis, v denotes the in-plane axial deformation of the x_2-axis, and w denotes the transverse deformation. With this in mind, the following equations for the total deformation are defined as follows:

$$\begin{aligned} u_1(x_1, x_2, t) &= u - x_3 w_{x_1} \\ u_2(x_1, x_2, t) &= v - x_3 w_{x_2} \\ u_3(x_1, x_2, t) &= w. \end{aligned}$$

2.2 Strain

Strain is for our purposes a dimensionless quantity that measures the compressing or stretching of an object based on a given load [3]. We will be using a nonlinear second order Green strain tensor defined below [1]:

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}),$$

where i,j = 1, 2, or 3 depending on the coordinate direction. Note the final term is the source of the geometric nonlinearity. Now we simply substitute in the deformation equations. It's relatively straightforward to see that all elements of this tensor where $i = 3$ or $j = 3$ are zero, so only the following elements of $E_{i,j}$ are nontrivial.

$$\begin{aligned} E_{11} &= \frac{1}{2}(2(u_{x_1} - x_3 w_{x_1 x_1}) + w_{x_1}^2) \\ E_{22} &= \frac{1}{2}(2(v_{x_2} - x_3 w_{x_2 x_2}) + w_{x_2}^2) \\ E_{12} &= \frac{1}{2}((u_{x_2} + v_{x_1} - 2x_3 w_{x_1 x_2}) + w_{x_1} w_{x_2}) \\ E_{21} &= \frac{1}{2}((u_{x_2} + v_{x_1} - 2x_3 w_{x_1 x_2}) + w_{x_1} w_{x_2}). \end{aligned}$$

2.3 Stress

Stress is a physical quantity measuring the internal forces affecting the physical system. The stress-strain relationship is materially linear and related to Hooke's law, where Y is the Young's modulus of elasticity and ν is the Poisson ratio. Note that both are determined by the material.

$$\sigma_{ij} = \frac{Y}{(1-\nu^2)} E_{ij}.$$

We then simply plug in our previous equations. As before, terms in which $i = 3$ or $j = 3$ are trivial.

$$\begin{aligned}
\sigma_{11} &= \frac{Y}{(1-\nu^2)} E_{11} \\
\sigma_{22} &= \frac{Y}{(1-\nu^2)} E_{22} \\
\sigma_{12} &= \frac{1-\nu}{2} \frac{Y}{(1-\nu^2)} E_{12} \\
\sigma_{21} &= \frac{1-\nu}{2} \frac{Y}{(1-\nu^2)} E_{12}.
\end{aligned}$$

2.4 Kinetic Energy

Kinetic energy, denoted T, is relatively straightforward, using the typical physics formulation.

$$T = \frac{1}{2} m ||V||^2.$$

We must account for all the mass. To do this, we assume a homogeneous plate density ρ so that T takes the form of an integral over the area. The squared norm of V in this case is identical to the dot product of V with itself, i.e., the sum of the squared derivatives of our displacement functions with respect to time.

$$T = \int_0^a \int_0^a \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{\rho}{2} ([u_1]_t^2 + [u_2]_t^2 + [u_3]_t^2) \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1.$$

We then plug in these derivatives, resulting in the following integral:

$$T = \frac{\rho}{2} \int_0^a \int_0^a \int_{\frac{-h}{2}}^{\frac{h}{2}} ([u - x_3 w_{x_1}]_t^2 + [v - x_3 w_{x_2}]_t^2 + [w]_t^2) \, \mathrm{d}x_3 \, \mathrm{d}x_2 \, \mathrm{d}x_1.$$

Expanding this out, we have:

$$
\begin{aligned}
T &= \frac{\rho}{2}\int_0^a\int_0^a\int_{\frac{-h}{2}}^{\frac{h}{2}}([u_t-[x_3]_t w_{x_1}-x_3 w_{x_1t}]^2+[v_t-[x_3]_t w_{x_2}-x_3 w_{x_2t}]^2+w_t^2)\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\\
&= \frac{\rho}{2}\int_0^a\int_0^a\int_{\frac{-h}{2}}^{\frac{h}{2}}((u_t^2-2u_t[x_3]_t w_{x_1}-2u_t x_3 w_{x_1t}+[x_3]_t^2 w_{x_1}^2+2[x_3]_t w_{x_1}x_3 w_{x_1t}+x_3^2 w_{x_1t}^2)\\
&+ (v_t^2-2v_t[x_3]_t w_{x_2}-2v_t x_3 w_{x_2t}+[x_3]_t^2 w_{x_2}^2+2[x_3]_t w_{x_2}x_3 w_{x_2t}+x_3^2 w_{x_2t}^2)+w_t^2)\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\\
&= \frac{\rho}{2}\int_0^a\int_0^a[x_3(u_t^2+v_t^2+w_t^2)+\frac{x_3^3}{3}(w_{x_1t}^2+w_{x_2t}^2)]_{x_3=\frac{-h}{2}}^{x_3=\frac{h}{2}}\,\mathrm{d}x_2\,\mathrm{d}x_1.
\end{aligned}
$$

As a result of the symmetrical limits of integration, many terms equal zero when we integrate with respect to x_3. In addition, the inertial term $\frac{h^3}{12}(w_{x_1t}^2+w_{x_2t}^2)$ is assumed to be zero [7].

$$
T = \int_0^a\int_0^a\frac{\rho h}{2}(u_t^2+v_t^2+w_t^2)\,\mathrm{d}x_2\,\mathrm{d}x_1.
$$

2.5 Potential Energy

The potential energy can be split into two major sources: energy caused by gravity and energy caused by the bent plate. Due to the method used to construct our coordinate system, the gravitational potential energy is assumed to be zero. The energy caused by the bent plate can be constructed as the energy stored in a bent spring defined as follows:

$$
U = \int_0^a\int_0^a\int_{\frac{-h}{2}}^{\frac{h}{2}}\frac{1}{2}(\sigma_{11}E_{11}+\sigma_{22}E_{22}+\sigma_{12}E_{12})\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1.
$$

Plugging in our terms, we have the following:

$$
\begin{aligned}
U &= \frac{1}{2}\int_0^a\int_0^a\int_{\frac{-h}{2}}^{\frac{h}{2}}(\frac{Y}{(1-\nu^2)}E_{11}^2+\frac{Y}{(1-\nu^2)}E_{22}^2+\frac{1-\nu}{2}\frac{Y}{(1-\nu^2)}E_{12}^2\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\\
&= \frac{1}{2}\int_0^a\int_0^a\int_{\frac{-h}{2}}^{\frac{h}{2}}(\frac{Y}{(1-\nu^2)}(\frac{1}{2}(2(u_{x_1}-x_3 w_{x_1x_1})+w_{x_1}^2)^2+\frac{Y}{(1-\nu^2)}(\frac{1}{2}(2(v_{x_2}-x_3 w_{x_2x_2})+w_{x_2}^2)^2\\
&+ \frac{1-\nu}{2}\frac{Y}{(1-\nu^2)}(\frac{1}{2}(u_{x_2}+v_{x_1}-2x_3 w_{x_1x_2}))^2\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1.
\end{aligned}
$$

Integrating with respect to x_3, we have the following:

$$\begin{aligned} U &= \frac{Y}{(1-\nu^2)}\int_0^a\int_0^a h((u_{x_1}+\frac{1}{2}w_{x_1}^2)^2+(v_{x_2}+\frac{1}{2}w_{x_2}^2)^2+\frac{1-\nu}{2}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})^2) \\ &+ \frac{h^3}{12}(w_{x_1}w_{x_1}^2+w_{x_2}w_{x_2}+w_{x_1}w_{x_2}^2)\,\mathrm{d}x_2\,\mathrm{d}x_1. \end{aligned}$$

However, we still require the potential energy of the external applied forces. Here, it will be defined as the negative of the work done by fluid forces acting on the plate. The f_{x_i} represent the fluid forces acting on the place, and K is a damping constant which acts on the first temporal derivative.

$$A = u(f_{x_1} - Ku_t) + v(f_{x_2} - Kv_t) + w(f_{x_3} - Kw_t)$$

2.6 Hamilton's Method

According to Hamilton's principle, the progression of all physical systems minimizes the time integral of the Lagrangian, which is to say the variation of the Lagrangian will always be zero, [6] i.e.

$$\delta\int_{t_0}^{t_1}[(T-U)+A]dt = 0.$$

Plugging in the kinetic and potential energies we have the following integral:

$$\begin{aligned} 0 &= \delta\int_{t_0}^{t_1}\int_0^a\int_0^a \frac{\rho h}{2}(u_t^2+v_t^2+w_t^2) \\ &- \frac{Yh}{(1-\nu^2)}((u_{x_1}+\frac{1}{2}w_{x_1}^2)^2+(v_{x_2}+\frac{1}{2}w_{x_2}^2)^2+\frac{1-\nu}{2}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})^2) \\ &+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1}w_{x_1}^2+w_{x_2}w_{x_2}+w_{x_1}w_{x_2}^2) \\ &+ u(f_{x_1}-Ku_t)+v(f_{x_2}-Kv_t)+w(f_{x_3}-Kw_t)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t. \end{aligned}$$

We then continue by calculus of variations, obtaining the following.

