Quiz 1, Propositional Logic

Date: September 7

1. Prove \((p \land q) \leftrightarrow \neg(p \rightarrow \neg q)\),

(a) (5pts) using truth tables.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(p \land q)</th>
<th>\neg q</th>
<th>(p \rightarrow \neg q)</th>
<th>(\neg(p \rightarrow \neg q))</th>
<th>((p \land q) \leftrightarrow \neg(p \rightarrow \neg q))</th>
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</thead>
<tbody>
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</tbody>
</table>

(b) (5pts) using algebra.

\[(p \land q) \leftrightarrow \neg(p \rightarrow \neg q)\]

\[
\equiv (p \land q) \leftrightarrow \neg(\neg p \lor \neg q) \quad \text{conditional law}
\]

\[
\equiv (p \land q) \leftrightarrow (\neg \neg p \land \neg \neg q) \quad \text{DeMorgan’s law}
\]

\[
\equiv (p \land q) \leftrightarrow (\neg p \land q) \quad \text{law of negation}
\]

\[
\equiv (p \land q) \leftrightarrow (p \land q) \quad \text{law of negation}
\]

\[
\equiv ((p \land q) \rightarrow (p \land q)) \land ((p \land q) \rightarrow (p \land q)) \quad \text{biconditional law}
\]

\[
\equiv (p \land q) \rightarrow (p \land q) \quad \text{idempotence}
\]

\[
\equiv \neg(p \land q) \lor (p \land q) \quad \text{conditional law}
\]

\[
\equiv TRUE \quad \text{excluded middle}
\]
Quiz 2, Rules of Inference
Date: September 14

1. (10pts) Prove \((p \land q) \leftrightarrow \neg(p \rightarrow \neg q)\), using inference rules.

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>([p \land q])</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>(p)</td>
<td>&amp; elimination, 1</td>
</tr>
<tr>
<td>3</td>
<td>(q)</td>
<td>&amp; elimination, 1</td>
</tr>
<tr>
<td>4</td>
<td>([p \rightarrow \neg q])</td>
<td>Assumption</td>
</tr>
<tr>
<td>5</td>
<td>(\neg q)</td>
<td>Modus ponens 4,2</td>
</tr>
<tr>
<td>6</td>
<td>(FALSE)</td>
<td>Contradiction 3,5</td>
</tr>
<tr>
<td>7</td>
<td>(\neg(p \rightarrow \neg q))</td>
<td>Reduction to absurdity 4,6</td>
</tr>
<tr>
<td>8</td>
<td>((p \land q) \rightarrow \neg(p \rightarrow \neg q))</td>
<td>(\rightarrow) introduction 1,7</td>
</tr>
<tr>
<td>9</td>
<td>([\neg(p \rightarrow \neg q)])</td>
<td>Assumption</td>
</tr>
<tr>
<td>10</td>
<td>([\neg(p \land q)])</td>
<td>Assumption</td>
</tr>
<tr>
<td>11</td>
<td>([p])</td>
<td>Assumption</td>
</tr>
<tr>
<td>12</td>
<td>([q])</td>
<td>Assumption</td>
</tr>
<tr>
<td>13</td>
<td>(p \land q)</td>
<td>&amp; introduction 11,12</td>
</tr>
<tr>
<td>14</td>
<td>(FALSE)</td>
<td>Contradiction 10,13</td>
</tr>
<tr>
<td>15</td>
<td>(\neg q)</td>
<td>Reduction to absurdity 12,14</td>
</tr>
<tr>
<td>16</td>
<td>(p \rightarrow \neg q)</td>
<td>(\rightarrow) introduction 11,15</td>
</tr>
<tr>
<td>17</td>
<td>(FALSE)</td>
<td>Contradiction 9,16</td>
</tr>
<tr>
<td>18</td>
<td>(\neg(p \land q))</td>
<td>Reduction to absurdity 10,17</td>
</tr>
<tr>
<td>19</td>
<td>((p \land q))</td>
<td>(\rightarrow) elimination 18</td>
</tr>
<tr>
<td>20</td>
<td>((p \rightarrow \neg q) \rightarrow (p \land q))</td>
<td>(\rightarrow) introduction 9,19</td>
</tr>
<tr>
<td>21</td>
<td>((p \land q) \leftrightarrow \neg(p \rightarrow \neg q))</td>
<td>\leftrightarrow introduction 8,20</td>
</tr>
</tbody>
</table>
Quiz 3, Predicate Logic

