Quiz 1, Propositional Logic

Date: February 3

1. (4pts) For \((p \land q) \lor \neg r\), give the truth table.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(p \land q)</th>
<th>(\neg r)</th>
<th>((p \land q) \lor \neg r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
<td>(F)</td>
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<td>(T)</td>
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</tr>
</tbody>
</table>

2. (3pts) Give in conjunctive normal form.

Use the fact that the *FALSE* values in the truth table occur at \((p, q, r) \in \{(T, F, T), (F, T, T), (F, F, T)\}\). This gives a CNF with three components, \((\neg p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg r) \land (p \lor q \lor \neg r)\).

Alternately, distribute the \(\neg r\) to get \((p \lor \neg r) \land (q \lor \neg r)\).
3. (3pts) Derive the CNF algebraically.

The short CNF form \((p \lor \neg r) \land (q \lor \neg r)\) can be derived immediately using the distributive law.

To get the longer three-component version, work backwards. Each step is reversible, so arriving at the starting point gives the solution.

\[
\begin{align*}
&\equiv (-p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg r) \land (p \lor q \lor \neg r) \\
&\equiv (-r \lor (-p \lor q)) \land ((-r \lor (p \lor \neg q)) \land (-r \lor (p \lor q))) \quad \text{(Commutative)} \\
&\equiv (-r \lor (-p \lor q)) \land (\neg r \lor ((p \lor \neg q) \land (p \lor q))) \quad \text{(Distributive)} \\
&\equiv (-r) \lor ((-p \lor q) \land (p \lor \neg q) \land (p \lor q)) \quad \text{(Distributive)} \\
&\equiv (-r) \lor ((-p \lor q) \land (p \lor FALSE)) \quad \text{(Contradiction)} \\
&\equiv (-r) \lor ((-p \lor q) \land p) \quad \text{(Prop. of FALSE)} \\
&\equiv (-r) \lor (p \land \neg p) \quad \text{(Commutative)} \\
&\equiv (-r) \lor ((p \land \neg p) \lor (p \land q)) \quad \text{(Distributive)} \\
&\equiv (-r) \lor (FALSE \lor (p \land q)) \quad \text{(Contradiction)} \\
&\equiv (-r) \lor (p \land q) \quad \text{(Prop. of FALSE)} \\
&\equiv (p \land q) \lor (\neg r) \quad \text{(Commutative)}
\end{align*}
\]
1. (10pts) Prove using inference rules, \((p \lor q \lor r) \rightarrow ((\neg p \land \neg q) \rightarrow r)\).

This problem was essentially impossible without some additional rules. In particular the “double negation introduction rule” is invoked even though not actually allowed! In order to explain our answer (others are possible) we will do it in three steps: first we will show the reasoning using a step called “vacuous proof” \((\neg \alpha \vdash (\alpha \rightarrow \gamma))\), second we show how this step can be justified using double negation introductions, and third we combine this into one (admittedly long) proof.

The proof on the test will not be so long, of course. Further, double negation introduction is allowed on the test.

\[
\begin{align*}
1 & \quad [p \lor q \lor r] & \text{Assumption} \\
2 & \quad [\neg p \land \neg q] & \text{Assumption} \\
3 & \quad [p \lor q] & \text{Assumption} \\
4 & \quad \neg p & \land \text{elimination 2} \\
5 & \quad p \rightarrow r & \text{vacuous proof 4} \\
6 & \quad \neg q & \land \text{elimination 2} \\
7 & \quad q \rightarrow r & \text{vacuous proof 6} \\
8 & \quad r & \text{case analysis 3,5,7} \\
9 & \quad (p \lor q) \rightarrow r & \rightarrow \text{introduction 3,8} \\
10 & \quad [r] & \text{Assumption} \\
11 & \quad r & \text{Copy of 10} \\
12 & \quad r \rightarrow r & \rightarrow \text{introduction 10,11} \\
13 & \quad r & \text{case analysis 1,9,12} \\
14 & \quad (\neg p \land \neg q) \rightarrow r & \rightarrow \text{introduction 2,13} \\
15 & \quad (p \lor q \lor r) \rightarrow ((\neg p \land \neg q) \rightarrow r) & \rightarrow \text{introduction 1,14}
\end{align*}
\]
The vacuous proof sequent used on lines 5 and 7 can be derived as follows:

