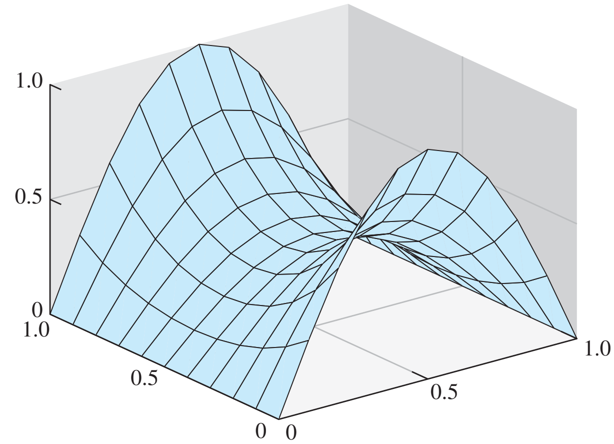

Figure 8.14 Electrostatic potential from the Laplace equation. Boundary conditions set in Example 8.9.

$$u(\tfrac{1}{4},\tfrac{7}{4}) = 1.1394 \quad u(\tfrac{2}{4},\tfrac{7}{4}) = 1.1977 \quad u(\tfrac{3}{4},\tfrac{7}{4}) = 1.2879$$
$$u(\tfrac{1}{4},\tfrac{6}{4}) = 0.8383 \quad u(\tfrac{2}{4},\tfrac{6}{4}) = 0.9163 \quad u(\tfrac{3}{4},\tfrac{6}{4}) = 1.0341$$
$$u(\tfrac{1}{4},\tfrac{5}{4}) = 0.4855 \quad u(\tfrac{2}{4},\tfrac{5}{4}) = 0.5947 \quad u(\tfrac{3}{4},\tfrac{5}{4}) = 0.7538$$

Since second-order finite difference formulas were used, the error of the Finite Difference Method `poisson.m` is second order in h and k. Figure 8.13(b) shows a more accurate approximate solution, for $h = k = 0.1$. The MATLAB code `poisson.m` is written for a rectangular domain, but changes can be made to shift to more general domains. ◀

For another example, we use the Laplace equation to compute a potential.

▶ EXAMPLE 8.9 Find the electrostatic potential on the square $[0,1] \times [0,1]$, assuming no charge in the interior and assuming the following boundary conditions:

$$u(x,0) = \sin \pi x$$
$$u(x,1) = \sin \pi x$$
$$u(0,y) = 0$$
$$u(1,y) = 0.$$

The potential u satisfies the Laplace equation with Dirichlet boundary conditions. Using mesh size $h = k = 0.1$, or $M = N = 10$ in `poisson.m` results in the plot shown in Figure 8.14. ◀

Reality Check 8 Heat distribution on a cooling fin

Heat sinks are used to move excess heat away from the point where it is generated. In this project, the steady-state distribution along a rectangular fin of a heat sink will be modeled. The heat energy will enter the fin along part of one side. The main goal will be to design the dimensions of the fin to keep the temperature within safe tolerances.

The fin shape is a thin rectangular slab, with dimensions $L_x \times L_y$ and width δ cm, where δ is relatively small. Due to the thinness of the slab, we will denote the temperature by $u(x, y)$ and consider it constant along the width dimension.

Heat moves in the following three ways: conduction, convection, and radiation. Conduction refers to the passing of energy between neighboring molecules, perhaps due to the movement of electrons, while in convection the molecules themselves move. Radiation, the movement of energy through photons, will not be considered here.

Conduction proceeds through a conducting material according to Fourier's first law

$$q = -KA\nabla u, \tag{8.41}$$

where q is heat energy per unit time (measured in watts), A is the cross-sectional area of the material, and ∇u is the gradient of the temperature. The constant K is called the **thermal conductivity** of the material. Convection is ruled by Newton's law of cooling,

$$q = -HA(u - u_b), \tag{8.42}$$

where H is a proportionality constant called the **convective heat transfer coefficient** and u_b is the ambient temperature, or **bulk temperature**, of the surrounding fluid (in this case, air).