Date: September 21


   “if $j$ is to the left of $i$, then the series increases going to the right of $j$”:
   $$(j < i) \rightarrow (A[j] < A[j + 1])$$

   “if $j$ is to the right of $i$, then the series decreases coming from the left of $j$”:
   $$(i < j) \rightarrow (A[j] > A[j + 1])$$

   “there is a point $i$ such that the series is increasing everywhere to the left of $i$, and decreasing everywhere to the right of $i$”:
   $$\exists i \in I_n : (\forall j \in I_n : (j < i) \rightarrow (A[j] < A[j + 1]))$$
   $$\land (\forall j \in I_n : (i < j) \rightarrow (A[j - 1] > A[j]))$$

   This assumes that either the first or second part can be a sequence of one element, i.e. the whole sequence can be just increasing or decreasing. Otherwise, insert “$i \neq 1 \land i \neq n$” after the first colon.

2. (5pts) Assert that a graph has 2 vertices such that every vertex is connected to one of the two. $G = V$ and $Edge(x, y)$.

   “a vertex $z$ is connected to either vertex $x$ or vertex $y$”:
   $$Edge(z, x) \lor Edge(z, y)$$

   “every vertex $z$, if it isn’t $x$ or $y$ itself, is connected to $x$ or $y$”:
   $$\forall z \in V : ((z \neq x) \land (z \neq y)) \rightarrow (Edge(z, x) \lor Edge(z, y))$$

   “there is a pair of vertices, $x$ and $y$, such that every vertex is connected to one of the two”:
   $$\exists x \in V : \exists y \in V : \forall z \in V : ((z \neq x) \land (z \neq y)) \rightarrow (Edge(z, x) \lor Edge(z, y))$$
1. (5 pts) Prove $2^{2n} - 1$ is divisible by 3, $n \geq 1$ (i.e. $\exists m \in \mathbb{N} : 2^{2n} - 1 = 3m$) (Hint: $4 = 3 + 1$).

When $n = 1$, $2^{2n} - 1 = 2^2 - 1 = 3$, which is divisible by 3, thus proving the base case.

Assume that for some $k \geq 1$, $2^{2k} - 1$ is divisible by 3, so for some integer $m_k$,

$$2^{2k} - 1 = 3m_k$$

We would like to prove the $k + 1$ case,

$$2^{2(k+1)} - 1 = 3m_{k+1} \quad \text{(for some integer } m_{k+1})$$

To do this, we begin with the left hand side, and work until we can substitute the inductive hypothesis,

$$2^{2(k+1)} - 1 = 2^{2k}2^2 - 1$$

$$= 4 \times 2^{2k} - 1$$

$$= 3 \times 2^{2k} + 3 \times 2^{2k} - 1 \quad \text{(from the hint)}$$

$$= 3 \times 2^{2k} + 3m_k \quad \text{(by the inductive hypothesis)}$$

$$= 3(2^{2k} + m_k) \quad \text{(by the inductive hypothesis)}$$

$$= 3m_{k+1} \quad \text{(if we let } m_{k+1} = 2^{2k} + m_k)$$

This proves the inductive conclusion, thus by mathematical induction, the theorem is proved.
2. (5 pts) Let \( S_{n+1} = 2S_n + 1, \) \( n \geq 0, \) \( S_0 = 0, \)
Prove \( S_n = 2^n - 1, \) \( n \geq 0. \)

When \( n = 0, \) we have

\[
S_n = S_0 = 0 = 2^0 - 1 = 2^0 - 1 = 1 - 1 = 0
\]

Assume that for some \( k \geq 0, \)

\[
S_k = 2^k - 1
\]

We would like to prove the \( k + 1 \) case,

\[
S_{k+1} = 2^{k+1} - 1
\]