1. \( \neg p \)  
   Given
2. \([\neg r]\)  
   Assumption
3. \( p \)  
   Copy of 1
4. \( \neg r \rightarrow \neg p \)  
   \( \rightarrow \) introduction 2,3
5. \( \neg\neg p \rightarrow \neg\neg r \)  
   Contraposition 4
6. \([p]\)  
   Assumption
7. \( \neg\neg p \)  
   Double negation introduction 6
8. \( \neg\neg r \)  
   Modus ponens 5,7
9. \( r \)  
   Double negation 8
10. \( p \rightarrow r \)  
   \( \rightarrow \) introduction 6,9
Thus, a more complete proof would look like this:

1. \[ p \lor q \lor r \] Assumption
2. \[ \neg p \land \neg q \] Assumption
3. \[ p \lor q \] Assumption
4. \[ \neg p \] \& elimination 2
5. \[ \neg r \] Assumption
6. \[ \neg p \] Copy of 4
7. \[ \neg r \rightarrow \neg p \] \rightarrow introduction 5,6
8. \[ \neg
\neg p \rightarrow \neg
\neg r \] Contrapositive 7
9. \[ p \] Assumption
10. \[ \neg
\neg p \] double negation introduction 9
11. \[ \neg
\neg r \] Modus ponens 8,10
12. \[ r \] double negation 11
13. \[ p \rightarrow r \] \rightarrow introduction 9,12
14. \[ \neg q \] \& elimination 2
15. \[ \neg r \] Assumption
16. \[ \neg q \] Copy of 14
17. \[ \neg r \rightarrow \neg q \] \rightarrow introduction 15,16
18. \[ \neg
\neg q \rightarrow \neg
\neg r \] Contrapositive 17
19. \[ q \] Assumption
20. \[ \neg
\neg q \] double negation introduction 19
21. \[ \neg
\neg r \] Modus ponens 18,20
22. \[ r \] double negation 21
23. \[ q \rightarrow r \] \rightarrow introduction 19,22
24. \[ r \] case analysis 3,13,23
25. \[ (p \lor q) \rightarrow r \] \rightarrow introduction 3,24
26. \[ r \] Assumption
27. \[ r \] Copy of 26
28. \[ r \rightarrow r \] \rightarrow introduction 26,27
29. \[ r \] case analysis 1,25,28
30. \[ (\neg p \land \neg q) \rightarrow r \] \rightarrow introduction 2,29
31. \[ (p \lor q \lor r) \rightarrow ((\neg p \land \neg q) \rightarrow r) \] \rightarrow introduction 1,30

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Quiz 3, Predicate Logic
Date: February 19

1. (2pts) Evaluate \( \exists i \in I^+ : \forall j \in I_i : j^2 = i \).

If \( i = 1 \), then the set \( I_i = \{1\} \), thus the only possible value of \( j \) in the universally quantified expression is 1. Since \( 1^2 = 1 \), the universally quantified expression \( \forall j \in I_i : j^2 = i \) is true when \( i = 1 \). Since there exists such an \( i \), the existentially quantified expression is true.

2. (4pts) Express \( A \neq B \) for two sets; use \( \notin \) but not \( \neg \).

If \( A \) and \( B \) are not equal, then one of the sets must contain an element that is not in the other one.

“There is something in \( A \) that is not in \( B \):”
\[ \exists a \in A : a \notin B \]

The above could be true for either \( A \) or for \( B \):
\[ (\exists a \in A : a \notin B) \lor (\exists b \in B : b \notin A) \]

Alternately:
\[
(A \subseteq B) \equiv (\forall a \in A : a \in B)
\]
\[
(A = B) \equiv (A \subseteq B) \land (B \subseteq A)
\]
\[
(A \neq B) \equiv \neg (A = B)
\]
\[
\equiv \neg (A \subseteq B) \lor \neg (B \subseteq A)
\]
\[
\equiv \neg (\forall a \in A : a \in B) \lor \neg (\forall a \in B : a \in A)
\]
\[
\equiv (\exists a \in A : a \notin B) \lor (\exists a \in B : a \notin A)
\]
3. (4pts) Using $\text{Edge}(x, y)$, express that $G(V, E)$ has at least one path of length 3.

