Mathematical Modeling of Large Deformations on a Non-Linear Plate

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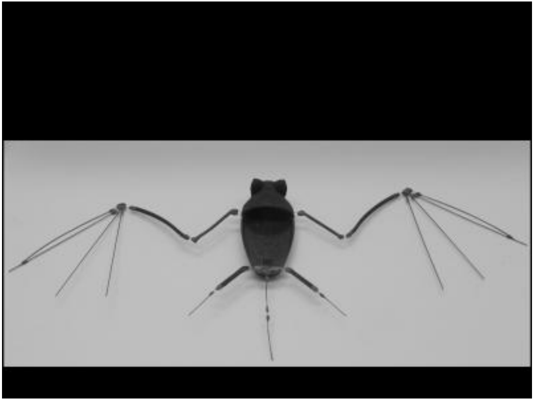
Rigid Wing

Rigid Wing

Multiple Rotor

Multiple Membrane Rotor

Biomimicry



Idea

First, we present the development of the mathematical model for the dynamic behavior of a nonlinear plate undergoing deformation both in transverse and axial directions using a Hamiltonian approach. We will end up with a system of coupled partial differential equations that equate to the fluid forces acting on the wing.

Kinematics

We use the Kirchhoff hypothesis for the deformation (u_i) of the plate.

Assumptions

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation

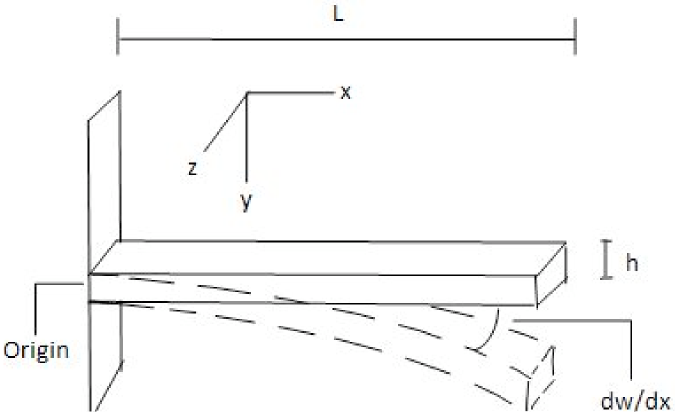
Equations

$$u_1 = u - x_3 w_{x_1}$$
$$u_2 = v - x_3 w_{x_2}$$
$$u_3 = w$$

Terms

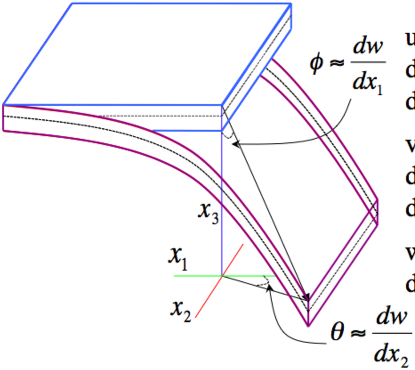
u = axial displacement in the x_1 direction
v = axial displacement in the x_2 direction
w = transverse displacement

This is an example from a beam model, which we will ultimately compare our model against:



[Hickman (2010)]

This is what we are actually working with:



u is axial displacement in x_1 direction.

v is axial displacement in x_2 direction.

w is transverse displacement.

Strain-Displacement

We use the Green strain tensor to relate strain (E_{ij}) and displacement as follows:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

$$E_{11} = \frac{1}{2}(2(u_{x_1} - x_3 w_{x_1 x_1}) + w_{x_1}^2)$$
$$E_{22} = \frac{1}{2}(2(v_{x_2} - x_3 w_{x_2 x_2}) + w_{x_2}^2)$$
$$E_{12} = E_{21} = \frac{1}{2}(u_{x_2} + v_{x_1} - 2x_3 w_{x_1 x_2}) + w_{x_1} w_{x_2})$$