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \frac{\rho h}{2}(u_t\delta u_t + v_t\delta v_t + w_t\delta w_t) \\
&- \frac{Yh}{(1-\nu^2)}((u_{x_1}+\frac{1}{2}w_{x_1}^2)\delta(u_{x_1}+\frac{1}{2}w_{x_1}^2) + (v_{x_2}+\frac{1}{2}w_{x_2}^2)\delta(v_{x_2}+\frac{1}{2}w_{x_2}^2) \\
&+ \frac{1-\nu}{2}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})\delta(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})) \\
&+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1x_1}\delta w_{x_1x_1} + w_{x_2x_2}\delta w_{x_2x_2} + w_{x_1x_2}\delta w_{x_1x_2}) \\
&+ \delta u f_{x_1} - K\delta u_t u + \delta v f_{x_2} - K\delta v_t v + \delta w f_{x_3} - K\delta w_t w \,\mathrm{d}x_2\,\mathrm{d}x_1.
\end{aligned}
$$

Using integration by parts to handle each term, the variation and the first spacial and temporal derivatives of the variation are zero at the limits of integration, therefore each boundary term is cancelled. After collecting all of the terms with contain δu, δv, δw, we can separate the integral into three parts as follows:

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta u\Bigg(-\rho h u_{tt} + \frac{Yh}{(1-\nu^2)}([u_{x_1}+\frac{1}{2}w_{x_1}^2]_{x_1} + \frac{1-\nu}{2}[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}]_{x_2}) \\
&+ Ku_t + f_{x_1}\Bigg)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t \\
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta v\Bigg(-\rho h v_{tt} + \frac{Yh}{(1-\nu^2)}([v_{x_2}+\frac{1}{2}w_{x_2}^2]_{x_2} + \frac{1-\nu}{2}[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}]_{x_1}) \\
&+ Kv_t + f_{x_2}\Bigg)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t \\
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta w\Bigg(-\rho h w_{tt} + \frac{Yh}{(1-\nu^2)}([w_{x_1}(u_{x_1}+\frac{1}{2}w_{x_1}^2)]_{x_1} + [w_{x_2}(v_{x_2} \\
&+ \frac{1}{2}w_{x_2}^2)]_{x_2} + \frac{1-\nu}{2}[w_{x_1}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})]_{x_2} + [w_{x_2}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})]_{x_1}) \\
&+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1x_1x_1x_1} + w_{x_2x_2x_2x_2} + 2w_{x_1x_1x_2x_2}) + Kw_t + f_{x_3}\Bigg)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t.
\end{aligned}
$$

The functions $f_{x_1}, f_{x_2}, f_{x_3}$ can be chosen arbitrarily to make the integral nonnegative, and this can only be true if the three integrands are identically zero, so the final model looks as follows:

$$
f_{x_1} = u_{tt} + Cu_t - D_1[u_{x_1} + \tfrac{1}{2}w_{x_1}^2]_{x_1} - B[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \tag{1}
$$

$$f_{x_2} = v_{tt} + Cv_t - D_1[v_{x_2} + \tfrac{1}{2}w_{x_2}^2]_{x_2} - B[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_1} \tag{2}$$

$$\begin{aligned} f_{x_3} &= w_{tt} + Cw_t - D_1([w_{x_1}(u_{x_1} + \frac{1}{2}w_{x_1}^2)]_{x_1} + [w_{x_2}(v_{x_2} + \frac{1}{2}w_{x_2}^2)]_{x_2}) \\ &- B[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} + [w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1} \\ &+ E[w_{x_1x_1x_1x_1} + w_{x_2x_2x_2x_2} + 2w_{x_1x_1x_2x_2}] \end{aligned} \tag{3}$$

where $C = \frac{K}{\rho h}$, $D_1 = \frac{(Y)}{\rho(1-\nu^2)}$, $B = \frac{Y}{2\rho(1+\nu)}$, and $E = \frac{Yh^2}{12(1-\nu^2)\rho}$.

3 Stability Result for the Mathematical Model

The above system of coupled partial differential equations (1), (2), and (3) provide the governing equations of motion for a plate being deformed by some external force with components f_{x_1}, f_{x_2}, and f_{x_3}. We will show for any transversal force f_3 the energy of the system changes proportionally to the force. In other words, our choice of initial conditions won't cause the system to experience *flutter* or other disastrous instabilities. To properly analyze the effect of the initial conditions on the stability of the system, we must simplify the system to an ordinary differential equation. The ultimate goal is to combine and re-separate the system based on elements containing temporal derivatives and all other elements. We will proceed using several lemmas, for details refer to the Appendix section.

3.1 Creating the Ordinary Differential Equation

For the purposes of unit conversion we introduce a constant $a \in (0, 1)$ into the system.

Multiply (1) by u_t and aCu respectively. By adding the resulting equations together and using the fact that,

$$\begin{aligned} u_{tt}u_t &= \frac{1}{2}\left[u_t\right]_t^2 \\ aC^2u_tu &= \frac{a}{2}C^2\left[u^2\right]_t \end{aligned}$$

we are left with,

$$\begin{aligned}\frac{1}{2}\left[u_t\right]_t^2 + Cu_t^2 + aCu_{tt}u &+ \frac{a}{2}C^2\left[u^2\right]_t \\ &= D_1\left[u_{x_1} + \frac{1}{2}(w_{x_1})^2\right]_{x_1} u_t + aD_1C\left[u_{x_1} + \frac{1}{2}(w_{x_1})^2\right]_{x_1} u \\ &\quad + E\left[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}\right]_{x_2} u_t \\ &\quad + aEC\left[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}\right]_{x_2} u + f_{x_1}(u_t + aCu) \end{aligned} \tag{4}$$

Apply the same method to (2) with v_t and aCv respectively. Add the resulting equations together and we get the following,

$$\begin{aligned}\frac{1}{2}\left[v_t\right]_t^2 + Cv_t^2 + aCv_{tt}v &+ \frac{a}{2}C^2\left[v^2\right]_t \\ &= D_1\left[v_{x_2} + \frac{1}{2}(w_{x_2})^2\right]_{x_2} v_t + aD_1C\left[v_{x_2} + \frac{1}{2}(w_{x_2})^2\right]_{x_2} v \\ &\quad + E\left[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}\right]_{x_1} v_t \\ &\quad + aEC\left[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}\right]_{x_1} v + f_{x_2}(v_t + aCv) \end{aligned} \tag{5}$$

Multiply (3) by w_t and $\frac{1}{2}aKw$, respectively. By adding the resulting equations together and using the fact that,

$$\begin{aligned} w_{tt}w_t &= \frac{1}{2}\left[w_t\right]_t^2 \\ \frac{aC^2}{2}w_tw &= \frac{a}{4}C^2\left[w^2\right]_t \end{aligned}$$

we are left with,

$$\begin{aligned}
\frac{1}{2}\left[w_t\right]_t^2 &+ Cw_t^2 + \frac{aC}{2}w_{tt}w + \frac{a}{4}C^2\left[w^2\right]_t \\
&= D_1\left[w_{x_1}\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)\right]_{x_1} w_t + D_1\left[w_{x_2}\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)\right]_{x_2} w_t \\
&\quad + \frac{aD_1C}{2}\left[w_{x_1}\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)\right]_{x_1} w + \frac{aD_1C}{2}\left[w_{x_2}\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)\right]_{x_2} w \\
&\quad + E\left[(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})w_{x_1}\right]_{x_2} w_t + E\left[(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})w_{x_2}\right]_{x_1} w_t \\
&\quad + \frac{aEC}{2}\left[(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})w_{x_1}\right]_{x_2} w + \frac{aEC}{2}\left[(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})w_{x_2}\right]_{x_1} w \\
&\quad - D\left[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}\right] w_t \\
&\quad - \frac{aDC}{2}\left[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}\right] w + f_{x_3}\left(w_t + \frac{aCw}{2}\right) \qquad (6)
\end{aligned}$$