Suppose that $G(V, E)$ has some path of length 3, and that it crosses the points $x, u, v,$ and $y$. That means that for the expression to be true, $x, u, v,$ and $y$ must all exist, and they must be connected by edges. Thus,

$$\exists x : \exists y : \exists u : \exists v : \text{Edge}(x, u) \land \text{Edge}(u, v) \land \text{Edge}(v, y).$$

Furthermore note that if the graph is undirected and $x$ and $y$ are connected by an edge, then $x, y, x, y$ is an automatic path of length 3. In order to avoid this sort of phenomenon, additional checks may be added:

$$\exists x : \exists y : \exists u : \exists v : \text{Edge}(x, u) \land \text{Edge}(u, v) \land \text{Edge}(v, y) \land x \neq v \land u \neq y.$$
1. (6pts) Prove by mathematical induction,

\[ \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}, \quad n \geq 0, \quad x \neq 1. \]

Let the nth case be defined by the predicate \( P(n) \),

\[ P(n) : \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}, \quad x \neq 1. \]

The goal is to show that \( P(n) \) is true for \( n \geq 0 \).

For the base case, \( P(0) \): \( \sum_{i=0}^{0} x^i = x^0 = 1 = \frac{x^{0+1} - 1}{x - 1} = \frac{x-1}{x-1} = 1 \)

For the inductive hypothesis, assume that \( k \geq 0 \), and assume that \( P(k) \) is true: \( \sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1}, \quad x \neq 1. \)

For the inductive conclusion, we will prove \( P(k+1) \) is true by starting with its left-hand side. Recall that \( P(k+1) \) is: \( \sum_{i=0}^{k+1} x^i = \frac{x^{(k+1)+1} - 1}{x - 1}, \quad x \neq 1. \)

Thus,

\[ \sum_{i=0}^{k+1} x^i = x^{k+1} + \sum_{i=0}^{k} x^i \]

Substituting the inductive hypothesis gives:

\[ x^{k+1} + \frac{x^{k+1} - 1}{x - 1} = \frac{x^{(k+1)(x-1)+x^{k+1}-1}}{x-1} = \frac{x^{(k+1)+1}-1}{x-1} \]

Thus, \( P(k) \rightarrow P(k+1) \) for all \( k \geq 0 \), which together with the base case proves the desired result.
2. (4pts) Give the formal outline of the proof of $\forall i \in N : P(i)$.

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<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P(0)$</td>
<td>Base case</td>
</tr>
<tr>
<td>2</td>
<td>$[k \in N]$</td>
<td>Assumption</td>
</tr>
<tr>
<td>3</td>
<td>$[P(k)]$</td>
<td>Assumption</td>
</tr>
<tr>
<td>4</td>
<td>$P(k + 1)$</td>
<td>proof of IC</td>
</tr>
<tr>
<td>5</td>
<td>$P(k) \rightarrow p(k + 1)$</td>
<td>$\rightarrow$ introduction 3,4</td>
</tr>
<tr>
<td>6</td>
<td>$\forall i \in N : P(i) \rightarrow P(i + 1)$</td>
<td>$\forall$ introduction 2,5</td>
</tr>
<tr>
<td>7</td>
<td>$\forall i \in N : P(i)$</td>
<td>Mathematical induction 1,6</td>
</tr>
</tbody>
</table>
1. Consider the following code:

\[
\begin{align*}
  i &\leftarrow 1 \\
  x &\leftarrow 1 \\
  \textbf{while } i < n \textbf{ do} \\
  & x \leftarrow x \cdot y \\
  & i \leftarrow i + 1
\end{align*}
\]

Assume \( y \) is a constant and \( n \geq 0 \)

(a) \((3pts)\) Give the loop invariant.

\[(i \leq n) \land (x = y^{i-1})\]

(b) \((4pts)\) Prove the loop invariant.

\[
\frac{(i \leq n) \land (x = y^i) \land (i < n)}{(i \leq n) \land (x = y^{i-1}) \land (i < n)}
\]

\[
\{ x \leftarrow x \cdot y; i \leftarrow i + 1 \} \quad (i \leq n) \land (x = y^{i-1})
\]

Once the loop terminates, \((i \leq n) \land (x = y^{i-1}) \land \neg(i < n)\) holds.