Stress-Strain

We use the a materially linear formulation to relate stress (σ_{ij}) and strain using Young's modulus (Y) and a Poisson ratio (ν) as follows:

$$\sigma_{11} = \frac{Y}{(1-\nu^2)}(E_{11} + \nu E_{22})$$
$$\sigma_{22} = \frac{Y}{(1-\nu^2)}(E_{22} + \nu E_{11})$$
$$\sigma_{12} = \sigma_{21} = \frac{1-\nu}{2}\frac{Y}{(1-\nu^2)}E_{12}$$

Kinetic Energy

For a homogeneous plate density ρ, to account for all the mass T takes the form of an integral over the area. We also drop the inertial term.

$$T = \frac{1}{2}m||V||^2$$
$$T = \int_0^a \int_0^a \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{\rho}{2}(\frac{\partial u_1}{\partial t}^2 + \frac{\partial u_2}{\partial t}^2 + \frac{\partial u_3}{\partial t}^2)dx_3 dx_2 dx_1$$

Terms

$$u_1 = u - x_3 w_{x_1}$$
$$u_2 = v - x_3 w_{x_2}$$
$$u_3 = w$$

$$T = \int_0^a \int_0^a \frac{\rho h}{2}(u_t^2 + v_t^2 + w_t^2)dx_2 dx_1$$

Potential Energy

The remaining potential energy is similar to a compressed spring, as follows:

$$
\begin{aligned}
U &= \int_0^a \int_0^a \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{1}{2}(\sigma_{11}E_{11} + \sigma_{22}E_{22} + \sigma_{12}E_{12})dx_3 dx_2 dx_1 \\
&= \frac{Y}{(1-\nu^2)} \int_0^a \int_0^a h((u_{x_1} + \frac{1}{2}w_{x_1}^2)^2 + (v_{x_2} + \frac{1}{2}w_{x_2}^2)^2 \\
&+ \frac{1-\nu}{2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2) \\
&+ \frac{h^3}{12}(w_{x_1}w_{x_1}^2 + w_{x_2}w_{x_2} + w_{x_1}w_{x_2}^2)dx_2 dx_1
\end{aligned}
$$

Potential Energy of External Applied Forces

The potential energy of the external applied forces will be defined as the negative of the work done by fluid forces acting on the plate, where K is a damping constant.

$$A = u(f_1 - Ku_t) + v(f_2 - Kv_t) + w(f_3 - Kw_t)$$

Hamiltonian Method

According to Hamilton's principle, the progression of all physical systems minimizes the time integral of the Lagrangian, which is to say the variation of the Lagrangian will always be zero, i.e.

$$\delta \int_{t_0}^{t_1} [(T - U) + A] dt = 0$$

Expanding this integral, we obtain the governing equations.

Hamiltonian, continued

$$
\begin{aligned}
0 &= \delta \int_{t_0}^{t_1} \int_0^a \int_0^a \frac{\rho h}{2}(u_t^2 + v_t^2 + w_t^2) \\
&- \frac{Yh}{(1-\nu^2)}((u_{x_1} + \frac{1}{2}w_{x_1}^2)^2 + (v_{x_2} + \frac{1}{2}w_{x_2}^2)^2 \\
&+ \frac{1-\nu}{2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2) \\
&+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1}w_{x_1}^2 + w_{x_2}w_{x_2} + w_{x_1}w_{x_2}^2) \\
&+ u(f_1 - Ku_t) + v(f_2 - Kv_t) + w(f_3 - Kw_t)dx_2dx_1dt
\end{aligned}
$$

Using integration by parts to handle each term, the variation and the first spacial and temporal derivatives of the variation are zero at the limits of integration, therefore each boundary term is cancelled. After collecting all of the terms with contain δu, δv, δw, we can separate the integral into three parts as follows:

Calculus of Variation - δu component

$$\begin{aligned} 0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta u(-\rho h u_{tt} + \frac{Yh}{(1-\nu^2)}([u_{x_1} + \frac{1}{2}w_{x_1}^2]_{x_1} \\ &+ \frac{1-\nu}{2}[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2}) + Ku_t + f_1)dx_2dx_1dt \end{aligned}$$

