Mathematical Modeling of Large Deformations on a Non-Linear Plate

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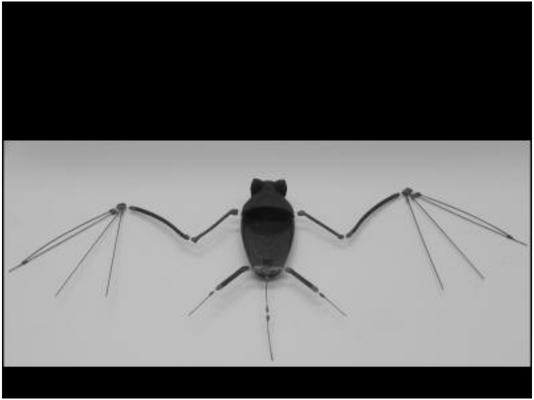
Rigid Wing

Rigid Wing

Multiple Rotor

Multiple Membrane Rotor

Biomimicry



Part 1: The Model

First, we present the development of the mathematical model for the dynamic behavior of a nonlinear plate undergoing deformation both in transverse and axial directions using a Hamiltonian approach.

Kinematics

We use the Kirchhoff hypothesis for the deformation (u_i) of the plate.

Assumptions

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation

Equations

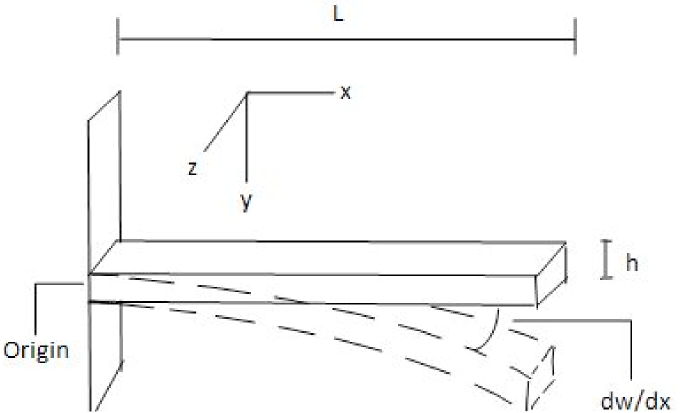
$$u_1 = u - x_3 w_{x_1}$$
$$u_2 = v - x_3 w_{x_2}$$
$$u_3 = w$$

Terms

u = axial displacement in the x_1 direction
v = axial displacement in the x_2 direction
w = transverse displacement

Visualization

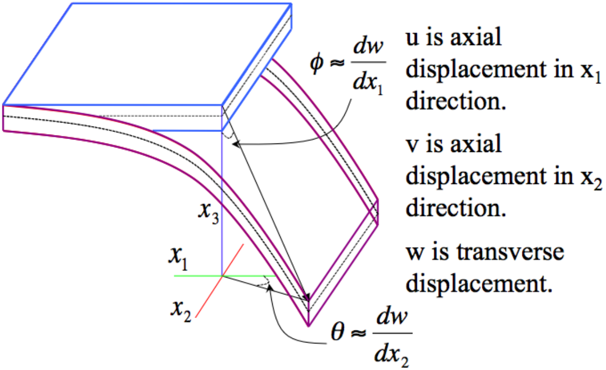
This is an example from a beam model, which we will ultimately compare our model against:



[Hickman (2010)]

Visualization

This is what we are actually working with:



Strain-Displacement

We use the Green strain tensor to relate strain (E_{ij}) and displacement as follows:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

$$E_{11} = \frac{1}{2}(2(u_{x_1} - x_3 w_{x_1 x_1}) + w_{x_1}^2)$$
$$E_{22} = \frac{1}{2}(2(v_{x_2} - x_3 w_{x_2 x_2}) + w_{x_2}^2)$$
$$E_{12} = E_{21} = \frac{1}{2}(u_{x_2} + v_{x_1} - 2x_3 w_{x_1 x_2}) + w_{x_1} w_{x_2})$$

We use the a materially linear formulation to relate stress (σ_{ij}) and strain using Young's modulus (Y) and a Poisson ratio (ν) as follows:

$$\sigma_{11} = \frac{Y}{(1-\nu^2)}(E_{11} + \nu E_{22})$$
$$\sigma_{22} = \frac{Y}{(1-\nu^2)}(E_{22} + \nu E_{11})$$
$$\sigma_{12} = \sigma_{21} = \frac{1-\nu}{2}\frac{Y}{(1-\nu^2)}E_{12}$$

Kinetic Energy

For a homogeneous plate density ρ, to account for all the mass T takes the form of an integral over the area. We also drop the inertial term.

$$T = \frac{1}{2}m||V||^2$$
$$T = \int_0^a \int_0^a \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{\rho}{2}(\frac{\partial u_1}{\partial t}^2 + \frac{\partial u_2}{\partial t}^2 + \frac{\partial u_3}{\partial t}^2)dx_3 dx_2 dx_1$$

