A Stability Estimate for a Nonlinear Beam

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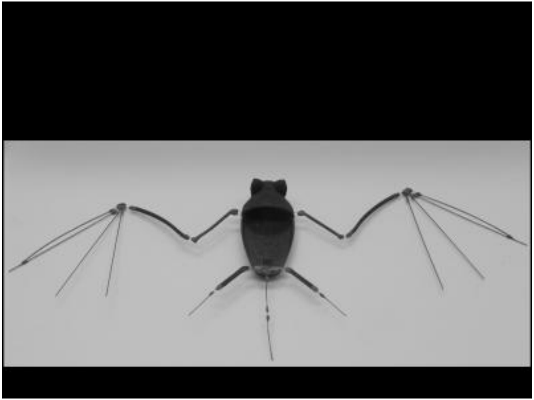
Rigid Wing

Rigid Wing

Multiple Rotor

Multiple Membrane Rotor

Biomimicry



Part 1: The Model

A quick summary of the development of the mathematical model for the dynamic behavior of a nonlinear beam undergoing deformation both in transverse and axial directions using a Hamiltonian approach.

We use the Kirchhoff hypothesis for the deformation (u_i) of the beam.

Assumptions

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the beam does not change during a deformation

Equations

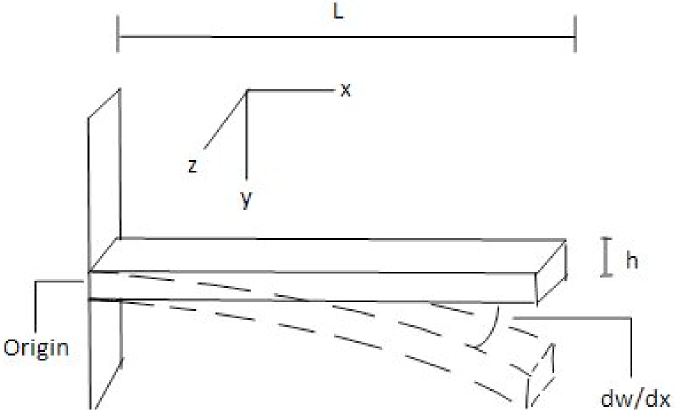
$$u_1 = u - yw_x$$
$$u_2 = w$$
$$u_3 = 0$$

Terms

u = axial displacement in the x direction

w = transverse displacement

Visualization



[Hickman (2010)]

Strain-Displacement

We use the geometrically nonlinear Green strain tensor to relate the strain tensor(E_{ij}) to displacement as follows:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$
$$E_{xx} = \frac{1}{2}(2(u_x - yw_{xx}) + w_x^2)$$

Stress-Strain

We use a materially linear elasticity tensor to relate stress (σ_{ij}) and strain using Young's modulus (Y) and a Poisson ratio (ν) as follows:

$$\sigma_{xx} = \frac{Y}{(1-\nu^2)} E_{xx}$$

Virtual Kinetic Energy

$$\delta K = \rho \int_0^L \int_{\frac{-h}{2}}^{\frac{h}{2}} \; [u_1]_t \delta [u_1]_t + [u_2]_t \delta [u_2]_t + [u_3]_t \delta [u_3]_t \; dx$$

Terms

$u_1 = u - y w_x$

$u_2 = w$

$u_3 = 0$

$$\delta K = \int_0^L \; \rho h (u_t \delta u_t + w_t \delta w_t) + \frac{\rho h^3}{12} (w_{xt} \delta w_{xt}) \; dx$$

Virtual Potential Energy

$$\delta U = \int_0^L \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{xx}\, \delta E_{xx}\, dx$$

$$\begin{aligned}\delta U &= \frac{E}{1-\nu^2}\int_0^L (h(u_x + \frac{1}{2}w_x^2)\delta u_x + (\frac{h^3}{12}w_{xx})\delta w_{xx} \\ &+ (h(w_x(u_x + \frac{1}{2}w_x^2)))\delta w_x \;\; dx\end{aligned}$$

Virtual Work

$$\delta V = \int_0^L (f_1 - k u_t)\delta u + (f_2 - k w_t)\delta w \;\; dx$$

Hamilton's Principle for Deformable Bodies

$$\int_0^T -\delta K + \delta U \ \ dt = \int_0^T \delta U \ \ dt$$

Final Model

$$
\begin{aligned}
f_1 &= \rho h u_{tt} + k u_t - \frac{Eh}{1-\nu^2}[u_x + \frac{1}{2} w_x^2)]_x \\
f_2 &= \rho h w_{tt} + k w_t - \frac{Eh}{1-\nu^2}[w_x(u_x + \frac{1}{2} w_x^2)]_x + \frac{Eh^3}{12(1-\nu^2)} w_{xxxx}
\end{aligned}
$$