Ultimately we will be integrating in each one of the equations and it is therefore in our best interest to rewrite many of the terms as derivatives allowing us to integrate by parts. We proceed in the following manner. Rewrite the left hand side of (4) as,

$$\begin{aligned}
\frac{1}{2}\left[u_t\right]_t^2 + Cu_t^2 + aCu_{tt}u &+ \frac{a}{2}C^2\left[u^2\right]_t \\
&= \frac{a}{2}\left[(u_t + Cu)^2\right]_t + \frac{1-a}{2}\left[u_t^2\right]_t + (1-a)Cu_t^2 \qquad (7)
\end{aligned}$$

rewrite the left hand side of (5) as,

$$\begin{aligned}
\frac{1}{2}\left[v_t\right]_t^2 + Cv_t^2 + aCv_{tt}v &+ \frac{a}{2}C^2\left[v^2\right]_t \\
&= \frac{a}{2}\left[(v_t + Cv)^2\right]_t + \frac{1-a}{2}\left[v_t^2\right]_t + (1-a)Cv_t^2 \qquad (8)
\end{aligned}$$

and finally rewrite the left hand side of (6) as,

$$\begin{aligned}
\frac{1}{2}\left[w_t\right]_t^2 + Cw_t^2 &+ \frac{aC}{2}w_{tt}w + \frac{a}{4}C^2\left[w^2\right]_t \\
&= \frac{a}{4}\left[(w_t + Cw)^2\right]_t + \left(\frac{1}{2} - \frac{a}{4}\right)\left[w_t^2\right]_t + \left(1 - \frac{a}{2}\right)Cw_t^2 \qquad (9)
\end{aligned}$$

Substituting the results from (7), (8) and (9) in (4), (5) and (6) respectively yields:

$$\begin{aligned}
\frac{a}{2}\left[(u_t+Cu)^2\right]_t &+ \frac{1-a}{2}\left[u_t^2\right]_t + (1-a)Cu_t^2 \\
&- D_1\left[u_{x_1}+\frac{1}{2}(w_{x_1})^2\right]_{x_1} u_t - aD_1C\left[u_{x_1}+\frac{1}{2}(w_{x_1})^2\right]_{x_1} u \\
&- E\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right]_{x_2} u_t \\
&- aEC\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right]_{x_2} u = f_{x_1}(u_t+aCu)
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{a}{2}\left[(v_t+Cv)^2\right]_t &+ \frac{1-a}{2}\left[v_t^2\right]_t + (1-a)Cv_t^2 \\
&- D_1\left[v_{x_2}+\frac{1}{2}(w_{x_2})^2\right]_{x_2} v_t - aD_1C\left[v_{x_2}+\frac{1}{2}(w_{x_2})^2\right]_{x_2} v \\
&- E\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right]_{x_1} v_t \\
&- aEC\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right]_{x_1} v = f_{x_2}(v_t+aCv)
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\frac{a}{4}\left[(w_t+Cw)^2\right]_t + \left(\frac{1}{2}-\frac{a}{4}\right)\left[w_t^2\right]_t + \left(1-\frac{a}{2}\right)Cw_t^2 \\
- &\ D_1\left[w_{x_1}\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)\right]_{x_1} w_t - D_1\left[w_{x_2}\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)\right]_{x_2} w_t \\
- &\ \frac{aD_1C}{2}\left[w_{x_1}\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)\right]_{x_1} w - \frac{aD_1C}{2}\left[w_{x_2}\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)\right]_{x_2} w \\
- &\ E\left[(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})w_{x_1}\right]_{x_2} w_t - E\left[(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})w_{x_2}\right]_{x_1} w_t \\
- &\ \frac{aEC}{2}\left[(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})w_{x_1}\right]_{x_2} w - \frac{aEC}{2}\left[(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})w_{x_2}\right]_{x_1} w \\
+ &\ D\left[w_{x_1x_1x_1x_1}+2w_{x_1x_1x_2x_2}+w_{x_2x_2x_2x_2}\right] w_t \\
+ &\ \frac{aDC}{2}\left[w_{x_1x_1x_1x_1}+2w_{x_1x_1x_2x_2}+w_{x_2x_2x_2x_2}\right] w = f_{x_3}\left(w_t+\frac{aCw}{2}\right)
\end{aligned} \tag{12}$$

As mentioned before we are interested in integrating these equations over our domain to get a measure of energy. To do this we make use of the steps taken in the previous few lines and inte-

grate many of the terms by parts and then sum each integrated equation yielding,

$$\begin{aligned}
\frac{d}{dt}\Bigg\{\int_0^L\int_0^L\Bigg[&\frac{a}{2}(u_t+Cu)^2+\frac{a}{2}(v_t+Cv)^2+\frac{a}{4}(w_t+Cw)^2+\frac{1-a}{2}u_t^2+\frac{1-a}{2}v_t^2+\left(\frac{1}{2}-\frac{a}{4}\right)w_t^2\\
&\frac{D_1}{2}\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+\frac{D_1}{2}\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2+\frac{E}{2}\left(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right)^2\\
&\frac{D}{2}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\Bigg]\,dx_1\,dx_2\Bigg\}\\
&+\int_0^L\int_0^L\Big[(1-a)Cu_t^2+(1-a)Cv_t^2+\left(1-\frac{a}{2}\right)Cw_t^2\\
&\quad+aCD_1\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+aCD_1+\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2\\
&\quad+aEC\left(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right)^2+\frac{aDC}{2}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\Big]\,dx_1\,dx_2\\
&\quad=\int_0^L\int_0^L\left[f_{x_1}(u_t+aCu)+f_{x_2}(v_t+aCv)+f_{x_3}\left(w_t+\frac{aCw}{2}\right)\right]\,dx_1\,dx_2
\end{aligned}$$

And we make the following definitions

$$\begin{aligned}
M_1=\int_0^L\int_0^L\Bigg[&\frac{a}{2}(u_t+Cu)^2+\frac{a}{2}(v_t+Cv)^2+\frac{a}{4}(w_t+Cw)^2+\frac{1-a}{2}u_t^2+\frac{1-a}{2}v_t^2+\left(\frac{1}{2}-\frac{a}{4}\right)w_t^2\\
&\frac{D_1}{2}\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+\frac{D_1}{2}\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2+\frac{E}{2}\left(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right)^2\\
&\frac{D}{2}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\Bigg]\,dx_1\,dx_2\\
M_2=\int_0^L\int_0^L\Big[&(1-a)Cu_t^2+(1-a)Cv_t^2+\left(1-\frac{a}{2}\right)Cw_t^2\\
&+aCD_1\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+aCD_1+\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2\\
&+aEC\left(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right)^2+\frac{aDC}{2}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\Big]\,dx_1\,dx_2\\
F=\int_0^L\int_0^L&\left[f_{x_1}(u_t+aCu)+f_{x_2}(v_t+aCv)+f_{x_3}\left(w_t+\frac{aCw}{2}\right)\right]\,dx_1\,dx_2
\end{aligned}$$

We then have the following differential equation.

$$\frac{dM_1}{dt}+M_2=F$$

Before we continue let us firstly note that the equations for M_1 and M_2 have fairly similar terms.

Both equations share terms share the same form of kinetic energy and take into account the way the system's energy is dependent upon the velocities of each direction and how the velocities are related to the direction. In addition note that the derivation up until the point has done under the assumption that $f_{x_1}, f_{x_2} \neq 0$. For the purposes of convenience we will now adopt this hypothesis.

From here we will proceed by putting a bound on a defined energy measure which has a form very similar to M_1. We will choose our energy measure in such a way that allows us to apply a bound to it, but also ensure that the contributions of the energy are not neglected.

As mentioned before for the purpose of convenience we will let,

$$F = \int_0^L \int_0^L \left[f_{x_3} \left(w_t + \frac{aCw}{2} \right) \right] \, dx_1 \, dx_2$$

and analyze the equation,

$$\frac{dM_1}{dt} + M_2 = F$$

3.2 Young's Inequality

We want to apply Young's Inequality to $f_{x_3} w$ and $f_{x_3} w_t$

We have by Young's Inequality that for $\epsilon_1 > 0$,

$$f_{x_3} w = \left(\frac{f_{x_3}}{\sqrt{2}} \right) \left(w\sqrt{2} \right) \leq \frac{1}{2\epsilon_1} \left(\frac{f_{x_3}}{\sqrt{2}} \right)^2 + \frac{\epsilon_1}{2} \left(w\sqrt{2} \right)^2 = \frac{f_{x_3}^2}{4\epsilon_1} + \epsilon_1 w^2$$

and because $\frac{aC}{2} > 0$ it follows that,

$$\frac{aC}{2} f_{x_3} w \leq \frac{aC}{2} \left(\frac{f_{x_3}^2}{4\epsilon_1} + \epsilon_1 w^2 \right)$$