Terms

$$u_1 = u - x_3 w_{x_1}$$
$$u_2 = v - x_3 w_{x_2}$$
$$u_3 = w$$

$$T = \int_0^a \int_0^a \frac{\rho h}{2}(u_t^2 + v_t^2 + w_t^2)dx_2 dx_1$$

Potential Energy

The remaining potential energy is similar to a compressed spring, as follows:

$$
\begin{aligned}
U &= \int_0^a \int_0^a \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{1}{2}(\sigma_{11}E_{11} + \sigma_{22}E_{22} + \sigma_{12}E_{12})dx_3 dx_2 dx_1 \\
&= \frac{Y}{(1-\nu^2)} \int_0^a \int_0^a h((u_{x_1} + \frac{1}{2}w_{x_1}^2)^2 + (v_{x_2} + \frac{1}{2}w_{x_2}^2)^2 \\
&+ \frac{1-\nu}{2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2) \\
&+ \frac{h^3}{12}(w_{x_1}w_{x_1}^2 + w_{x_2}w_{x_2} + w_{x_1}w_{x_2}^2)dx_2 dx_1
\end{aligned}
$$

Potential Energy of External Applied Forces

The potential energy of the external applied forces will be defined as the negative of the work done by fluid forces acting on the plate, where K is a damping constant.

$$A = u(f_1 - Ku_t) + v(f_2 - Kv_t) + w(f_3 - Kw_t)$$

Hamiltonian Method

According to Hamilton's principle, the progression of all physical systems minimizes the time integral of the Lagrangian, which is to say the variation of the Lagrangian will always be zero, i.e.

$$\delta \int_{t_0}^{t_1} [(T - U) + A] dt = 0$$

Expanding this integral, we obtain the governing equations.

Hamiltonian, continued

$$
\begin{aligned}
0 &= \delta \int_{t_0}^{t_1} \int_0^a \int_0^a \frac{\rho h}{2}(u_t^2 + v_t^2 + w_t^2) \\
&- \frac{Yh}{(1-\nu^2)}((u_{x_1} + \frac{1}{2}w_{x_1}^2)^2 + (v_{x_2} + \frac{1}{2}w_{x_2}^2)^2 \\
&+ \frac{1-\nu}{2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2) \\
&+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1}w_{x_1}^2 + w_{x_2}w_{x_2} + w_{x_1}w_{x_2}^2) \\
&+ u(f_1 - Ku_t) + v(f_2 - Kv_t) + w(f_3 - Kw_t)dx_2dx_1dt
\end{aligned}
$$

Using integration by parts to handle each term, the variation and the first spacial and temporal derivatives of the variation are zero at the limits of integration, therefore each boundary term is cancelled. After collecting all of the terms with contain δu, δv, δw, we can separate the integral into three parts as follows:

Calculus of Variation - δu component

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta u(-\rho h u_{tt} + \frac{Yh}{(1-\nu^2)}([u_{x_1} + \frac{1}{2}w_{x_1}^2]_{x_1} \\
&+ \frac{1-\nu}{2}[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2}) + Ku_t + f_1)dx_2dx_1dt
\end{aligned}
$$

Calculus of Variation - δv component

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta v\Big(-\rho h v_{tt} + \frac{Yh}{(1-\nu^2)}\big([v_{x_2} + \frac{1}{2}w_{x_2}^2]_{x_2} \\
&+ \frac{1-\nu}{2}[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_1}\big) + Kv_t + f_2\Big)dx_2dx_1dt
\end{aligned}
$$

Calculus of Variation - δw component

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1}\int_0^a\int_0^a \delta w\Big(-\rho h w_{tt} + \frac{Yh}{(1-\nu^2)}\Big([w_{x_1}(u_{x_1} + \frac{1}{2}w_{x_1}^2)]_{x_1} \\
&+ [w_{x_2}(v_{x_2} + \frac{1}{2}w_{x_2}^2)]_{x_2} + \frac{1-\nu}{2}[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} \\
&+ [w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}\Big) \\
&+ \frac{Yh^3}{12(1-\nu^2)}(w_{x_1x_1x_1x_1} + w_{x_2x_2x_2x_2} + 2w_{x_1x_1x_2x_2}) \\
&+ Kw_t + f_3\Big)dx_2dx_1dt
\end{aligned}
$$

Final Model

$$
\begin{aligned}
f_1 &= u_{tt} + Cu_t - D_1[u_{x_1} + \frac{1}{2}(w_{x_1})^2]_{x_1} - E[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2} \\
f_2 &= v_{tt} + Cv_t - D_1[v_{x_2} + \frac{1}{2}(w_{x_2})^2]_{x_2} - E[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_1} \\
f_3 &= w_{tt} + Cw_t + D[w_{x_1x_1x_1x_1} + 2w_{x_1x_1x_2x_2} + w_{x_2x_2x_2x_2}] \\
&\quad - D_1[w_{x_1}(u_{x_1} + \frac{1}{2}(w_{x_1})^2)]_{x_1} - D_1[w_{x_2}(v_{x_2} + \frac{1}{2}(w_{x_2})^2)]_{x_2} \\
&\quad - E[w_{x_1}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_2} - E[w_{x_2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})]_{x_1}
\end{aligned}
$$

where C, D, E, and D_1 are all constants depending on the system.