Part 2: A prioi bounds and stability

Assumptions:

$$\begin{aligned} f_1 &= 0 \\ f_2 &= f \end{aligned}$$

Opening Move

Divide both equations by ρh, so that we have:

$$\begin{aligned} 0 &= u_{tt} + K_1 u_t - D_1[u_x + \frac{1}{2}w_x^2)]_x \\ q &= w_{tt} + K_1 w_t - D_1[w_x(u_x + \frac{1}{2}w_x^2)]_x + D_2 w_{xxxx} \end{aligned}$$

Momentum Equations

Multiply the first equation by u_t and $aK_1 u$ and the second by w_t and $aK_1 w$; where $0 \leq a \leq 1$.

$$
\begin{aligned}
0 &= u_{tt} u_t + K_1 u_t^2 - D_1[u_x + \frac{1}{2} w_x^2)]_x u_t \\
0 &= aK_1 u_{tt} u + aK_1^2 u_t u - aK_1 D_1[u_x + \frac{1}{2} w_x^2)]_x u \\
qw_t &= w_{tt} w_t + K_1 w_t^2 - D_1[w_x(u_x + \frac{1}{2} w_x^2)]_x w_t + D_2 w_{xxxx} w_t \\
aK_1 qw &= aK_1 w_{tt} w + aK_1^2 w_t w - aK_1 D_1[w_x(u_x + \frac{1}{2} w_x^2)]_x w \\
&+ aK_1 D_2 w_{xxxx} w
\end{aligned}
$$

Momentum Equations

Sum the previous equations, together with the following relations:

$$
\begin{aligned}
&u_{tt}u_t + K_1 u_t^2 + aK_1 u_{tt} u + aK_1^2 u_t u \\
&= \frac{1}{2}[u_t^2]_t + K_1 u_t^2 + aK_1 u_{tt} u + \frac{aK_1^2}{2}[u^2]_t \\
&= \frac{a}{2}[(u_t + K_1 u)^2]_t + \frac{1-a}{2}[u_t^2]_t + (1-a)K_1 u_t^2
\end{aligned}
$$

$$
\begin{aligned}
&w_{tt}w_t + K_1 w_t^2 + aK_1 w_{tt} w + aK_1^2 w_t w \\
&= \frac{1}{2}[w_t^2]_t + K_1 w_t^2 + aK_1 w_{tt} w + \frac{aK_1^2}{4}[w^2]_t \\
&= \frac{a}{4}[(w_t + K_1 w)^2]_t + (\frac{1}{2} - \frac{a}{4})[w_t^2]_t + (1 - \frac{a}{2})K_1 w_t^2
\end{aligned}
$$

Momentum Equation

Then integrate with respect to x, and use integration by parts to get:

$$\begin{aligned}\frac{d}{dt}\int_0^L \quad & \frac{a}{2}((u_t + K_1 u)^2) + \frac{a}{4}(w_t + K_1 w)^2) \\ & + \frac{1-a}{2}(u_t^2) + (\frac{1}{2} - \frac{a}{4})(w_t^2) \\ & + \frac{D_1}{2}(u_x + \frac{1}{2}(w_x^2))^2 + \frac{D_2}{2} w_{xx}^2 dx\end{aligned}$$

$$\begin{aligned}+\int_0^L \quad & (1-a)K_1 u_t^2 + (1 - \frac{a}{2})K_1 w_t^2 \\ & + aK_1 D_1 (u_x + \frac{1}{2}(w_x^2))^2 + \frac{aK_1 D_2}{2} w_{xx}^2 dx \\ & = \int_0^L q(\frac{aK_1 w}{2} + w_t)dx\end{aligned}$$

ODE

$$
\begin{aligned}
V_1(t) &= \int_0^L \frac{a}{2}((u_t + K_1 u)^2) + \frac{a}{4}(w_t + K_1 w)^2) \\
&+ \frac{1-a}{2}(u_t^2) + (\frac{1}{2} - \frac{a}{4})(w_t^2) \\
&+ \frac{D_1}{2}(u_x + \frac{1}{2}(w_x^2))^2 + \frac{D_2}{2} w_{xx}^2 dx
\end{aligned}
$$

$$
\begin{aligned}
V_2(t) &= \int_0^L (1-a) K_1 u_t^2 + (1 - \frac{a}{2}) K_1 w_t^2 \\
&+ a K_1 D_1 (u_x + \frac{1}{2}(w_x^2))^2 + \frac{a K_1 D_2}{2} w_{xx}^2 dx
\end{aligned}
$$