To do this, we begin with the left hand side,

\[
S_{k+1} = 2S_k + 1 \quad \text{(from the recursive definition)}
\]

\[
= 2(2^k - 1) + 1 \quad \text{(from the inductive hypothesis)}
\]

\[
= 2 \times 2^k - 2 + 1
\]

\[
= 2^{k+1} - 1
\]

This proves the inductive conclusion, thus by mathematical induction, the theorem is proved.
Quiz 5, Program Verification

Date: October 10

1. (5pts) State, prove, and use the loop invariant for the following code, assuming \( n \geq 0 \).

\[
\begin{align*}
  &i \leftarrow 0 \\
  &s \leftarrow 1 \\
  \text{while } i < n \text{ do} \\
  &\quad s \leftarrow \frac{5}{2} \times s \\
  &\quad i \leftarrow i + 1 \\
  &\quad s \leftarrow 6 \times s
\end{align*}
\]

Solution:

\[
\begin{align*}
  &i \leftarrow 0 \\
  &s \leftarrow 1 \\
  // (s = 15^i) \land (i \leq n) \\
  \text{while } i < n \text{ do} \\
  // (s = 15^i) \land (i \leq n) \land (i < n) \\
  &\quad s \leftarrow \frac{5}{2} \times s \\
  // (s = \frac{5}{2} \times 15^i) \land (i \leq n) \land (i < n) \\
  &\quad i \leftarrow i + 1 \\
  // (s = \frac{5}{2} \times 15^{i-1}) \land (i \leq n) \\
  &\quad s \leftarrow 6 \times s \\
  // (s = 6 \times \frac{5}{2} \times 15^{i-1} = 15 \times 15^{i-1} = 15^i) \land (i \leq n) \\
  // (s = 15^i) \land (i \leq n) \land \neg(i < n)
\end{align*}
\]

Note that \((i \leq n) \land \neg(i < n)\) implies \(i = n\), so \(s = 15^n\) at the end.
2. (5pts) State, prove, and use the loop invariant for the following code, assuming \( n \geq 0 \).

\[
\begin{align*}
m &\leftarrow n \\
y &\leftarrow 1 \\
z &\leftarrow x \\
\text{while } m < 0 \text{ do} \\
&\quad \text{if } ODD(m) \text{ then } y \leftarrow y \cdot z \\
&\quad \quad z \leftarrow z \cdot z \\
&\quad m \leftarrow \text{FLOOR}(m/2)
\end{align*}
\]

Solution:

\[
\begin{align*}
m &\leftarrow n \\
y &\leftarrow 1 \\
z &\leftarrow x \\
&\quad \text{while } m < 0 \text{ do} \\
&\quad \quad \text{// } (y^z \cdot m = x^n) \land (m \geq 0) \\
&\quad \quad \text{while } m < 0 \text{ do} \\
&\quad \quad \quad \text{// } (y^z \cdot m = x^n) \land (m \geq 0) \land (m > 0) \\
&\quad \quad \quad \quad \text{if } ODD(m) \text{ then } y \leftarrow y \cdot z \\
&\quad \quad \quad \quad \quad z \leftarrow z \cdot z \\
&\quad \quad \quad \quad \quad \text{// } (y^z \cdot m = x^n) \land (m \geq 0) \land (m > 0) \\
&\quad \quad \quad \quad \quad m \leftarrow \text{FLOOR}(m/2) \\
&\quad \quad \quad \quad \quad \text{// } (y^z \cdot m = x^n) \land (m \geq 0) \\
&\quad \quad \quad \quad \quad \text{// } (y^z \cdot m = x^n) \land (m \geq 0) \\
&\quad \quad \quad \quad \quad \text{// } (y^z \cdot m = x^n) \land (m \geq 0) \land \neg(m > 0)
\end{align*}
\]

Note that \((m \geq 0) \land \neg(m > 0)\) implies \( m = 0 \), so \( y = x^n \) at the end.
Quiz 6, Mathematical Induction II
Date: October 24

1. (5pts) Prove $\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$, $a \neq 1$, $n \geq 0$.

When $n = 0$, $\sum_{i=0}^{n} a^i = \sum_{i=0}^{0} a^i = a^0 = 1$, and

$\frac{a^{n+1} - 1}{a - 1} = \frac{a^{0+1} - 1}{a - 1} = \frac{a - 1}{a - 1} = 1$, which proves the base case.