\((i \leq n) \land \neg(i < n)\) is the same as \((i \leq n) \land (i \geq n)\), which implies \(i = n\). Combining with \(x = y^{i-1}\) gives \(x = y^{n-1}\).
1. (3pts) How many strings of length 8 are in \{aa, bbb\}^*?

Note that the number of aa has an even number of characters, while bbb has an odd number of characters. So if one bbb appears, then another bbb must also appear. The four possible strings are:
\{aaaaaaa, bbbbbbea, bbbaabbh, aabbbbb\}

2. (4pts) Prove that if \(L_0 \subseteq L_1 \subseteq L_2\), then \(L_1 \cup L_2 \subseteq L_1 \cup L_2\).

\(L_0 = \{\Lambda\}\), so if \(L_0 \subseteq L_1 \subseteq L_2\), then \(\Lambda \in L_1 \subseteq L_2\). Thus \(\Lambda\) must be part of both \(L_1\) and \(L_2\).

Since \(L_1^1 = L_2 = \{\Lambda\} L_2\), and \(\{\Lambda\} \subseteq L_1\), then \(L_1^1 = \{\Lambda\} L_2 \subseteq L_1 \subseteq L_2\), proving the result.

3. (3pts) Is this a law? \(L_1 \cup (L_2 L_3) \equiv (L_1 \cup L_2)(L_1 \cup L_3)\)

Let \(L_1 = \{0\}, L_2 = \{1\}, L_3 = \{1\}\).
\(L_1 \cup (L_2 L_3) = \{0\} \cup (\{11\}) = \{0, 11\}\).
\((L_1 \cup L_2)(L_1 \cup L_3) = (\{0, 1\})(\{0, 1\}) = \{00, 01, 10, 11\}\).

The two expressions are not the same, therefore the expression is not a law.
Quiz 7, Finite Automata
Date: April 7

1. (10pts) $\Sigma = \{a, b\}$, $L = \{x \mid \text{before any } aa \text{ there is } bb\}$. Give a state transition diagram for $L$.

Examples:
$\text{abbaabaa} \in L$
$\text{abbaabaa} \in L$
$\text{a} \in L$

$q_0$: empty string
$q_1$: last character was an $a$, but neither $aa$ nor $bb$ have been seen
$q_2$: last character was a $b$, but neither $aa$ nor $bb$ have been seen
$q_3$: $bb$ was seen first
Quiz 8, Cross Product Technique

Date: April 14

1. (10pts) Using cross product, give a DFA for 
   \( L = \{ x \mid x \text{ contains } ab \text{ and } x \text{ contains } bba \} \).

   \( M_1: \ L_1 = \{ x \mid x \text{ contains } ab \} \)

   \[
   \begin{array}{c}
   \text{start} \\
   A \\
   \end{array}
   \begin{array}{c}
   a \\
   \rightarrow \\
   B \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   C \\
   \end{array}
   \]

   \( M_2: \ L_2 = \{ x \mid x \text{ contains } bba \} \)

   \[
   \begin{array}{c}
   \text{start} \\
   X \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   Y \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   Z \\
   \begin{array}{c}
   a \\
   \rightarrow \\
   W \\
   \end{array}
   \end{array}
   \end{array}
   \]