Proposition 1

$$[V_1]_t + V_2 = \int_0^L q(\frac{aK_1 w}{2} + w_t)dx$$

Propostion 1: For $a = 0$ the rate of change of the sum of the beam kinetic and potential energies ($\rho h[V_1]_t$) plus the dissipated power ($\rho h V_2$) equals the flux of the energy given to the beam system from the fluid flow.

$$\frac{d}{dt}y(t) + x(t) = f(t)$$

Goal:

$$\begin{aligned}
\frac{d}{dt}[y(t)] + y(t) &\leq f(t) \\
\frac{d}{dt}[e^t y(t)] &\leq f(t)e^t \\
e^T y(T) - y(0) &\leq \int_0^T f(t)e^t dt \\
y(T) &\leq y(0)e^{-T} + e^{-T}\int_0^T f(t)e^t dt
\end{aligned}$$

Cauchy Inequality

Cauchy's Inequality: $\forall \varepsilon > 0$, we have $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$

$$q(\frac{aK_1 w}{2} + w_t) \leq \frac{aK_1}{2}(\frac{q^2}{4\varepsilon_1} + \varepsilon_1 w^2) + \frac{q^2}{4\varepsilon_2} + \varepsilon_2 w_t^2$$

Cauchy Inequality

$$\begin{aligned}
[V_1]_t &+ \int_0^L (1-a)K_1 u_t^2 + ((1-\frac{a}{2})K_1 - \varepsilon_2)w_t^2 \\
&+ aK_1 D_1(u_x + \frac{1}{2}(w_x^2))^2 + (\frac{aK_1 D_2}{2} w_{xx}^2 - \frac{aK_1\varepsilon_1}{2} w^2)dx \\
&\leq \int_0^L q^2(\frac{aK_1}{8\varepsilon_1} + \frac{1}{4\varepsilon_2})dx
\end{aligned}$$

Poincaré Inequality

Poincaré Inequality: Assume that $1 \leq p \leq \infty$ and that Ω is a bounded connected open subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with a Lipschitz boundary (i.e., Ω is a Lipschitz domain). Then there exists a constant C, depending only on Ω and p, such that for every function u in the Sobolev space $W^{1,p}(\Omega)$:

$$||u - u_\Omega||_{L^p(\Omega)} \leq C_{\Omega,p}||\nabla u||_{L^p(\Omega)}$$

Poincaré Inequality

$$C_F^2 \int_0^L w^2 dx \leq \int_0^L w_{xx}^2 dx$$

Since we are in $\mathbb{R}$, we can set $C_F = \frac{1}{L^2}$.

Cauchy Inequality

Let $\varepsilon_1 = \frac{D_2}{2L^4} = \frac{C_F^2 D_2}{2}$:

$$
\begin{aligned}
[V_1]_t &+ \int_0^L (1-a)K_1 u_t^2 + ((1-\frac{a}{2})K_1 - \varepsilon_2)w_t^2 \\
&+ aK_1 D_1 (u_x + \frac{1}{2}(w_x^2))^2 + (\frac{aK_1 D_2}{2} w_{xx}^2)dx \\
&\leq \int_0^L q^2 (\frac{aK_1}{8\varepsilon_1} + \frac{1}{4\varepsilon_2})dx
\end{aligned}
$$

Final Inequality

$$V_2^* \leq 2V_1$$

therefore

$$\int_0^T V_2^*(t)dt \leq \frac{V_1(0)}{K_1}(1 - e^{-2K_1 T}) + 2(\frac{aK_1}{8\varepsilon_1} + \frac{1}{4\varepsilon_2})e^{-2K_1 T} \int_0^T e^{2K_1 T} \int_0^t ||q||_{L^2}^2 d\tau dt$$

Theorem

Theorem: Let the nonlinear beam be excited by a distributed transversal load q, then the energy functional V_2^* satisfies the above inequality.

Stability Results

Remark 1: Assume $||q||_{L^2}$ to be bounded by C for all time, then:

$$\begin{aligned} \int_0^T V_2^*(t)dt &\leq \frac{V_1(0)}{K_1}(1 - e^{-2K_1 T}) \\ &+ (\frac{aK_1}{8\varepsilon_1}\frac{C}{K_1}(T + e^{-2K_1 T} - 1) \\ &\leq C_0 + C_1 T \end{aligned}$$

Additionally, if $\lim_{t\to\infty} V_2^*(t) = A$, then $A < \infty$.