

Changing the rules for continued fraction expansions and their natural extension domains

Claire Merriman

November 14, 2020

The Ohio State University

Table of contents

1. Regular continued fractions
2. Odd integer continued fractions
3. α -odd continued fractions, (Boca-M, 2019)
4. Natural extensions

Regular continued fractions

Standard algorithm

Let $x > 0$, then a_0 is the largest integer less than x . $x = a_0 + \frac{1}{r_1}$ for

$1 < r_1$. Let a_1 be the largest integer less than r_1 . $x = a_0 + \frac{1}{a_1 + \frac{1}{r_2}}$ for

$1 < r_2$. Continue this process,

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} .$$

- This process terminates if and only if x is rational.
- $n = n - 1 + \frac{1}{1}$, so we require that the last digit is greater than 1.
- Then every real number has a unique continued fraction expansion.

- $$\frac{1}{[a_0; a_1, a_2, \dots]} = \frac{1}{a_0 + [0; a_1, a_2, \dots]} = [0; a_0, a_1, a_2, \dots].$$

- $$\frac{1}{[0; a_1, a_2, \dots]} = a_1 + \frac{1}{a_2 + [0; a_3, \dots]} = [a_0; a_1, a_2, \dots].$$

Define $T : [0, 1) \rightarrow [0, 1)$ by

$$T(x) = \begin{cases} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k} \right] \\ 0 & \text{if } x = 0 \end{cases}.$$

Note that $T([0; a_1, a_2, a_3, \dots]) = [0; a_2, a_3, \dots]$

Natural Extension of the regular Gauss map

Define $\bar{T} : [0, 1)^2 \rightarrow [0, 1)^2$ by

$$\bar{T}(x, y) = \begin{cases} \left(\frac{1}{x} - k, \frac{1}{y+k}\right) & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ (0, y) & \text{if } x = 0 \end{cases}.$$

$$\bar{T}([0; a_1, a_2, \dots], [0; b_1, b_2, \dots]) = ([0; a_2, a_3, \dots], [0; a_1, b_1, b_2, \dots]).$$

Facts about the regular Gauss map

- The dynamical system $([0, 1), T, \mathcal{B}_{[0,1)}, \mu)$ is ergodic for
$$d\mu = \frac{1}{\log 2} \frac{dx}{x+1}$$
- The dynamical system $([0, 1)^2, \bar{T}, \mathcal{B}_{[0,1)^2}, \nu)$ is ergodic for
$$d\nu = \frac{1}{\log 2} \frac{dx dy}{(1+xy)^2}$$

Odd integer continued fractions

Odd integer algorithm

What if instead of rounding down in the Euclidean algorithm, we found the nearest odd integer?

1. How do we deal with even integers?
2. How do we get all positive real numbers?

Odd integers, an example

Let $x = \frac{36}{13}$.

- Since $x \in (2, 3)$, $x = 3 - r_1$ for $r_1 \in (0, 1)$. Here, $r_1 = \frac{3}{13}$.
- $x = 3 - \frac{1}{\frac{13}{3}} = 3 - \frac{1}{5 - r_2}$ for $r_2 \in (0, 1)$. Here, $r_2 = \frac{2}{3}$.
- $x = 3 - \frac{1}{5 - \frac{1}{\frac{3}{2}}} = 3 - \frac{1}{5 - \frac{1}{1 + r_3}}$ for $r_3 \in (0, 1)$. Here, $r_3 = \frac{1}{2}$.
- $x = 3 - \frac{1}{5 - \frac{1}{1 + \frac{1}{1 + 1}}} = 3 - \frac{1}{5 - \frac{1}{1 + \frac{1}{3 - 1}}}$

Another example

$$G = \frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} = \frac{1}{1 - \frac{1}{3 - \frac{1}{3 - \dots}}}$$

Odd Continued fractions

Rieger (1980) defined a new type of continued fractions

$$x = a_0 + \frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \frac{\epsilon_3}{\ddots}}}$$
$$= a_0 + \langle\langle (\epsilon_1/a_1)(\epsilon_2/a_2), \dots \rangle\rangle_o$$

where the a_i are odd, $\epsilon_i = \pm 1$, and $a_i + \epsilon_{i+1} \geq 2$. That is, you never have a 1 followed by subtraction.