We will show for any transversal force f_3 the energy of the system changes proportionally to the force. In other words, our choice of initial conditions won't cause the system to experience *flutter* or other disastrous instabilities.

Multiply the first equation by u_t and aCu, the second by v_t and aCv, and the third by w_t and aCw; where $0 \leq a \leq 1$.

After this, add the resulting equations. For example, the first equation becomes:

$$\frac{1}{2}\left[u_t\right]_t^2 + Cu_t^2 + aCu_{tt}u + \frac{a}{2}C^2\left[u^2\right]_t =$$
$$D_1[u_{x_1} + \frac{1}{2}w_{x_1}^2]_{x_1}u_t + aD_1Cu_{x_1} + \frac{1}{2}w_{x_1}^2]_{x_1}u +$$
$$E[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2}u_t + aEC[u_{x_2} + v_{x_1} + w_{x_1}w_{x_2}]_{x_2}u +$$
$$f_1(u_t + aCu)$$

The overall goal of what we are doing is to reduce our now *more* complicated system into a (relatively) simple ordinary differential equation. Using integration by parts and some algebra tricks, this is possible, though it takes a lot of work. If you want the full derivation, email me.

Ordinary Differential Equation

$$[M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) dx_1 dx_2$$

$$M_1 = \int_0^L \int_0^L \frac{a}{2}(u_t + Cu)^2 + \frac{a}{2}(v_t + Cv)^2 + \frac{a}{4}(w_t + Cw)^2 + \frac{1-a}{2} u_t^2 + \frac{1-a}{2} v_t^2$$

$$+ \left(\frac{1}{2} - \frac{a}{4}\right) w_t^2 + \frac{D_1}{2}\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + \frac{D_1}{2}\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2$$

$$+ \frac{E}{2}(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})^2 + \frac{D}{2}(w_{x_1 x_1}^2 + 2w_{x_1 x_2}^2 + w_{x_2 x_2}^2) dx_1 dx_2$$

$$M_2 = \int_0^L \int_0^L (1-a) C u_t^2 + (1-a) C v_t^2 + (1 - \frac{a}{2}) C w_t^2$$

$$+ aCD_1\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aCD_1\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2$$

$$+ aEC(u_{x_2} + v_{x_1} + w_{x_1} w_{x_2})^2 + \frac{aCD}{2}(w_{x_1 x_1}^2 + 2w_{x_1 x_2}^2 + w_{x_2 x_2}^2) dx_1 dx_2$$

Bounding the Equation

$$[M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) dx_1 dx_2$$

One thing to notice is this equation looks a lot like,

$$\frac{d}{dt}[y(t)] + x(t) = f(t)$$

And if we can get that to look like,

$$\frac{d}{dt}[y(t)] + y(t) \leq f(t)$$

$$\frac{d}{dt}\left[e^t y(t)\right] \leq f(t)e^t$$

$$e^T y(T) - y(0) \leq \int_0^T f(t)e^t dt$$

$$y(T) \leq y(0)e^{-T} + e^{-T} \int_0^T f(t)e^t dt$$

$$[M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) dx_1 dx_2$$

It would be ideal to find a function M_2^* so that M_2^* is bounded above by some factor of M_1 because that would provide us with the inequality we seek. Thus we begin our search for the elusive M_2^* by repeatedly throwing inequalities at our differential equation.

Constructing M_2^*

Using Young's Inequality and the Poincaré Inequality we get,

$$[M_1]_t + C\int_0^L\int_0^L (1-a)u_t^2 + (1-a)v_t^2 + \frac{1}{2}(1-\frac{a}{2})w_t^2$$

$$+aD_1\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aD_1\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2$$

$$+aE(u_{x_2} + v_{x_1} + w_{x_1x_2})^2 + \frac{aD}{4}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2)\,dx_1dx_2$$

$$\leq \int_0^L\int_0^L f_3(w_t + \frac{aCw}{2})dx_1dx_2$$

or setting M_2^* to be the terms in red,

$$[M_1]_t + CM_2^* \leq f_3^2\left(\frac{aC}{8\varepsilon_1} + \frac{1}{4\varepsilon_2}\right)dx_1dx_2$$

Goal: We will use M_2^* as the energy function to bound changes in the solutions in order to prove our stability result.