The fin is a rectangle $[0, L_x] \times [0, L_y]$ by δ cm in the z direction, as illustrated in Figure 8.15(a). Energy equilibrium in a typical $\Delta x \times \Delta y \times \delta$ box interior to the fin, aligned along the x and y axes, says that the energy entering the box per unit time equals the energy leaving. The heat flux into the box through the two $\Delta y \times \delta$ sides and two $\Delta x \times \delta$ sides is by conduction, and through the two $\Delta x \times \Delta y$ sides is by convection, yielding the steady-state equation

$$\begin{aligned} -K\Delta y\delta u_x(x, y) + K\Delta y\delta u_x(x + \Delta x, y) - K\Delta x\delta u_y(x, y) \\ + K\Delta x\delta u_y(x, y + \Delta y) - 2H\Delta x\Delta y u(x, y) = 0. \end{aligned} \tag{8.43}$$

Here, we have set the bulk temperature $u_b = 0$ for convenience; thus, u will denote the difference between the fin temperature and the surroundings.

Dividing through by $\Delta x \Delta y$ gives

$$K\delta\frac{u_x(x + \Delta x, y) - u_x(x, y)}{\Delta x} + K\delta\frac{u_y(x, y + \Delta y) - u_y(x, y)}{\Delta y} = 2Hu(x, y),$$

and in the limit as $\Delta x, \Delta y \to 0$, the elliptic partial differential equation

$$u_{xx} + u_{yy} = \frac{2H}{K\delta}u \tag{8.44}$$

results.

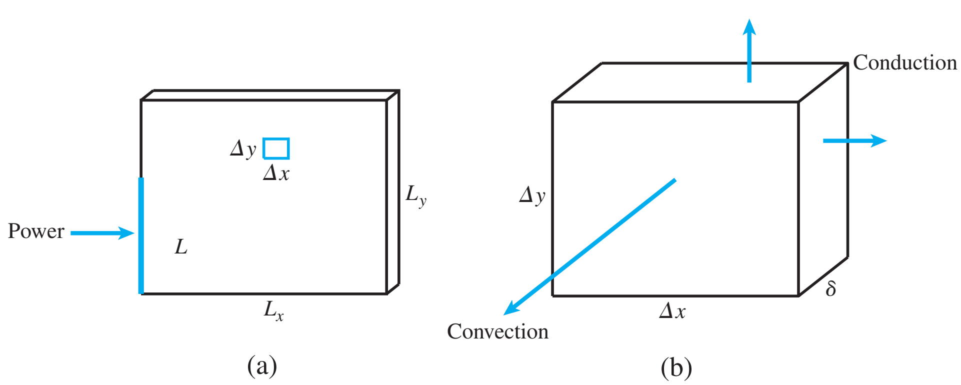

Figure 8.15 Cooling fin in Reality Check 8. (a) Power input occurs along interval $[0, L]$ on left side of fin. (b) Energy transfer in small interior box is by conduction along the x and y directions, and by convection along the air interface.

Similar arguments imply the **convective** boundary condition

$$K u_{\text{normal}} = H u$$

where u_{normal} is the partial derivative with respect to the outward normal direction $\vec{n}$. The convective boundary condition is known as a **Robin** boundary condition, one that involves both the function value and its derivative. Finally, we will assume that power enters the fin along one side according to Fourier's law,

$$u_{\text{normal}} = \frac{P}{L\delta K},$$

where P is the total power and L is the length of the input.

On a discrete grid with step sizes h and k, respectively, the finite difference approximation (5.8) can be used to approximate the PDE (8.44) as

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2} = \frac{2H}{K\delta} u_{ij}.$$

This discretization is used for the interior points (x_i, y_j) where $1 < i < m$, $1 < j < n$ for integers m, n. The fin edges obey the Robin conditions using the first derivative approximation

$$f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} + O(h^2).$$

To apply this approximation to the fin edges, note that the outward normal direction translates to

$$\begin{aligned}
u_{\text{normal}} &= -u_y \text{ on bottom edge} \\
u_{\text{normal}} &= u_y \text{ on top edge} \\
u_{\text{normal}} &= -u_x \text{ on left edge} \\
u_{\text{normal}} &= u_x \text{ on right edge}
\end{aligned}$$