The extended odd Gauss Map

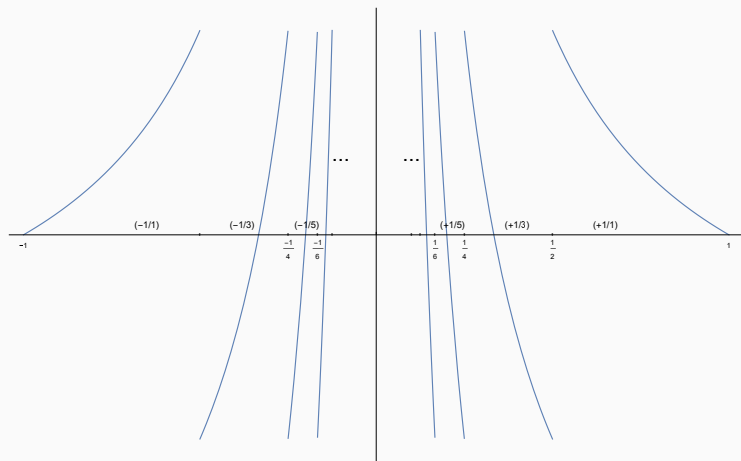
Define T_o on $[-1, 1)$

$$T_o(x) = \frac{1}{|x|} - (2k + 1) \text{ for } |x| \in \left(\frac{1}{2k+2}, \frac{1}{2k} \right]$$

Then the first digit of x is $(\text{sign}(x)/(2k + 1))$.

$$\epsilon_i = \text{sign } T_0^{i-1}(x), a_i = 2k + 1 \text{ for } |T_0^{i-1}(x)| \in \left(\frac{1}{2k+2}, \frac{1}{2k} \right]$$

Odd Gauss Map



$$T_o(\langle\langle(\epsilon_1/a_1)(\epsilon_2/a_2), \dots\rangle\rangle_o) = \langle\langle(\epsilon_2/a_2)(\epsilon_3/a_3), \dots\rangle\rangle_o.$$

Grotesque Continued Fractions

Rieger (1980) defined the grotesque continued fractions

$$y = \frac{\epsilon_0}{a_0 + \frac{\epsilon_{-1}}{a_{-1} + \frac{\epsilon_{-2}}{\ddots}}}} = \langle\langle (\epsilon_1/a_1)(\epsilon_2/a_2), \dots \rangle\rangle_{grot}$$

where the a_i are odd, $\epsilon_i = \pm 1$, and $a_i + \epsilon_i \geq 2$.

- $\frac{-1}{1+z}$ is only allowed in the odd continued fractions
- $\frac{1}{1-z}$ is only permitted in grotesque continued fractions

Grotesque Gauss map

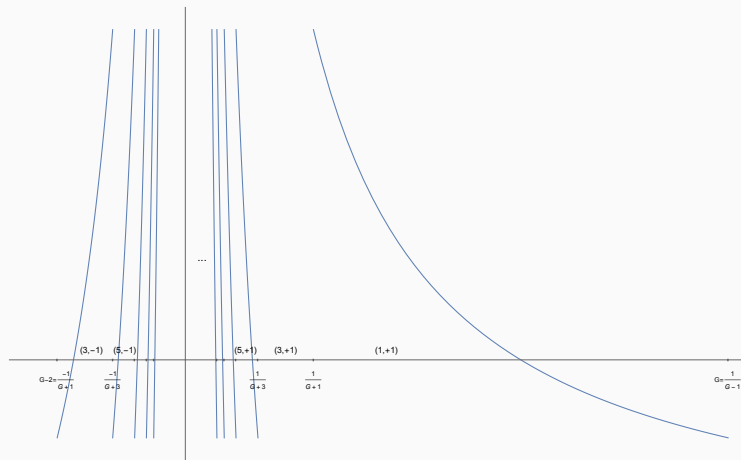
Define $T_{grot} : [G - 2, G) \rightarrow [G - 2, G)$ where $G = \frac{1+\sqrt{5}}{2}$ by

$$T_{grot}(x) = \frac{1}{|x|} - (2k + 1) \text{ for } |x| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G} \right]$$

Then the first digit of x is $(\text{sign}(x)/(2k + 1))$

$$\epsilon_i = \text{sign } T_{grot}^{i-1}(x), a_i = 2k + 1 \text{ for } |T_{grot}^{i-1}(x)| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G} \right]$$

Grotesque Gauss Map



$$T_o(\langle\langle(\epsilon_1/a_1)(\epsilon_2/a_2), \dots\rangle\rangle_{grot}) = \langle\langle(\epsilon_2/a_2)(\epsilon_3/a_3), \dots\rangle\rangle_{grot}.$$

What are we doing?

The odd continued fractions and grotesque continued fractions are *dual continued fraction expansions*.

Goals:

1. Define a way to interpolate between the grotesque continued fractions and the (extended) odd continued fractions.
2. Find invariant measures and natural extensions of the relevant Gauss maps.
3. Extend these results to other dynamical systems that generate continued fractions with odd denominators.