Calculus of Variation - δv component

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta v\Big(-\rho h v_{tt} + \frac{Yh}{(1-\nu^2)}\Big([v_{x_2} + \frac{1}{2}w_{x_2}^2]_{x_2} \\
&+ \frac{1-\nu}{2}[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_1}\Big) + Kv_t + f_2\Big)dx_2\,dx_1\,dt
\end{aligned}
$$

Calculus of Variation - δw component

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta w(-\rho h w_{tt} + \frac{Yh}{(1-\nu^2)}([w_{x_1}(u_{x_1} + \frac{1}{2}w_{x_1}^2)]_{x_1} \\
&+ [w_{x_2}(v_{x_2} + \frac{1}{2}w_{x_2}^2)]_{x_2} + \frac{1-\nu}{2}[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} \\
&+ [w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}) \\
&+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1x_1x_1x_1} + w_{x_2x_2x_2x_2} + 2w_{x_1x_1x_2x_2}) \\
&+ Kw_t + f_3)dx_2dx_1dt
\end{aligned}
$$

Final Model

$$
\begin{aligned}
f_1 &= u_{tt} + Cu_t - D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} \\
f_2 &= v_{tt} + Cv_t - D_1[v_{x_2} + \frac{1}{2}(w_{x_2})^2]_{x_2} - E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_1} \\
f_3 &= w_{tt} + Cw_t + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -E[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_2} - E[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})]_{x_1}
\end{aligned}
$$

where C, D, E, and D_1 are all constants depending on the system.

Idea

We will show for any transversal force f_3 the energy of the system changes proportionally to the force. In other words, our choice of initial conditions won't cause the system to experience *flutter* or other disastrous instabilities. This will be proven for a class of boundary conditions by my collaborator, Charles Daly, though you can email us for the theorem, corollaries, and their respective proofs.

Idea

We now want to reduce our coupled model to an uncoupled model, to make discretizing it and analyzing it numerically much simpler. We will conclude by discussing a model for the transversal force f_3, as before.

Assumptions

$$
\begin{aligned}
h_1(t) &= u_{x_1} + \frac{1}{2}(w_{x_1})^2 \\
h_2(t) &= v_{x_2} + \frac{1}{2}(w_{x_2})^2 \\
h_3(t) &= u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}
\end{aligned}
$$

[Ferguson 2006]

Consequences for Axial Forces

The axial forces are symmetric, so we will only consider f_1.

$$
\begin{aligned}
f_1 &= u_{tt} + Cu_t - D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - E[u_{x_2} + v_{x_1} + w_{x_1} w_{x_2}]_{x_2} \\
f_1 &= u_{tt} + Cu_t - D_1[h_1(t)]_{x_1} - E[h_3(t)]_{x_2} \\
f_1 &= u_{tt} + Cu_t
\end{aligned}
$$

The damping term should reduce the fluid force to zero as time goes to infinity.

Consequences for Transverse Force

$$
\begin{aligned}
f_3 &= w_{tt} + Cw_t + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad -E[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - E[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}
\end{aligned}
$$

$$
\begin{aligned}
f_3 &= w_{tt} + Cw_t + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -D_1[w_{x_1}h_1(t)]_{x_1} - D_1[w_{x_2}h_2(t)]_{x_2} \\
&\quad -E[w_{x_1}h_3(t)]_{x_2} - E[w_{x_2}h_3(t)]_{x_1}
\end{aligned}
$$

Consequences for Transverse Force

We assume we can rearrange the mixed partial derivatives because the system is symmetric.

$$
\begin{aligned}
f_3 &= w_{tt} + Cw_t + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad -D_1h_1(t)[w_{x_1x_1} - D_1h_2(t)[w_{x_2x_2}] - 2Eh_3(t)[w_{x_1x_2}]
\end{aligned}
$$

Future Work

- Continue numerical validation using finite difference method and finite element method
- Error convergence
- Nonlinear material constitutive law
- Approximation of other nonlinearities
- Parameter identification studies to validate the model
- Allow f_1 and f_2 to be nontrivial
- Allow $\int_0^t ||f_3||_{L_2}^2 d\tau$ to be bounded or constant
- Couple the structural model to a fluid model

References

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Contact Information

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