Equivalently with $f_{x_3} w_t$ we have that for any $\epsilon_2 > 0$,

$$f_{x_3} w_t = \left(\frac{f_{x_3}}{\sqrt{2}} \right) \left(w_t\sqrt{2} \right) \leq \frac{1}{2\epsilon_2} \left(\frac{f_{x_3}}{\sqrt{2}} \right)^2 + \frac{\epsilon_2}{2} \left(w_t\sqrt{2} \right)^2 = \frac{f_{x_3}^2}{4\epsilon_2} + \epsilon_2 w_t^2$$

and thus it follows that,

$$\frac{aC}{2} f_{x_3} w + f_{x_3} w_t = f_{x_3} \left(w_t + \frac{aCw}{2} \right) \leq \frac{aC}{2} \left(\frac{f_{x_3}^2}{4\epsilon_1} + \epsilon_1 w^2 \right) + \frac{f_{x_3}^2}{4\epsilon_2} + \epsilon_2 w_t^2$$

leaving us with the following inequality for our original equation,

$$[M_1]_t + M_2 = \int_0^L \int_0^L \left[f_{x_3} \left(w_t + \frac{aCw}{2} \right) \right] \, dx_1 \, dx_2 \leq \int_0^L \int_0^L \left[\frac{aC}{2} \left(\frac{f_{x_3}^2}{4\epsilon_1} + \epsilon_1 w^2 \right) + \frac{f_{x_3}^2}{4\epsilon_2} + \epsilon_2 w_t^2 \right] \, dx_1 \, dx_2$$

3.3 Poincaré Inequality

With the introduction of ϵ_1 and ϵ_2 we are free to equate them to any two positive real numbers we desire. We will choose them in a fashion that allows us to relate M_1 and M_2.

Choose ϵ_1 and ϵ_2 accordingly from the Poincaré Inequality:

By rearranging the previous inequality we see that,

$$[M_1]_t + M_2 - \epsilon_1 \frac{aC}{2} w^2 - \epsilon_2 w_t^2 \leq \int_0^L \int_0^L \left[\frac{aC}{2} \frac{f_{x_3}^2}{4\epsilon_1} + \frac{f_{x_3}^2}{4\epsilon_2} \right] \, dx_1 \, dx_2$$

From which we may replace the equation of M_2 yielding,

$$\begin{aligned}
[M_1]_t + \int_0^L \int_0^L \Bigg(& (1-a)Cu_t^2 + (1-a)Cv_t^2 + \left[(1-\frac{a}{2})C - \epsilon_2 \right] w_t^2 \\
& + aCD_1 \left(u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right)^2 + aCD_1 \left(v_{x_2} + \frac{1}{2}(w_{x_2})^2 \right)^2 \\
& + aEC(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})^2 \\
& + \frac{aC}{2} \left[D(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2) - \epsilon_1 w^2 \right] \Bigg) \, dx_1 \, dx_2 \\
& \leq \int_0^L \int_0^L f_{x_3}^2 \left(\frac{aC}{8\epsilon_1} + \frac{1}{4\epsilon_2} \right) \, dx_1 \, dx_2
\end{aligned} \tag{13}$$

Note by the special case of the Poincaré Inequality where w vanishes on the boundary, it follows that there exists K_1 so that,

$$\int_0^L \int_0^L w^2 \, dx_1 \, dx_2 \leq K_1^2 \int_0^L \int_0^L w_{x_1}^2 + w_{x_2}^2 \, dx_1 \, dx_2 = K_1^2 \int_0^L \int_0^L w_{x_1}^2 \, dx_1 \, dx_2 + K_1^2 \int_0^L \int_0^L w_{x_2}^2 \, dx_1 \, dx_2$$

And by application of the inequality to both individual integrals on the right hand side, we guarantee the existence of a K_2 and K_3 so that,

$$\int_0^L \int_0^L w^2 \, dx_1 \, dx_2 \leq K_2^2 \int_0^L \int_0^L \left(w_{x_1x_1}^2 + w_{x_1x_2}^2 \right) dx_1 \, dx_2 + K_3^2 \int_0^L \int_0^L \left(w_{x_2x_2}^2 + w_{x_1x_2}^2 \right) dx_1 \, dx_2$$

If we let $K = \max\{K_1, K_2\}$ then we receive the following inequality,

$$\int_0^L \int_0^L w^2 \, dx_1 \, dx_2 \leq K^2 \int_0^L \int_0^L \left(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2 \right) dx_1 \, dx_2$$

It follows that because $\frac{aC\epsilon_1}{2} > 0$ that,

$$-\frac{aC\epsilon_1}{2}K^2\int_0^L\int_0^L \left(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right) dx_1\, dx_2 \leq -\frac{aC\epsilon_1}{2}\int_0^L\int_0^L w^2\, dx_1\, dx_2$$

where we may apply this to (13) to yield,

$$\begin{aligned}
[M_1]_t + \int_0^L\int_0^L \Bigg(& (1-a)Cu_t^2 + (1-a)Cv_t^2 + \left[(1-\frac{a}{2})C - \epsilon_2\right] w_t^2 \\
& + aCD_1\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aCD_1\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2 \\
& + aEC(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2 \\
& + \frac{aC}{2}\left(D - \epsilon_1 K^2\right)\left(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right)\Bigg) dx_1\, dx_2 \\
& \leq \int_0^L\int_0^L f_{x_3}^2\left(\frac{aC}{8\epsilon_1} + \frac{1}{4\epsilon_2}\right) dx_1\, dx_2
\end{aligned}$$

And by letting $\epsilon_1 = \frac{D}{2K^2}$ to imitate the form of M_1 we see,

$$\begin{aligned}
[M_1]_t + \int_0^L\int_0^L \Bigg(& (1-a)Cu_t^2 + (1-a)Cv_t^2 + \left[(1-\frac{a}{2})C - \epsilon_2\right] w_t^2 \\
& + aCD_1\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aCD_1\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2 \\
& + aEC(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2 \\
& + \frac{aCD}{4}\left(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right)\Bigg) dx_1\, dx_2 \\
& \leq \int_0^L\int_0^L f_{x_3}^2\left(\frac{aCK^2}{4D} + \frac{1}{4\epsilon_2}\right) dx_1\, dx_2
\end{aligned}$$

To shape M_2 into the form of M_1 the choice of ϵ_2 is much simpler. Let $\epsilon_2 = \frac{(1-\frac{a}{2})C}{2}$ then,

$$\begin{aligned}
[M_1]_t + C\int_0^L\int_0^L \Bigg(& (1-a)u_t^2 + (1-a)v_t^2 + \frac{1}{2}\left(1-\frac{a}{2}\right) w_t^2 \\
& + aD_1\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aD_1\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2 \\
& + aE(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2 \\
& + \frac{aD}{4}\left(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right)\Bigg) dx_1\, dx_2
\end{aligned}$$

$$\leq \int_0^L \int_0^L f_{x_3}^2 \left(\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)} \right) \, dx_1 \, dx_2$$

We now define the integral on the left hand side to be M_2*, and if one compares M_1 with M_2^* it is clear that,

$$M_2^* \leq 2M_1$$

3.4 Bounding the Equation

From here we will apply the inequality we have just found to solve for a bound M_2^* in terms of the L_2 norm of f_{x_3} as a function of time:

Recall that currently have the following inequality,

$$[M_1]_t + CM_2^* \leq \int_0^L \int_0^L f_{x_3}^2 \left(\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)} \right) \, dx_1 \, dx_2$$

Which we may integrate in time from zero to some time t to yield,

$$M_1(t) - M_1(0) + C \int_0^t M_2^*(\tau) \, d\tau \leq \left(\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)} \right) \int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau$$

where $|| \cdot ||_{L_2}$ is the L-2 norm. We may apply the fact that $M_2^* \leq 2M_1$ and see,

$$\frac{1}{2} M_2^*(t) - M_1(0) + C \int_0^t M_2^*(\tau) \, d\tau \leq \left(\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)} \right) \int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau$$

and simply rearranging a few terms produces this first order linear differential inequality.

$$M_2^*(t) + 2C \int_0^t M_2^*(\tau) \, d\tau \leq 2M_1(0) + \left(\frac{aCK^2}{2D} + \frac{2}{C\,(2-a)} \right) \int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau$$

Note we may solve this by means of multiplying by the integrating factor. This method produces,

$$\frac{d}{dt} \left[e^{2Ct} \int_0^t M_2^*(\tau) \, d\tau \right] \leq 2M_1(0) e^{2Ct} + e^{2Ct} \left(\frac{aCK^2}{2D} + \frac{2}{C\,(2-a)} \right) \int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau$$

and integrating in time from 0 to some time T gives us,

$$\left[e^{2Ct} \int_0^t M_2^*(\tau) \, d\tau \right]_0^T \leq \left[\frac{M_1(0)}{C} e^{2Ct} \right]_0^T + \int_0^T \left(e^{2Ct} \left(\frac{aCK^2}{2D} + \frac{2}{C\,(2-a)} \right) \int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau \right) \, dt$$

$$e^{2CT}\int_0^T M_2^*(\tau)\,d\tau \le \frac{M_1(0)}{C}\left[e^{2CT}-1\right]+\left[\frac{aCK^2}{2D}+\frac{2}{C\,(2-a)}\right]\int_0^T e^{2Ct}\left(\int_0^t ||f_{x_3}||_{L_2}^2\,d\tau\right)\,dt$$

where we may solve for $\int_0^T M_2^*(\tau)\,d\tau$ to yield,

$$\int_0^T M_2^*(\tau)\,d\tau \le \frac{M_1(0)}{C}\left[1-e^{-2CT}\right]+e^{-2CT}\left[\frac{aCK^2}{2D}+\frac{2}{C\,(2-a)}\right]\int_0^T e^{2Ct}\left(\int_0^t ||f_{x_3}||_{L_2}^2 d\tau\right)\,dt$$

and this gives us our theorem.