By Construction

$$M_2^* \leq 2M_1$$

$$M_1 = \int_0^L \int_0^L \frac{a}{2}(u_t + Cu)^2 + \frac{a}{2}(v_t + Cv)^2 + \frac{a}{4}(w_t + Cw)^2 + \frac{1-a}{2}u_t^2 + \frac{1-a}{2}v_t^2$$

$$+ \left(\frac{1}{2} - \frac{a}{4}\right) w_t^2 + \frac{D_1}{2}\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + \frac{D_1}{2}\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2$$

$$+ \frac{E}{2}(u_{x_2} + v_{x_1} + w_{x_1}w_{x_2})^2 + \frac{D}{2}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2)dx_1dx_2$$

$$M_2^* = \int_0^L \int_0^L (1-a)u_t^2 + (1-a)v_t^2 + \frac{1}{2}(1 - \frac{a}{2})w_t^2$$

$$+aD_1\left(u_{x_1} + \frac{1}{2}(w_{x_1})^2\right)^2 + aD_1\left(v_{x_2} + \frac{1}{2}(w_{x_2})^2\right)^2$$

$$+aE(u_{x_2} + v_{x_1} + w_{x_1x_2})^2 + \frac{aD}{4}(w_{x_1x_1}^2 + 2w_{x_1x_2}^2 + w_{x_2x_2}^2)\,dx_1dx_2$$

First Order Linear Differential Equation

$$[M_1]_t + M_2 = \int_0^L \int_0^L f_3(w_t + \frac{aCw}{2}) dx_1 dx_2$$

$$[M_1]_t + CM_2^* \leq \int_0^L \int_0^L f_3^2 \left(\frac{aC}{8\varepsilon_1} + \frac{1}{4\varepsilon_2} \right) dx_1 dx_2$$

$$\frac{1}{2} M_2^*(t) - M_1(0) + C \int_0^t M_2^*(\tau) d\tau \leq \left(\frac{aC}{8\varepsilon_1} + \frac{1}{4\varepsilon_2} \right) \int_0^t ||f_3||_{L_2}^2 d\tau$$

$$M_2^*(t) + 2C \int_0^t M_2^*(\tau) d\tau \leq 2M_1(0) + \left(\frac{aC}{4\varepsilon_1} + \frac{1}{2\varepsilon_2} \right) \int_0^t ||f_3||_{L_2}^2 d\tau$$

$$\frac{d}{dt} \left[e^{2Ct} \int_0^t M_2^*(\tau) d\tau \right] \leq 2M_1(0) e^{2Ct} + e^{2Ct} \left(\frac{aC}{4\varepsilon_1} + \frac{1}{2\varepsilon_2} \right) \int_0^t ||f_3||_{L_2}^2 d\tau$$

Our Theorem

For any non-linear plate and any transversal force, f_3, the energy measure, M_2^*, satisfies the following inequality for any given time.

$$\int_0^T M_2^*(\tau)d\tau \leq \frac{M_1(0)}{C}\left[1-e^{-2CT}\right]+e^{-2CT}\left[\frac{aC}{4\varepsilon_1}+\frac{1}{2\varepsilon_2}\right]\int_0^T e^{2Ct}\left(\int_0^t ||f_3||_{L_2}^2 d\tau\right) dt$$

$$\text{where} \quad M_2^* = \int_0^L\int_0^L (1-a)u_t^2+(1-a)v_t^2+\frac{1}{2}(1-\frac{a}{2})w_t^2$$

$$+aD_1\left(u_{x_1}+\frac{1}{2}(w_{x_1})^2\right)^2+aD_1\left(v_{x_2}+\frac{1}{2}(w_{x_2})^2\right)^2$$

$$+aE(u_{x_2}+v_{x_1}+w_{x_1x_2})^2+\frac{aD}{4}(w_{x_1x_1}^2+2w_{x_1x_2}^2+w_{x_2x_2}^2)\,dx_1dx_2$$

where u, v, are axial displacements and w is transverse displacement, $\varepsilon_1, \varepsilon_2, C, M_1(0)$ are real numbers dependent on the system.

The next step, now that we have proven the system to be analytically stable, is to show numerical stability and use parameter identification studies to validate our model using an explicit FTCS finite difference method. At the moment, we are working on proofs of stability similar to the classic "Von Neumann stability" to get convergence results.

Future Work

- Continue numerical validation using finite difference method and finite element method
- Error convergence
- Nonlinear material constitutive law
- Approximation of other nonlinearities
- Parameter identification studies to validate the model
- Allow f_1 and f_2 to be nontrivial
- Allow $\int_0^t ||f_3||_{L_2}^2 d\tau$ to be bounded or constant
- Couple the structural model to a fluid model

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