Assume that for some $k \geq 0$, $\sum_{i=0}^{k} a^i = \frac{a^{k+1} - 1}{a - 1}$ with $a \neq 1$.

We would like to prove the $k + 1$ case, $\sum_{i=0}^{k+1} a^i = \frac{a^{(k+1)+1} - 1}{a - 1}$.

To do this, we begin with the left hand side, and work until we can substitute the inductive hypothesis,

$\sum_{i=0}^{k+1} a^i = a^{k+1} + \sum_{i=0}^{k} a^i$

$= a^{k+1} + \frac{a^{k+1} - 1}{a - 1}$ (from the inductive hypothesis)

$= \frac{a^{k+1}(a - 1) + a^{k+1} - 1}{a - 1}$

$= \frac{a^{k+1+1} - 1}{a - 1}$

This proves the inductive conclusion, thus by mathematical induction, the theorem is proved.
2. (5pts) Let $S_{n+1} = S_n + \left(\frac{1}{2}\right)^n$, $n \geq 0$, $S_0 = 0$.
Prove $S_n = 2 - \left(\frac{1}{2}\right)^{n-1}$, $n \geq 0$.

When $n = 0$, $2 - \left(\frac{1}{2}\right)^{0-1} = 2 - \left(\frac{1}{2}\right)^{-1} = 2 - 2 = 0 = S_0 = S_n$, which proves the base case.

Assume that for some $k \geq 0$, $S_k = 2 - \left(\frac{1}{2}\right)^{k-1}$.

We would like to prove the $k+1$ case, $S_{k+1} = 2 - \left(\frac{1}{2}\right)^{(k+1)-1} = 2 - \left(\frac{1}{2}\right)^k$.

To do this, we begin with the left hand side, and work until we can substitute the inductive hypothesis,

\[ S_{k+1} = S_k + \left(\frac{1}{2}\right)^k \quad \text{(from the recursive definition)} \]
\[ = 2 - \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k \quad \text{(from the inductive hypothesis)} \]
\[ = 2 - 2 \times \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k \]
\[ = 2 - \left(\frac{1}{2}\right)^k \]

This proves the inductive conclusion, thus by mathematical induction, the theorem is proved.
Quiz 7, Regular Expressions

Date: November 2

1. (3pts) Write all strings of length 6 in $L(r)$, $r = ((a + ab)^*ba^*)$.

   $L(r) = \{baaaaa, abaaaa, abbaaa, aabaaa, ababaa,
    aabbaa, ababba, aaabaa, ababa,
    ababab, aaabba, abaabb, aababb, aaaaba,
    abaaab, aabaab, aaabab, aaaabb, aaaaab\}$

2. (3pts) Simplify $(0 + 1)^*0(0 + 1)^* + (0 + 1)^*00(0 + 1)^*$.

   Note that $(0 + 1)^*00(0 + 1)^* \subseteq (0 + 1)^*0(0 + 1)^*$, which allows us to simplify the sum to $(0 + 1)^*0(0 + 1)^*$. Since the string must have a first zero, this expression can be further simplified to $1^*0(0 + 1)^*$.

3. (4pts) Give $r$, $L(r) = \{x \mid x \text{ contains } aba \text{ but not } aa\}$.

   The set of all strings of $a$s and $b$s is given by $(a + b)^*$, so the set of strings without consecutive $a$s which does not end on an $a$ is given by $(ab + b)^*$. Similarly, the set of strings without consecutive $a$s which does not begin with an $a$ is given by $(ba + b)^*$. Combining with the required $aba$ gives the solution:

   $$r = (ab + b)^*aba(ba + b)^*$$
1. (6pts) Convert \((a + b)^*\) into a regular grammar with unit productions.

\[
\begin{align*}
P_1 &= \{S_1 \to aA_1, A_1 \to \Lambda\} \\
P_2 &= \{S_2 \to bA_2, A_2 \to \Lambda\} \\
P_3 &= \{S_3 \to S_1, S_3 \to S_2, S_1 \to aA_1, A_1 \to \Lambda, S_2 \to bA_2, A_2 \to \Lambda\} \\
P_4 &= \{S_4 \to \Lambda, S_4 \to S_3, S_3 \to S_1, S_3 \to S_2, S_1 \to aA_1, A_1 \to S_4, S_2 \to bA_2, A_2 \to S_4\}
\end{align*}
\]

Using \(P_4\) as the final answer, the start symbol is \(S_4\).
2. \((4 pts)\) Convert into a regular grammar:
\[\{S \to aA, S \to B, A \to aA, A \to bB, B \to \Lambda, B \to A\}\].

