Multiplication Operators on the Lipschitz Space of a Tree

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Given a complex Banach space X consisting of functions defined on a set Ω and a complex valued function ψ defined on Ω, the operator

$$M_\psi f = \psi f \text{ for all } f \in X$$

is the multiplication operator with symbol ψ.

Ω is an infinite tree T, which is a metric space with distance counting the number of edges linking two vertices

X is the space $\mathcal{L}$ of complex valued Lipschitz functions on T

$\mathcal{L}$ is a discrete analog of the Banach space of analytic functions on $\mathbb{D}$, the Bloch space $\mathcal{B}$

Bloch Space

$\mathcal{B}(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ such that

$$(1-|z|^2)|f'(z)| < \infty$$

$||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$

$\mathcal{B}_0(\mathbb{D})$ is the set of Bloch functions such that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$$

Lipschitz Functions

Where do Lipschitz functions come in?

$\forall z, w \in \mathbb{D}, \;\; f \in \mathcal{B}(\mathbb{D}) \iff \exists C > 0 \mid |f(z) - f(w)| \leq C\rho(z, w)$

ρ is the Bergman metric

$\rho(x, y) = \inf(\mathit{length}(\gamma))$, where $\gamma(0) = x$ and $\gamma(1) = y$ and $\gamma \in C^1$

Infinite Trees

Infinite trees are essentially discretizations of the hyperbolic disk in $\mathbb{C}$.

$\mathcal{L}$ is the space of all complex valued functions f on T such that

$$|f(v) - f(u)| \leq Cd(v, u)$$

Recall $d(v, u)$ is the number of edges linking u and v.

Originality

Harmonic functions in this case mean satisfying the mean value property.

Homogeneous trees are trees with each vertex having the same number of neighbors.

The Lipschitz space of T can be though of as the space $\mathcal{L}$ of the functions f acting on T such that $Df \in L^{\infty}$.

For a Lipschitz function f on T, let $||f||_{\mathcal{L}} = |f(o)| + ||Df||_{\infty}$.

$Df(v) = |f(v) - f(v^{-})|$ for $v \neq o$

Theorem 2.1

The correspondence $f \in \mathcal{L} \mapsto ||f||_{\mathcal{L}}$ is a norm which endows $\mathcal{L}$ with a complex Banach space structure.

Proof for Norm

First, show $||f||_{\mathcal{L}}$ is a norm.

The mapping $f \mapsto ||Df||_\infty$ is a semi-norm.

We must show $||f||_{\mathcal{L}} = 0 \iff f = 0$.

If $||f||_{\mathcal{L}} = 0$, then $||Df||_\infty = 0$ and $|f(o)| = 0$, so f must vanish.

Clearly, $||0||_{\mathcal{L}} = 0$.

Therefore $||f||_{\mathcal{L}}$ is a norm.

Proof Sketch for Completeness

Now, show $\mathcal{L}$ is complete, since $\mathcal{L}$ is a complex vector space with a norm.

Let f_n be a Cauchy sequence of functions in $\mathcal{L}$.

We show f_n converges pointwise to some function f on T.

Then we show f is Lipschitz.

Finally, we show $||f_n - f||_{\mathcal{L}} \to 0$ as $n \to \infty$.

Little Lipschitz Space

The little Lipschitz space is the subspace $\mathcal{L}_0$ of $\mathcal{L}$ consisting of all functions f on T such that $\lim_{|v|\to\infty} Df(v) = 0$.

Theorem 2.3

$\mathcal{L}_0$ is a closed and separable subspace of $\mathcal{L}$

Lemma 3.1

For $v \in T$, $\sup_{f \in \mathcal{L}}(|f(v)| : f(o) = 0, ||f||_{\mathcal{L}} \leq 1) = |v|$

This is also true for $f \in \mathcal{L}_0$

Boundedness of Multiplication Operators

A Banach space is functional if $\forall w \in \Omega,\ \exists C > 0$ such that $|f(w)| \leq C$, for all $f \in X$ such that $||f|| = 1$.

Lemma 3.2

Let X be a functional Banach space on the set Ω and let ψ be a complex valued function on Ω such that $M_\psi : X \mapsto X$. Then M_ψ is bounded on X and $\forall w \in \Omega,\ \ |\psi(w)| \leq |M_\psi|$. In particular ψ is bounded.

Proposition 3.3

$\mathcal{L}$ is a functional Banach space

Let $||f||_{\mathcal{L}} = 1$, and let $g(v) = f(v) - f(o)$. Then $g(o) = 0$, so we apply Lemma 3.1.

$|f(v)| \leq |f(o)| + |g(v)| \leq 1 + |v|$

Lemma 3.4

For $f \in \mathcal{L}$, $|f(v)| \leq |f(o)| + |v| * ||Df||_{\infty}$

If $||f||_{\mathcal{L}} \leq 1$, then $|f(v)| \leq |v|$.

For $f \in \mathcal{L}_0$, then $\lim_{|v| \to \infty} \frac{f(v)}{|v|} = 0$.

Boundedness of Multiplication Operators

Define $\sigma_\psi = \sup_{v \neq o}(|v| D\psi(v))$. Note that $\sigma_\psi < \infty \implies \psi \in \mathcal{L}_0$.

Theorem 3.6

Let T be a tree and ψ a function on T. Then the following are equivalent:

M_ψ is bounded on $\mathcal{L}$
M_ψ is bounded on $\mathcal{L}_0$
$\psi \in L^\infty$ and $\sigma_\psi < \infty$

Theorem 4.1

If M_ψ is a bounded multiplication operator on $\mathcal{L}$ or $\mathcal{L}_0$, then

$$max(||\psi||_{\mathcal{L}}, ||\psi||_\infty) \leq ||M_\psi|| \leq ||\psi||_\infty + \sigma_\psi.$$

These estimates are sharp.

Bibliography

[1] Colonna, Flavia and Easley, Glenn R. "Multiplication Operators on the Lipschitz Space of a Tree." Integral Equations and Operator Theory, 68 (2010), 391-411