Second, note that the second-order first derivative approximation above yields

$$\begin{aligned}
u_y &\approx \frac{-3u(x,y) + 4u(x,y+k) - u(x,y+2k)}{2k} \text{ on bottom edge} \\
u_y &\approx \frac{-3u(x,y) + 4u(x,y-k) - u(x,y-2k)}{-2k} \text{ on top edge} \\
u_x &\approx \frac{-3u(x,y) + 4u(x+h,y) - u(x+2h,y)}{2h} \text{ on left edge} \\
u_x &\approx \frac{-3u(x,y) + 4u(x-h,y) - u(x-2h,y)}{-2h} \text{ on right edge}
\end{aligned}$$

Putting both together, the Robin boundary condition leads to the difference equations

$$\begin{aligned}
\frac{-3u_{i1} + 4u_{i2} - u_{i3}}{2k} &= -\frac{H}{K} u_{i1} \text{ on bottom edge} \\
\frac{-3u_{in} + 4u_{i,n-1} - u_{i,n-2}}{2k} &= -\frac{H}{K} u_{in} \text{ on top edge} \\
\frac{-3u_{1j} + 4u_{2j} - u_{3j}}{2h} &= -\frac{H}{K} u_{1j} \text{ on left edge} \\
\frac{-3u_{mj} + 4u_{m-1,j} - u_{m-2,j}}{2h} &= -\frac{H}{K} u_{mj} \text{ on right edge.}
\end{aligned}$$

If we assume that the power enters along the left side of the fin, Fourier's law leads to the equation

$$\frac{-3u_{1j} + 4u_{2j} - u_{3j}}{2h} = -\frac{P}{L\delta K}. \tag{8.45}$$

There are mn equations in the mn unknowns u_{ij}, $1 \le i \le m$, $1 \le j \le n$ to solve.

Assume that the fin is composed of aluminum, whose thermal conductivity is $K = 1.68$ W/cm °C (watts per centimeter-degree Celsius). Assume that the convective heat transfer coefficient is $H = 0.005$ W/cm^2 °C, and that the room temperature is $u_b = 20$°C.

Suggested activities:

1. Begin with a fin of dimensions 2 × 2 cm, with 1 mm thickness. Assume that 5W of power is input along the entire left edge, as if the fin were attached to dissipate power from a CPU chip with $L = 2$ cm side length. Solve the PDE (8.44) with $M = N = 10$ steps in the x and y directions. Use the `mesh` command to plot the resulting heat distribution over the xy-plane. What is the maximum temperature of the fin, in °C ?
2. Increase the size of the fin to 4 × 4 cm. Input 5W of power along the interval [0, 2] on the left side of the fin, as in the previous step. Plot the resulting distribution, and find the maximum temperature. Experiment with increased values of M and N. How much does the solution change?
3. Find the maximum power that can be dissipated by a 4 × 4 cm fin while keeping the maximum temperature less than 80°C. Assume that the bulk temperature is 20°C and the power input is along 2 cm, as in steps 1 and 2.
4. Replace the aluminum fin with a copper fin, with thermal conductivity $K = 3.85$ W/cm °C. Find the maximum power that can be dissipated by a 4 × 4 cm fin with the 2 cm power input in the optimal placement, while keeping the maximum temperature below 80°C.
5. Plot the maximum power that can be dissipated in step 4 (keeping maximum temperature below 80 degrees) as a function of thermal conductivity, for $1 \le K \le 5$ W/cm°C.
6. Redo step 4 for a water-cooled fin. Assume that water has a convective heat transfer coefficient of $H = 0.1$ W/cm^2 °C, and that the ambient water temperature is maintained at 20°C.
7. Cut a rectangular notch from the right side of the fin, and redo step 4. Does the notched fin dissipate more, or less, power than the original?

The design of cooling fins for desktop and laptop computers is a fascinating engineering problem. To dissipate ever greater amounts of heat, several fins are needed in a small space, and fans are used to enhance convection near the fin edges. The addition of fans to complicated fin geometry moves the simulation into the realm of computational fluid dynamics, a vital area of modern applied mathematics. ✔

8.3.2 Finite Element Method for elliptic equations

A somewhat more flexible approach to solving partial differential equations arose from the structural engineering community in the mid-20th century. The Finite Element Method converts the differential equation into a variational equivalent called the weak form of the equation, and uses the powerful idea of orthogonality in function spaces to stabilize its calculations. Moreover, the resulting system of linear equations can have considerable symmetry in its structure matrix, even when the underlying geometry is complicated.