Solution:

\[
\begin{align*}
S & \to aA \\
S & \to B \\
A & \to aA \\
A & \to bB \\
B & \to \Lambda \\
B & \to A
\end{align*}
\]
Quiz 9, Regular Grammar Conversion

Date: November 14

1. (6pts) Convert \{S \rightarrow aS, S \rightarrow bB, A \rightarrow aB, A \rightarrow aS, B \rightarrow bA, B \rightarrow \Lambda\} into a regular expression.

First add \(S', H\), and missing loopbacks.

\[
\begin{align*}
&\text{} & S' & \rightarrow S & S & \rightarrow bB & A & \rightarrow aB & B & \rightarrow bA & H & \rightarrow \Lambda \\
&\text{} & S & \rightarrow aS & A & \rightarrow aS & B & \rightarrow H \\
&\text{} & A & \rightarrow A & B & \rightarrow B \\
\end{align*}
\]

To remove \(S\) then \(A\) then \(B\), begin by removing \(S\).

\[
\begin{align*}
&\text{} & S' & \rightarrow S / S & \rightarrow aS / S & \rightarrow bB & : & S' & \rightarrow a^* bB \\
&\text{} & A & \rightarrow aS / S & \rightarrow aS / S & \rightarrow bB & : & A & \rightarrow a a^* bB \\
\end{align*}
\]

After removing \(S\) the remaining productions are:

\[
\begin{align*}
&\text{} & S' & \rightarrow a^* bB & A & \rightarrow a + a a^* bB & B & \rightarrow bA & H & \rightarrow \Lambda \\
&\text{} & A & \rightarrow A & B & \rightarrow H \\
&\text{} & B & \rightarrow B \\
\end{align*}
\]

Remove \(A\)

\[
\begin{align*}
&\text{} & B & \rightarrow bA / A & \rightarrow A / A & \rightarrow a + a a^* bB & : & B & \rightarrow b(a + a a^* b)B \\
\end{align*}
\]

After removing \(A\), the remaining productions are:

\[
\begin{align*}
&\text{} & S' & \rightarrow a^* bB & B & \rightarrow H & H & \rightarrow \Lambda \\
&\text{} & B & \rightarrow \Lambda + b(a + a a^* b)B \\
\end{align*}
\]

Remove \(B\)

\[
\begin{align*}
&\text{} & S' & \rightarrow a^* bB / B & \rightarrow \Lambda + b(a + a a^* b)B / B & \rightarrow H & : \\
\end{align*}
\]

Regular expression: \(a^* b(b(a + a a^* b))^*\)
2. (6pts) Convert \{S \to aS, S \to bB, A \to aB, A \to aS, B \to bA, B \to \Lambda\} into a deterministic regular grammar.

\[
\begin{align*}
V_{\{S\}} & \rightarrow aV_{\{S\}} & V_{\{A\}} & \rightarrow aV_{\{S,B\}} & V_{\{B\}} & \rightarrow aV_{\emptyset} \\
V_{\{S\}} & \rightarrow bV_{\{B\}} & V_{\{A\}} & \rightarrow bV_{\emptyset} & V_{\{B\}} & \rightarrow bV_{\{A\}} \\
V_{\{S,B\}} & \rightarrow aV_{\{S\}} & V_{\{A,B\}} & \rightarrow aV_{\{S,B\}} & V_{\{B\}} & \rightarrow \Lambda \\
V_{\{S,B\}} & \rightarrow bV_{\{A,B\}} & V_{\{A,B\}} & \rightarrow bV_{\{A\}} & V_{\{S,B\}} & \rightarrow \Lambda \\
V_{\{A,B\}} & \rightarrow \Lambda \\
V_{\{A,B\}} & \rightarrow \Lambda
\end{align*}
\]
1. (5pts) Build a DFA for $\Sigma = \{a, b, c\}$, 
$L = \{x \mid x \text{ contains at most 2 } a \text{s and at least two } c \text{s}\}$. 