α -odd continued fractions,
(Boca-M, 2019)

Comparing Gauss maps

If we ignore the rules for generating the digits, both T_o and T_{grot} take

$$x \mapsto \frac{1}{|x|} - (2k + 1)$$

$$\langle\langle (\epsilon_1/a_1)(\epsilon_2/a_2) \dots \rangle\rangle \mapsto \langle\langle (\epsilon_2/a_2)(\epsilon_3/a_3) \dots \rangle\rangle$$

$T_o : [-1, 1) \rightarrow [-1, 1)$		$T_{grot} : [G - 2, G) \rightarrow [G - 2, G)$
Intervals $ x \in \left(\frac{1}{2k+2}, \frac{1}{2k}\right]$		Intervals $ x \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G}\right]$
$(\epsilon_i/1)$ always followed by $(+1/a_{i+1})$		Never $(-1/1)$.

Define a new map $\varphi_\alpha : [\alpha - 2, \alpha) \rightarrow [\alpha - 2, \alpha)$ by

$$\varphi_\alpha(x) = \frac{1}{|x|} - (2k + 1) \text{ for } |x| \in \left(\frac{1}{2k + 1 + \alpha}, \frac{1}{2k - 1 + \alpha}\right]$$

We will consider $\alpha \in [g, G)$ for $g = \frac{-1+\sqrt{5}}{2} = G - 1$.

$\varphi_\alpha : [\alpha - 2, \alpha) \rightarrow [\alpha - 2, \alpha)$ by

$$\varphi_\alpha(x) = \frac{1}{|x|} - (2k + 1) \text{ for } |x| \in \left(\frac{1}{2k+1+\alpha}, \frac{1}{2k-1+\alpha} \right]$$

Again, we define $\epsilon_i = \text{sign}(\varphi_\alpha^{i-1}(x))$, $a_i = 2k + 1$ such that

$$|\varphi_\alpha^{i-1}(x)| \in \left(\frac{1}{2k+1+\alpha}, \frac{1}{2k-1+\alpha} \right]$$

$$\varphi_\alpha(\langle\langle (\epsilon_1/a_1)(\epsilon_2/a_2)(\epsilon_3/a_3) \dots \rangle\rangle_\alpha) = \langle\langle (\epsilon_2/a_2)(\epsilon_3/a_3) \dots \rangle\rangle_\alpha$$

Let $d_\alpha(x) = 2 \lfloor \frac{1}{2|x|} - \frac{1-\alpha}{2} \rfloor + 1 = a_1(x)$ emphasizing the α .

Lemma

1. When $g < \alpha \leq G$ we have $d_\alpha(\alpha) = 1$ and when $g \leq \alpha < G$ we have $d_\alpha(\alpha - 2) = 1$.
2. When $\alpha \in (g, 1) \cup (1, G)$ we have

$$\frac{1}{\varphi_\alpha(\alpha)} + \frac{1}{\varphi_\alpha(\alpha - 2)} = -2$$

3. When $g < \alpha \leq 1$ we have

$$-\frac{1}{3+\alpha} < \varphi_\alpha(\alpha-2) \leq 0 \leq \varphi_\alpha(\alpha) < \frac{1}{1+\alpha},$$
$$d_\alpha(\varphi_\alpha(\alpha)) = d_\alpha(\varphi_\alpha(\alpha-2)) - 2 \tag{1}$$

$$\text{and } \varphi_\alpha^2(\alpha) = \varphi_\alpha^2(\alpha-2).$$

4. When $1 \leq \alpha < G$ we have

$$-\frac{1}{1+\alpha} < \varphi_\alpha(\alpha) \leq 0 \leq \varphi_\alpha(\alpha-2) < \alpha,$$
$$d_\alpha(\varphi_\alpha(\alpha)) = d_\alpha(\varphi_\alpha(\alpha-2)) + 2 \tag{2}$$

$$\text{and } \varphi_\alpha^2(\alpha) = \varphi_\alpha^2(\alpha-2).$$

Natural extensions

Natural extension of the α -odd Gauss map

We define $\bar{\varphi}_\alpha : \Omega_\alpha \rightarrow \Omega_\alpha$ by

$$\left(\frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \dots}}, \frac{1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}} \right) \mapsto \left(\frac{\epsilon_2}{a_2 + \frac{\epsilon_3}{a_3 + \dots}}, \frac{1}{a_1 + \frac{\epsilon_1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}}} \right)$$

Or

$$(x, y) \mapsto \left(\frac{1}{|x|} - (2k + 1), \frac{1}{2k + 1 + \epsilon(x)y} \right)$$

for $|x| \in \left(\frac{1}{2k + 1 + \alpha}, \frac{1}{2k - 1 + \alpha} \right], \epsilon(x) = \text{sign}(x)$.