   \( M = M_1 \times M_2: \)

   \[
   \begin{array}{c}
   \text{start} \\
   AX \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   AY \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   AZ \\
   \begin{array}{c}
   a \\
   \rightarrow \\
   AW \\
   \end{array}
   \end{array}
   \end{array}
   \begin{array}{c}
   a \\
   \rightarrow \\
   BX \\
   \begin{array}{c}
   a \\
   \rightarrow \\
   BY \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   BZ \\
   \begin{array}{c}
   a \\
   \rightarrow \\
   BW \\
   \end{array}
   \end{array}
   \end{array}
   \end{array}
   \begin{array}{c}
   a \\
   \rightarrow \\
   CX \\
   \begin{array}{c}
   a \\
   \rightarrow \\
   CY \\
   \begin{array}{c}
   b \\
   \rightarrow \\
   CZ \\
   \begin{array}{c}
   a \\
   \rightarrow \\
   CW \\
   \end{array}
   \end{array}
   \end{array}
   \end{array}
   \]

   Note that states \( AW, BY, \) and \( BZ \) are unreachable.
Quiz 9, Regular Expressions
Date: April 21

1. Give a regular expression over $\Sigma = \{0, 1\}$.

   (a) (3pts) $L_1 = \{x \mid x \text{ contains 010}\}$

   Build $L_1$ by taking 010, and putting any string, $(0 + 1)^*$, before and after it.
   $(0 + 1)^*010(0 + 1)^*$

   (b) (3pts) $L_2 = \{x \mid x \text{ does not contain 010}\}$

   In order to avoid 010, every 1 must appear as two or more consecutive 1’s, $(111^*)$, or else appear at the beginning or end of the string. Thus,
   $1^*(0 + 111^*)^*1^*$

   (c) (4pts) $L_3 = \{x \mid x \text{ contains exactly one 010}\}$

   Begin with the string 010, then apply the logic of $L_2$ when adding strings to the beginning and end.
   $1^*(0 + 111^*)^*010(0 + 111^*)^*1^*$
Quiz 10. Regular Expression Conversion
Date: April 28

1. (10pts) Convert the following machine into a regular expression, deleting states in the order $B$ then $A$ then $C$.

![Machine Diagram]

Solution:

![Solution Diagram]

Final regular expression:

$$(a + bd^* c)^* bd^* + (a + bd^* c)^* bd^* e (f(a + bd^* c)^* bd^* e)^* f(a + bd^* c)^* bd^*$$
Quiz 11, Regular Grammars

Date: May 5

1. (7pts) Convert \( a + bc \) into a regular grammar with unit productions.

   Grammar for \( r_a = a \):
   
   \( (\{A, A_1\}, \{a,b,c\}, A_1, \{A_1 \rightarrow aA, A \rightarrow \Lambda\}) \)

   Grammar for \( r_b = b \):
   
   \( (\{B, B_1\}, \{a,b,c\}, B_1, \{B_1 \rightarrow bB, B \rightarrow \Lambda\}) \)

   Grammar for \( r_c = c \):
   
   \( (\{C, C_1\}, \{a,b,c\}, C_1, \{C_1 \rightarrow cC, C \rightarrow \Lambda\}) \)

   Grammar for \( r_{bc} = bc \):
   
   \( (\{B, B_1, C, C_1\}, \{a,b,c\}, B_1, \{B_1 \rightarrow bB, B \rightarrow C_1, C_1 \rightarrow cC, C \rightarrow \Lambda\}) \)

   Grammar for \( r = a + bc \):
   
   \( (\{S, A, A_1, B, B_1, C, C_1\}, \{a,b,c\}, S,
   S \rightarrow A_1,
   A_1 \rightarrow aA,
   A \rightarrow \Lambda,
   S \rightarrow B_1,
   B_1 \rightarrow bB,
   B \rightarrow C_1,
   C_1 \rightarrow cC,
   C \rightarrow \Lambda}) \)

2. (3pts) Convert into a regular grammar without unit productions:

   \[
   \begin{align*}
   S & \rightarrow A \\
   S & \rightarrow aB \\
   A & \rightarrow C \\
   B & \rightarrow bS \\
   B & \rightarrow \Lambda
   \end{align*}
   \]

   Solution:

   \[
   \begin{align*}
   S & \rightarrow aB \\
   B & \rightarrow bS \\
   B & \rightarrow \Lambda
   \end{align*}
   \]