3.5 Theorem

We have shown that for any transversal force, f_{x_3}, the defined energy measure of our system satifies the following inequality for any given time T,

$$\int_0^T M_2^*(\tau)\,d\tau \le \frac{M_1(0)}{C}\left[1-e^{-2CT}\right]+e^{-2CT}\left[\frac{aCK^2}{2D}+\frac{2}{C\,(2-a)}\right]\int_0^T e^{2Ct}\left(\int_0^t ||f_{x_3}||_{L_2}^2 d\tau\right)\,dt$$

where $M_2^*(t)$ is defined as,

$$M_2^*(t)=\int_0^L\int_0^L\left((1-a)u_t^2+(1-a)v_t^2+\frac{1}{2}\left(1-\frac{a}{2}\right)w_t^2+aD_1\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+aD_1\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2\right.$$
$$\left.+aE(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})^2\frac{aD}{4}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\right)dx_1\,dx_2$$

This guarantees the stability of our system and accounts for many of the contributions of our system including most importantly the kinetic energies of all the separate components. With some conditioning on the force being applied we may immediately deduce two corollaries.

3.6 Corollary 1

If $||f_{x_3}||_{L_2}$ is bounded in time by a constant, F, then for all time T we have that,

$$\int_0^T M_2^*(\tau)\,d\tau \le \left(\frac{M_1(0)}{C}-\frac{F^2}{4C^2}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]\right)+\frac{F^2}{2C}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]T$$

By hypothesis we have that for all time $||f_{x_3}||_{L_2}$ is bounded is bounded by F and thus,

$$\int_0^t ||f_{x_3}||_{L_2}^2\,d\tau \le \int_0^t F^2\,d\tau = tF^2$$

And multiplying the above inequality by e^{2Ct} we see that for any time t,

$$e^{2Ct}\int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau \leq tF^2e^{2Ct}$$

Applying the same method as in the first step of the corollary we see that for all T,

$$\int_0^T e^{2Ct}\left(\int_0^t ||f_{x_3}||_{L_2}^2 \, d\tau\right) dt \leq \int_0^T tF^2e^{2Ct}\, dt = \frac{F^2}{4C^2}\left(e^{2CT}(2CT-1)+1\right)$$

We may now apply the theorem and receive that,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \frac{M_1(0)}{C}\left[1-e^{-2CT}\right] + e^{-2CT}\left[\frac{aCK^2}{2D}+\frac{2}{C\,(2-a)}\right]\frac{F^2}{4C^2}\left(e^{2CT}(2CT-1)+1\right)$$

And by rearranging a few term we find that,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \left(\frac{M_1(0)}{C}-\frac{F^2}{4C^2}\left[\frac{aCK^2}{2D}+\frac{2}{C\,(2-a)}\right]\right)\left(1-e^{-2CT}\right)+\frac{F^2}{2C}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]T$$

In addition we have $(1-e^{-2CT}) \leq 1$ for all $T \geq 0$ and therefore,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \left(\frac{M_1(0)}{C}-\frac{F^2}{4C^2}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]\right)+\frac{F^2}{2C}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]T$$

3.7 Corollary 2

If $\int_0^\infty ||f_{x_3}||_{L_2}^2\, dt = F$ then,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \left(\frac{M_1(0)}{C}+\frac{F}{2C}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]\right)$$

By applying the equality hypothesis to the theorem we have that,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \frac{M_1(0)}{C}\left[1-e^{-2CT}\right]+e^{-2CT}\left[\frac{aCK^2}{4D}+\frac{1}{C\,(2-a)}\right]\int_0^T Fe^{2Ct}\, dt$$

And by integrating we find that,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \frac{M_1(0)}{C}\left[1 - e^{-2CT}\right] + e^{-2CT}\left[\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)}\right]\frac{F}{2C}\left(e^{2CT} - 1\right)$$

By rearranging the terms we find,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \left(\frac{M_1(0)}{C} + \frac{F}{2C}\left[\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)}\right]\right)\left(1 - e^{-2CT}\right)$$

Therefore it follows that,

$$\int_0^T M_2^*(\tau)\, d\tau \leq \left(\frac{M_1(0)}{C} + \frac{F}{2C}\left[\frac{aCK^2}{4D} + \frac{1}{C\,(2-a)}\right]\right)$$

4 Appendix

This section is dedicated to many of the steps that were omitted in the previous sections for the purpose of clarity. The proofs follow in order of appearance throughout the derivation. Due to the symmetry of equations (1) and (2) we will omit the steps taken on (2) due to the similarity and symmetry that equations (1) and (2) share. In addition for the first two lemmas we have isolated the left hand sides to the terms including the force and the coefficient only for the purposes of avoiding Step 5 as it is no more than simple substitution and rearrangement.

4.1 Lemma 1

Multiply (1) by u_t and aCu respectively. This yields the following two equations.

$$\begin{aligned} aCuf_{x_1} &= aCuu_{tt} + aC^2uu_t - aCD_1u[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - aCEu[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \\ u_tf_{x_1} &= u_tu_{tt} + Cu_t^2 - D_1u_t[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - Eu_t[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \end{aligned}$$

From here we may add the equations together and simplify.

$$\begin{aligned} aCuf_{x_1} + u_tf_{x_1} &= aCuu_{tt} + u_tu_{tt} + aC^2uu_t + Cu_t^2 \\ &\quad -aCD_1u[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - D_1u_t[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} \\ &\quad -aCEu[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} - Eu_t[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \end{aligned}$$

$$\begin{aligned} f_{x_1}(aCu + u_t) &= aCuu_{tt} + \frac{1}{2}[u_t^2]_t + \frac{a}{2}C^2[u^2]_t + Cu_t^2 \\ &\quad -aCD_1u[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - D_1u_t[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} \\ &\quad -aCEu[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} - Eu_t[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \end{aligned}$$

4.2 Lemma 2

Multiply (3) by w_t and $\frac{1}{2}aKw$, respectively. This yields the following two equations.

$$\begin{aligned}
\frac{1}{2}aCwf_{x_3} &= \frac{1}{2}aCww_{tt} + \frac{1}{2}aC^2ww_t + \frac{1}{2}aCDw[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -\frac{1}{2}aCD_1w[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - \frac{1}{2}aCD_1w[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -\frac{1}{2}aCEw[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - \frac{1}{2}aCEw[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1} \\
w_tf_{x_3} &= w_tw_{tt} + Cw_t^2 + Dw_t[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -D_1w_t[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1w_t[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -Ew_t[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - Ew_t[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}
\end{aligned}$$

From here we may add the equations together and simplify.

$$\begin{aligned}
\frac{1}{2}aCwf_{x_3} + w_tf_{x_3} &= \frac{1}{2}aCww_{tt} + w_tw_{tt} + \frac{1}{2}aC^2ww_t + Cw_t^2 \\
&\quad +\frac{1}{2}aCDw[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad +Dw_t[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -\frac{1}{2}aCD_1w[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - \frac{1}{2}aCD_1w[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -D_1w_t[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1w_t[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -\frac{1}{2}aCEw[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - \frac{1}{2}aCEw[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1} \\
&\quad -Ew_t[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - Ew_t[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}
\end{aligned}$$

$$\begin{aligned}
f_{x_3}\left(\frac{1}{2}aCw + w_t\right) &= \frac{1}{2}aCww_{tt} + \frac{1}{2}[w_t^2]_t + \frac{1}{4}aC^2[w^2]_t + Cw_t^2 \\
&\quad +\frac{1}{2}aCDw[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad +Dw_t[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -\frac{1}{2}aCD_1w[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - \frac{1}{2}aCD_1w[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -D_1w_t[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1w_t[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -\frac{1}{2}aCEw[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - \frac{1}{2}aCEw[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1} \\
&\quad -Ew_t[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - Ew_t[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}
\end{aligned}$$

4.3 Lemma 3

Rewrite the left hand side of (4).