![DFA Diagram]
2. (5pts) Build a DFA for $\Sigma = \{a, b\}$, $L = \{x \mid x \text{ contains } aba \text{ before any } bab\}$.

$q_0$ - no $a$s or $b$s seen
$q_a$ - last seen $a$ (but not $ba$), no $aba$ or $bab$ yet
$q_{ba}$ - last pair seen is $ba$, no $aba$ or $bab$ yet
$q_b$ - last seen $b$ (but not $ab$), no $aba$ or $bab$ yet
$q_{ab}$ - last pair seen is $ab$, no $aba$ or $bab$ yet
$q_s$ - $aba$ seen first
$q_t$ - $bab$ seen first (trap state; does not need to be shown)
Quiz 11, Finite Automata II

Date: November 30

1. (3pts) Give a DFA for \( L(P) \), \( P = abbabbb \).

2. (4pts) Give a DFA for \( L(r) \), \( r = (abb + ab)^*aba \).

3. (3pts) Prove \( L = \{a^ib^j \mid i > j \} \) is not regular.

Let \( S \) be the set \( S = \{a^i \mid i \geq 0\} \). Pick any distinct pair \( x = a^i \) and \( y = a^j \) from \( S \), and without loss of generality assume that \( i > j \) (since \( x \) and \( y \) are different, then one of them has to be the longer one, so we’ll call \( x \) the longer one).

Let \( z = b^j \), so \( z \) is the same length as \( y \) and shorter than \( x \). It follows that \( xz = a^ib^j \in L \) but \( yz = a^jb^j \notin L \). This means that no matter which distinct pair of elements are picked from \( S \), there is some string which can be appended to each of them which will allow them to be distinguished by a state machine.

Since every string in \( S \) is distinguishable by a state machine, any state machine which recognizes the language must have an infinite number of states to accommodate for each of the strings in \( S \). No finite automata can have an infinite number of states, therefore the language \( L \) is not regular.
Quiz 12, Context-Free Grammars

Date: December 7

1. (5pts) Give a CFG for \{a^i b^j c^k \mid j = 2i + 3k \}.

Since \( j = 2i + 3k \), the resulting string can be rewritten as
\( a^{i} b^{2i+3k} c^{k} = (a^{i} b^{2i})(b^{3k} c^{k}) \).

Let \( A \) generate strings of the form \( a^{i} b^{2i} \) and \( C \) generate strings of the form \( b^{3k} c^{k} \). Then the following CFG will generate the language:

\[
S \rightarrow AC
A \rightarrow aAbb \mid \Lambda
C \rightarrow bbbCc \mid \Lambda
\]

2. (5pts) Give a CFG for \{a^i b^j c^k \mid i + j > k \}.

We note that since \( i + j > k \), we can find some \( i' \) and \( j' \) such that
\( i' + j' = k \) and both \( i \geq i' \) and \( j \geq j' \). Thus, \( i = i' + n \) and \( j = j' + m \)
for some non-negative \( n \) and \( m \). Furthermore,
\[
i + j = (i' + n) + (j' + m) = (i' + j') + (n + m) = k + (n + m),
\]
so \( n + m \) must be greater than zero.

The resulting string can be rewritten \( a^{i'+n} b^{j'+m} c^{i'+j'} = a^{i'} (a^{n} b^{m}) (b^{j'} c^{j'}) c^{j'} \).

Let \( A \) generate strings of the form \( a^{n} \), \( B \) generate strings of the form \( b^{m} \), \( X \) generate nonempty strings of the form \( a^{n} b^{m} \), and \( C \) generate strings of the form \( b^{j'} c^{j'} \). Then the following CFG will generate the language:

\[
S \rightarrow aSc \mid XC
A \rightarrow aA \mid \Lambda
B \rightarrow bB \mid \Lambda
X \rightarrow AaB \mid AbB
C \rightarrow bCc \mid \Lambda
\]