Natural extension of the grotesque Gauss map

We define $\overline{T}_{grot} : [G - 2, G) \times [0, 1) \rightarrow [G - 2, G) \times [0, 1)$ by

$$\left(\frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \dots}}, \frac{1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}} \right) \mapsto \left(\frac{\epsilon_2}{a_2 + \frac{\epsilon_3}{a_3 + \dots}}, \frac{1}{a_1 + \frac{\epsilon_1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}}} \right)$$

Or

$$(x, y) \mapsto \left(\frac{1}{|x|} - (2k + 1), \frac{1}{2k + 1 + \epsilon(x)y} \right)$$

for $|x| \in \left(\frac{1}{2k + 1 + G}, \frac{1}{2k - 1 + G} \right]$, $\epsilon(x) = \text{sign}(x)$.

Natural extension of the extended odd Gauss map

We define $\bar{T}_0 : ([-1, 0) \times [0, 2 - G)) \cup ([0, 1) \times [0, G)) \rightarrow$
 $([-1, 0) \times [0, 2 - G)) \cup ([0, 1) \times [0, G))$ by

$$\left(\frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \dots}}, \frac{1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}} \right) \mapsto \left(\frac{\epsilon_2}{a_2 + \frac{\epsilon_3}{a_3 + \dots}}, \frac{1}{a_1 + \frac{\epsilon_1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}}} \right)$$

Or

$$(x, y) \mapsto \left(\frac{1}{|x|} - (2k + 1), \frac{1}{2k + 1 + \epsilon(x)y} \right)$$

for $|x| \in \left(\frac{1}{2k + 2}, \frac{1}{2k} \right], \epsilon(x) = \text{sign}(x)$.

Natural extension domains

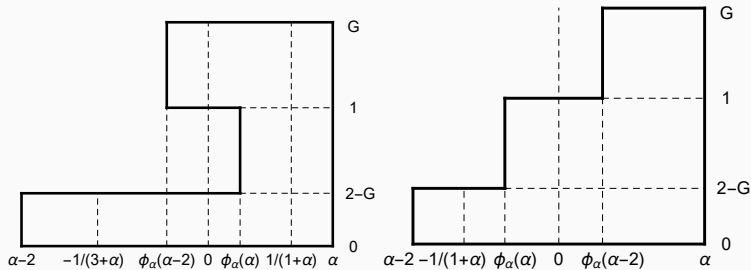


Figure 1: The natural extension domains Ω_α for $g \leq \alpha \leq 1$ (left) and $1 \leq \alpha \leq G$ (right)

(Boca-M, 2019)

- The measure $\mu_\alpha = \frac{1}{3 \log G} \frac{dx dy}{(1+xy)^2}$ is Φ_α -invariant on Ω_α .
- $d\nu_\alpha = h_\alpha d\lambda$ is φ_α -invariant with h_α coming from integrating μ_α over the y -values of the natural extension domain.
- $(\Omega_\alpha, \mathcal{B}_{\Omega_\alpha}, \tilde{\mu}_\alpha, \Phi_\alpha)$ gives the natural extension of $(I_\alpha, \mathcal{B}_{I_\alpha}, \nu_\alpha, \varphi_\alpha)$.
- $(I_\alpha, \mathcal{B}_{I_\alpha}, \nu_\alpha, \varphi_\alpha)$ is ergodic, exact, and mixing of all orders.

Standard consequences of ergodicity

Let $\frac{p_n}{q_n} = \langle\langle (\epsilon_1/a_1) \dots (\epsilon_n/a_n) \rangle\rangle_\alpha$,

For every $\alpha \in [g, G]$ and almost every $x \in \alpha$:

$$(i) \lim_n \frac{1}{n} \log |q_n(x; \alpha)x - p_n(x; \alpha)| = -\frac{\pi^2}{18 \log G}.$$

$$(ii) \lim_n \frac{1}{n} \log q_n(x; \alpha) = \frac{\pi^2}{18 \log G}.$$

$$(iii) \lim_n \frac{1}{n} \log \left| x - \frac{p_n(x; \alpha)}{q_n(x; \alpha)} \right| = -\frac{\pi^2}{9 \log G}.$$

For every $\alpha \in [g, G]$ the entropy of φ_α with respect to ν_α is

$$H(\alpha) = -2 \int_{\alpha-2}^{\alpha} \log |x| h_\alpha(x) dx = \frac{\pi^2}{9 \log G}.$$