$$\begin{aligned}
aCuu_{tt} + \frac{1}{2}\left[u_t^2\right]_t + \frac{a}{2}C^2\left[u^2\right]_t + Cu_t^2 &= aCuu_{tt} + u_tu_{tt} + aC^2uu_t + Cu_t^2 \\
&= (u_{tt} + Cu_t)(aCu + u_t) \\
&= (u_{tt} + Cu_t)(au_t + aCu + u_t - au_t) \\
&= a(u_t + Cu)(u_{tt} + Cu_t) + (1-a)u_t(u_{tt} + Cu_t) \\
&= a(u_t + Cu)(u_{tt} + Cu_t) + (1-a)(u_tu_{tt} + Cu_t^2) \\
&= a(u_t + Cu)(u_{tt} + Cu_t) + (1-a)u_tu_{tt} + \\
&\quad (1-a)Cu_t^2 \\
&= a(u_t + Cu)\left[u_t + Cu\right]_t + (1-a)u_t[u_t]_t + \\
&\quad (1-a)Cu_t^2 \\
&= \frac{a}{2}\left[(u_t + Cu)^2\right]_t + \frac{1-a}{2}\left[u_t^2\right]_t + (1-a)Cu_t^2
\end{aligned}$$

4.4 Lemma 4

Rewrite the left hand side of (6).

$$\begin{aligned}
\frac{1}{2}aCww_{tt} + \frac{1}{2}[w_t^2]_t + \frac{1}{4}aC^2[w^2]_t + Cw_t^2 &= \frac{1}{2}aCww_{tt} + w_tw_{tt} + \frac{1}{2}aC^2ww_t + Cw_t^2 \\
&= (w_{tt} + Cw_t)(\frac{1}{2}aCw + w_t) \\
&= (w_{tt} + Cw_t)(\frac{1}{2}aw_t + \frac{1}{2}aCw + w_t - \frac{1}{2}aw_t) \\
&= \frac{a}{2}(w_t + Cw)(w_{tt} + Cw_t) + \left(1 - \frac{a}{2}\right)w_t(w_{tt} + Cw_t) \\
&= \frac{a}{2}(w_t + Cw)(w_{tt} + Cw_t) + \left(1 - \frac{a}{2}\right)w_tw_{tt} \\
&\quad + \left(1 - \frac{a}{2}\right)Cw_t^2 \\
&= \frac{a}{4}\left[(w_t + Cw)^2\right]_t + \frac{1}{2}\left(1 - \frac{a}{2}\right)[w_t^2]_t + \left(1 - \frac{a}{2}\right)Cw_t^2
\end{aligned}$$

The integration will be done and applied to each equation separately, then summed together. Consider firstly (10).

4.5 Lemma 5

$$\begin{aligned}
&\int_0^L \int_0^L -aCD_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1}u\,dx_1\,dx_2 \\
&\quad = -\int_0^L \left(\int_0^L aCD_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1}u dx_1\right)dx_2
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^L \left(\left[aCD_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] u \right]_0^L - \int_0^L aCD_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] u_{x_1} dx_1 \right) dx_2 \\
&= -\int_0^L \left(-\int_0^L aCD_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] u_{x_1} dx_1 \right) dx_2 \\
&= \int_0^L \int_0^L aCD_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] u_{x_1} \, dx_1 \, dx_2
\end{aligned}
$$

4.6 Lemma 6

$$
\begin{aligned}
&\int_0^L \int_0^L -D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} u_t \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \int_0^L D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} u_t \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\left[D_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] u_t \right]_0^L - \int_0^L D_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] [u_t]_{x_1} \right) dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(-\int_0^L D_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] [u_t]_{x_1} \, dx_1 \right) dx_2 \\
&\quad = \int_0^L \int_0^L D_1 \left[u_{x_1} + \frac{1}{2}(w_{x_1})^2 \right] [u_t]_{x_1} \, dx_1 \, dx_2
\end{aligned}
$$

4.7 Lemma 7

$$
\begin{aligned}
&\int_0^L \int_0^L -aCE[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \int_0^L aCE[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u \, dx_2 \, dx_1 \\
&\quad = -\int_0^L \left(\left[aCE \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] u \right]_0^L - \int_0^L aCE \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] u_{x_1} \right) dx_2 \, dx_1 \\
&\quad = -\int_0^L \left(-\int_0^L aCE \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] u_{x_2} dx_2 \right) dx_1 \\
&\quad = \int_0^L \int_0^L aCE \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] u_{x_2} \, dx_2 \, dx_1 \\
&\quad = \int_0^L \int_0^L aCE \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] u_{x_2} \, dx_1 \, dx_2
\end{aligned}
$$

4.8 Lemma 8

$$
\begin{aligned}
&\int_0^L \int_0^L -E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u_t \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u_t dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(\left[E \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] u_t \right]_0^L - \int_0^L E \left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2} \right] [u_t]_{x_2} \right) dx_2 \, dx_1
\end{aligned}
$$

$$= -\int_0^L \left(-\int_0^L E\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right] \left[u_t\right]_{x_2} dx_2 \right) dx_1$$
$$= \int_0^L \int_0^L E\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right] \left[u_t\right]_{x_2} \, dx_2 \, dx_1$$
$$= \int_0^L \int_0^L E\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right] \left[u_t\right]_{x_2} \, dx_1 \, dx_2$$

4.9 Lemma 9

By taking the integral of (2) we get,

$$\int_0^L \int_0^L f_{x_1}(aCu + u_t)\, dx_1\, dx_2 =$$
$$\int_0^L \int_0^L \left(\frac{a}{2}\left[(u_t + Cu)^2\right]_t + \frac{1-a}{2}\left[u_t^2\right]_t + (1-a)Cu_t^2 \right.$$
$$-aCD_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} u - D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} u_t$$
$$\left. -aCE[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u - E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} u_t \right) dx_1\, dx_2$$

and by the previous lemmas this is equal to,

$$\int_0^L \int_0^L f_{x_1}(aCu + u_t)\, dx_1\, dx_2 =$$
$$\int_0^L \int_0^L \left(\frac{a}{2}\left[(u_t + Cu)^2\right]_t + \frac{1-a}{2}\left[u_t^2\right]_t + (1-a)Cu_t^2 \right.$$
$$+aCD_1\left[u_{x_1} + \frac{1}{2}(w_{x_1})^2\right] u_{x_1} + D_1\left[u_{x_1} + \frac{1}{2}(w_{x_1})^2\right]\left[u_t\right]_{x_1}$$
$$\left. aCE\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right] u_{x_2} + E\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right]\left[u_t\right]_{x_2} \right) dx_1\, dx_2$$

4.10 Lemma 10

Noticing that the equations (10) and (11) are the same with the exception of u and v this leads us to the conclusion that,

$$\int_0^L \int_0^L f_{x_2}(aCv + v_t)\, dx_1\, dx_2 =$$
$$= \int_0^L \int_0^L \left(\frac{a}{2}\left[(v_t + Cv)^2\right]_t + \frac{1-a}{2}\left[v_t^2\right]_t + (1-a)Cv_t^2 \right.$$
$$aCD_1\left[v_{x_2} + \frac{1}{2}(w_{x_2})^2\right] v_{x_2} + D_1\left[v_{x_2} + \frac{1}{2}(w_{x_2})^2\right]\left[v_t\right]_{x_2}$$
$$\left. +aCE\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right] v_{x_1} + E\left[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}\right]\left[v_t\right]_{x_1} \right) dx_1\, dx_2$$

4.11 Lemma 11

This integration will be done and applied to equation (12).

$$
\begin{aligned}
&\int_0^L \int_0^L \frac{1}{2} aCD[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}]w \, dx_1 \, dx_2 \\
&= \frac{1}{2} aCD \int_0^L \int_0^L \left[w_{x_1x_1x_1}\right]_{x_1} w \, dx_1 \, dx_2 + 2D \int_0^L \int_0^L \left[w_{x_1x_1x_2}\right]_{x_2} w \, dx_2 \, dx_1 \\
&\quad + D \int_0^L \int_0^L \left[w_{x_2x_2x_2}\right]_{w_2} w \, dx_2 \, dx_1 \\
&= \frac{1}{2} aCD \int_0^L \left(\left[\left[w_{x_1x_1x_1}\right] w\right]_0^L - \int_0^L \left[w_{x_1x_1x_1}\right] \left[w\right]_{x_1} dx_1 \right) dx_2 \\
&\quad + aCD \int_0^L \left(\left[\left[w_{x_1x_1x_2}\right] w\right]_0^L - \int_0^L \left[w_{x_1x_1x_2}\right] \left[w\right]_{x_2} dx_2 \right) dx_1 \\
&\quad + \frac{1}{2} aCD \int_0^L \left(\left[\left[w_{x_2x_2x_2}\right] w\right]_0^L - \int_0^L \left[w_{x_2x_2x_2}\right] \left[w\right]_{x_2} dx_2 \right) dx_1 \\
&= -\frac{1}{2} aCD \int_0^L \int_0^L \left[w_{x_1x_1}\right]_{x_1} \left[w\right]_{x_1} \, dx_1 \, dx_2 - aCD \int_0^L \int_0^L \left[w_{x_1x_2}\right]_{x_1} \left[w\right]_{x_2} \, dx_1 \, dx_2 \\
&\quad - \frac{1}{2} aCD \int_0^L \int_0^L \left[w_{x_2x_2}\right]_{x_2} \left[w\right]_{x_2} \, dx_2 \, dx_1 \\
&= -\frac{1}{2} aCD \int_0^L \left(\left[w_{x_1x_1}\right] \left[w_{x_1}\right]_0^L - \int_0^L \left[w_{x_1x_1}\right] \left[w_{x_1x_1}\right] dx_1 \right) dx_2 \\
&\quad - aCD \int_0^L \left(\left[w_{x_1x_2}\right] \left[w_{x_2}\right]_0^L - \int_0^L \left[w_{x_1x_2}\right] \left[w_{x_1x_2}\right] dx_1 \right) dx_2 \\
&\quad - \frac{1}{2} aCD \int_0^L \left(\left[w_{x_2x_2}\right] \left[w_{x_2}\right]_0^L - \int_0^L \left[w_{x_2x_2}\right] \left[w_{x_2x_2}\right] dx_2 \right) dx_1 \\
&= \frac{1}{2} aCD \int_0^L \int_0^L \left[w_{x_1x_1}\right] \left[w_{x_1x_1}\right] \, dx_1 \, dx_2 + aCD \int_0^L \int_0^L \left[w_{x_1x_2}\right] \left[w_{x_1x_2}\right] \, dx_1 \, dx_2 \\
&\quad + \frac{1}{2} aCD \int_0^L \int_0^L \left[w_{x_2x_2}\right] \left[w_{x_2x_2}\right] \, dx_1 \, dx_2 \\
&= aCD \int_0^L \int_0^L \left(\frac{1}{2} \left[w_{x_1x_1}^2\right] + \left[w_{x_1x_2}^2\right] + \frac{1}{2} \left[w_{x_2x_2}^2\right] \right) \, dx_1 \, dx_2 \\
&= \int_0^L \int_0^L \frac{1}{2} aCD \left[w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right] \, dx_1 \, dx_2
\end{aligned}
$$

4.12 Lemma 12

$$
\begin{aligned}
&\int_0^L \int_0^L D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}]w_t \, dx_1 \, dx_2 \\
&\quad = D\int_0^L \int_0^L \left[w_{x_1x_1x_1}\right]_{x_1} w_t \, dx_1 \, dx_2 + 2D\int_0^L \int_0^L \left[w_{x_1x_1x_2}\right]_{x_2} w_t \, dx_2 \, dx_1 + D\int_0^L \int_0^L \left[w_{x_2x_2x_2}\right]_{w_2} w_t \, dx_2 \, dx_1 \\
&\quad = D\int_0^L \left(\left[\left[w_{x_1x_1x_1}\right] w_t\right]_0^L - \int_0^L \left[w_{x_1x_1x_1}\right] \left[w_t\right]_{x_1} dx_1 \right) dx_2 \\
&\qquad +2D\int_0^L \left(\left[\left[w_{x_1x_1x_2}\right] w_t\right]_0^L - \int_0^L \left[w_{x_1x_1x_2}\right] \left[w_t\right]_{x_2} dx_2 \right) dx_1 \\
&\qquad\quad +D\int_0^L \left(\left[\left[w_{x_2x_2x_2}\right] w_t\right]_0^L - \int_0^L \left[w_{x_2x_2x_2}\right] \left[w_t\right]_{x_2} dx_2 \right) dx_1 \\
&\quad = -D\int_0^L \int_0^L \left[w_{x_1x_1}\right]_{x_1} \left[w_t\right]_{x_1} \, dx_1 \, dx_2 \\
&\qquad -2D\int_0^L \int_0^L \left[w_{x_1x_2}\right]_{x_1} \left[w_t\right]_{x_2} \, dx_1 \, dx_2 \\
&\qquad\quad -D\int_0^L \int_0^L \left[w_{x_2x_2}\right]_{x_2} \left[w_t\right]_{x_2} \, dx_2 \, dx_1 \\
&\quad = -D\int_0^L \left(\left[w_{x_1x_1}\right] \left[w_{tx_1}\right]_0^L - \int_0^L \left[w_{x_1x_1}\right] \left[w_{tx_1x_1}\right] dx_1 \right) dx_2 \\
&\qquad -2D\int_0^L \left(\left[w_{x_1x_2}\right] \left[w_{tx_2}\right]_0^L - \int_0^L \left[w_{x_1x_2}\right] \left[w_{tx_1x_2}\right] dx_1 \right) dx_2 \\
&\qquad\quad -D\int_0^L \left(\left[w_{x_2x_2}\right] \left[w_{tx_2}\right]_0^L - \int_0^L \left[w_{x_2x_2}\right] \left[w_{tx_2x_2}\right] dx_2 \right) dx_1 \\
&\quad = D\int_0^L \int_0^L \left[w_{x_1x_1}\right] \left[w_{x_1x_1}\right]_t \, dx_1 \, dx_2 \\
&\qquad +2D\int_0^L \int_0^L \left[w_{x_1x_2}\right] \left[w_{x_1x_2}\right]_t \, dx_1 \, dx_2 \\
&\qquad\quad +D\int_0^L \int_0^L \left[w_{x_2x_2}\right] \left[w_{x_2x_2}\right]_t \, dx_1 \, dx_2 \\
&\quad = D\int_0^L \int_0^L \left(\frac{1}{2}\left[w_{x_1x_1}^2\right]_t + \left[w_{x_1x_2}^2\right]_t + \frac{1}{2}\left[w_{x_2x_2}^2\right]_t \right) dx_1 \, dx_2 \\
&\quad = \int_0^L \int_0^L \frac{D}{2}\left[w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right]_t \, dx_1 \, dx_2
\end{aligned}
$$

4.13 Lemma 13

$$\begin{aligned}
&\int_0^L \int_0^L -\frac{1}{2} aCD_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} w \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L \frac{1}{2} aCD_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} w dx_1 \right) dx_2 \\
&\quad = -\int_0^L \left(\left[\frac{1}{2} aCD_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] w \right]_0^L - \int_0^L \frac{1}{2} aCD_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] [w]_{x_1} \, dx_1 \right) dx_2 \\
&\quad = -\int_0^L \left(-\int_0^L \frac{1}{2} aCD_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] [w]_{x_1} \, dx_1 \right) dx_2 \\
&\quad = \int_0^L \int_0^L \frac{1}{2} aCD_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] [w]_{x_1} \, dx_1 \, dx_2
\end{aligned}$$

4.14 Lemma 14

$$\begin{aligned}
&\int_0^L \int_0^L -\frac{1}{2} aCD_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} w \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L \frac{1}{2} aCD_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} w dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(\left[\frac{1}{2} aCD_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] w \right]_0^L - \int_0^L \frac{1}{2} aCD_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w]_{x_2} \, dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(-\int_0^L \frac{1}{2} aCD_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w]_{x_2} \, dx_2 \right) dx_1 \\
&\quad = \int_0^L \int_0^L \frac{1}{2} aCD_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w]_{x_2} \, dx_2 \, dx_1 \\
&\quad = \int_0^L \int_0^L \frac{1}{2} aCD_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w]_{x_2} \, dx_1 \, dx_2
\end{aligned}$$

4.15 Lemma 15

$$\begin{aligned}
&\int_0^L \int_0^L -D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} w_t \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} w_t dx_1 \right) dx_2 \\
&\quad = -\int_0^L \left(\left[D_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] w_t \right]_0^L - \int_0^L D_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] [w_t]_{x_1} \, dx_1 \right) dx_2 \\
&\quad = -\int_0^L \left(-\int_0^L D_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] [w_t]_{x_1} \, dx_1 \right) dx_2 \\
&\quad = \int_0^L \int_0^L D_1 \left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2) \right] [w_t]_{x_1} \, dx_1 \, dx_2
\end{aligned}$$

4.16 Lemma 16

$$
\begin{aligned}
&\int_0^L \int_0^L -D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} w_t \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} w_t dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(\left[D_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] w_t \right]_0^L - \int_0^L D_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w_t]_{x_2} \, dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(- \int_0^L D_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w_t]_{x_2} \, dx_2 \right) dx_1 \\
&\quad = \int_0^L \int_0^L D_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w_t]_{x_2} \, dx_2 \, dx_1 \\
&\quad = \int_0^L \int_0^L D_1 \left[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2) \right] [w_t]_{x_2} \, dx_1 \, dx_2
\end{aligned}
$$

4.17 Lemma 17

$$
\begin{aligned}
&\int_0^L \int_0^L -\frac{1}{2} aCE[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_2} w \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L \frac{1}{2} aCE[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_2} w dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(\left[\frac{1}{2} aCE \left[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] w \right]_0^L - \int_0^L \frac{1}{2} aCE \left[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_2} \, dx_2 \right) dx_1 \\
&\quad = -\int_0^L \left(- \int_0^L \frac{1}{2} aCE \left[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_2} \, dx_2 \right) dx_1 \\
&\quad = \int_0^L \int_0^L \frac{1}{2} aCE \left[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_2} \, dx_2 \, dx_1 \\
&\quad = \int_0^L \int_0^L \frac{1}{2} aCE \left[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_2} \, dx_1 \, dx_2
\end{aligned}
$$

4.18 Lemma 18

$$
\begin{aligned}
&\int_0^L \int_0^L -\frac{1}{2} aCE[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_1} w \, dx_1 \, dx_2 \\
&\quad = -\int_0^L \left(\int_0^L \frac{1}{2} aCE[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_1} w dx_1 \right) dx_2 \\
&\quad = -\int_0^L \left(\left[\frac{1}{2} aCE \left[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] w \right]_0^L - \int_0^L \frac{1}{2} aCE \left[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_1} \, dx_1 \right) dx_2 \\
&\quad = -\int_0^L \left(- \int_0^L \frac{1}{2} aCE \left[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_1} \, dx_1 \right) dx_2
\end{aligned}
$$

$$= \int_0^L \int_0^L \frac{1}{2} aCE \left[w_{x_2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w]_{x_1} \, dx_1 \, dx_2$$

4.19 Lemma 19

$$\int_0^L \int_0^L -E[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_2} w_t \, dx_1 \, dx_2$$
$$= -\int_0^L \left(\int_0^L E[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_2} w_t dx_2 \right) dx_1$$
$$= -\int_0^L \left(\left[E \left[w_{x_1} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] w_t \right]_0^L - \int_0^L E \left[w_{x_1} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_2} \, dx_2 \right) dx_1$$
$$= -\int_0^L \left(- \int_0^L E \left[w_{x_1} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_2} \, dx_2 \right) dx_1$$
$$= \int_0^L \int_0^L E \left[w_{x_1} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_2} \, dx_2 \, dx_1$$
$$= \int_0^L \int_0^L E \left[w_{x_1} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_2} \, dx_1 \, dx_2$$

4.20 Lemma 20

$$\int_0^L \int_0^L -E[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_1} w_t \, dx_1 \, dx_2$$
$$= -\int_0^L \left(\int_0^L E[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_1} w_t dx_1 \right) dx_2$$
$$= -\int_0^L \left(\left[E \left[w_{x_2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] w_t \right]_0^L - \int_0^L E \left[w_{x_2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_1} \, dx_1 \right) dx_2$$
$$= -\int_0^L \left(- \int_0^L E \left[w_{x_2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_1} \, dx_1 \right) dx_2$$
$$= \int_0^L \int_0^L E \left[w_{x_2} (u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}) \right] [w_t]_{x_1} \, dx_1 \, dx_2$$

4.21 Lemma 21

By taking the integral of (3) we get,

$$\int_0^L \int_0^L f_{x_3}\left(\frac{1}{2}aCw + w_t\right) dx_1\, dx_2 =$$
$$\int_0^L \int_0^L \left(\frac{a}{4}\left[(w_t + Cw)^2\right]_t + \frac{1}{2}\left(1 - \frac{a}{2}\right)[w_t^2]_t + \left(1 - \frac{a}{2}\right)Cw_t^2\right.$$
$$+\frac{1}{2}aCD[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}]w + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}]w_t$$
$$-\frac{1}{2}aCD_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1}w - \frac{1}{2}aCD_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2}w$$
$$-D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1}w_t - D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2}w_t$$
$$-\frac{1}{2}aCE[w_{x_1}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})]_{x_2}w - \frac{1}{2}aCE[w_{x_2}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})]_{x_1}w$$
$$\left.-E[w_{x_1}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})]_{x_2}w_t - E[w_{x_2}(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})]_{x_1}w_t\right) dx_1\, dx_2$$

and by the previous lemmas this is equal to,

$$\int_0^L \int_0^L f_{x_3}\left(\frac{1}{2}aCw + w_t\right) dx_1\, dx_2 =$$
$$\int_0^L \int_0^L \left(\frac{a}{4}\left[(w_t + Cw)^2\right]_t + \frac{1}{2}\left(1 - \frac{a}{2}\right)[w_t^2]_t + \left(1 - \frac{a}{2}\right)Cw_t^2\, dx_1\, dx_2\right.$$
$$+\frac{1}{2}aCD\left[w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right] + \frac{D}{2}\left[w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2\right]_t$$
$$+\frac{1}{2}aCD_1\left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)\right][w_{x_1} + w_{x_2}] + D_1\left[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)\right][w_{x_1} + w_{x_2}]_t$$
$$\left.+aCE(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})[w_{x_1}w_{x_2}] + E(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})[w_{x_1}w_{x_2}]_t\right) dx_1\, dx_2$$

4.22 Lemma 22

Applying lemmas (9),(10), and (21) and adding the equations together factoring out the time derivative we see that,

$$\int_0^L\int_0^L f_{x_1}(aCu+u_t)+f_{x_2}(aCv+v_t)+f_{x_3}\left(\frac{1}{2}aCw+w_t\right)dx_1\,dx_2=$$
$$\int_0^L\int_0^L\left(\frac{d}{dt}\left[\frac{a}{2}(u_t+Cu)^2+\frac{1-a}{2}u_t^2+\frac{a}{2}(v_t+Cv)^2+\frac{1-a}{2}v_t^2+\frac{a}{4}(w_t+Cw)^2+\frac{1}{2}\left(1-\frac{a}{2}\right)w_t^2\right]\right.$$
$$(1-a)Cu_t^2+(1-a)Cv_t^2+\left(1-\frac{a}{2}\right)Cw_t^2+\frac{1}{2}aCD\left[w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right]$$
$$+\frac{D}{2}\left[w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right]_t$$
$$+aCD_1\left[u_{x_1}+\frac{1}{2}(w_{x_1})^2\right]u_{x_1}+aCD_1\left[v_{x_2}+\frac{1}{2}(w_{x_2})^2\right]v_{x_2}+\frac{1}{2}aCD_1\left[w_{x_1}(u_{x_1}+\frac{1}{2}(w_{x_1})^2)\right][w_{x_1}+w_{x_2}]$$
$$+D_1\left[u_{x_1}+\frac{1}{2}(w_{x_1})^2\right][u_t]_{x_1}+D_1\left[v_{x_2}+\frac{1}{2}(w_{x_2})^2\right][v_t]_{x_2}+D_1\left[w_{x_1}(u_{x_1}+\frac{1}{2}(w_{x_1})^2)\right][w_{x_1}+w_{x_2}]_t$$
$$+aCE\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right]u_{x_2}+aCE\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right]v_{x_1}+aCE(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})\left[w_{x_1}w_{x_2}\right]$$
$$\left.+E\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right][u_t]_{x_2}+E\left[u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right][v_t]_{x_1}+E(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2})\left[w_{x_1}w_{x_2}\right]_t\right)dx_1\,dx_2$$

After factoring and applying the reverse chain rule we are ultimately left with,

$$\frac{d}{dt}\left\{\int_0^L\int_0^L\left[\frac{a}{2}(u_t+Cu)^2+\frac{a}{2}(v_t+Cv)^2+\frac{a}{4}(w_t+Cw)^2+\frac{1-a}{2}u_t^2+\frac{1-a}{2}v_t^2+\left(\frac{1}{2}-\frac{a}{4}\right)w_t^2\right.\right.$$
$$\frac{D_1}{2}\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+\frac{D_1}{2}\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2+\frac{E}{2}\left(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right)^2$$
$$\left.\left.\frac{D}{2}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\right]dx_1\,dx_2\right\}$$
$$+\int_0^L\int_0^L\left[(1-a)Cu_t^2+(1-a)Cv_t^2+\left(1-\frac{a}{2}\right)Cw_t^2\right.$$
$$+aCD_1\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+aCD_1+\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2$$
$$\left.+aEC\left(u_{x_2}+v_{x_1}+w_{x_1}w_{x_2}\right)^2+\frac{aDC}{2}\left(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2\right)\right]dx_1\,dx_2$$
$$=\int_0^L\int_0^L\left[f_{x_1}(u_t+aCu)+f_{x_2}(v_t+aCv)+f_{x_3}\left(w_t+\frac{aCw}{2}\right)\right]dx_1\,dx_2$$

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