Hyperbolic surfaces, cutting sequences, and continued fractions

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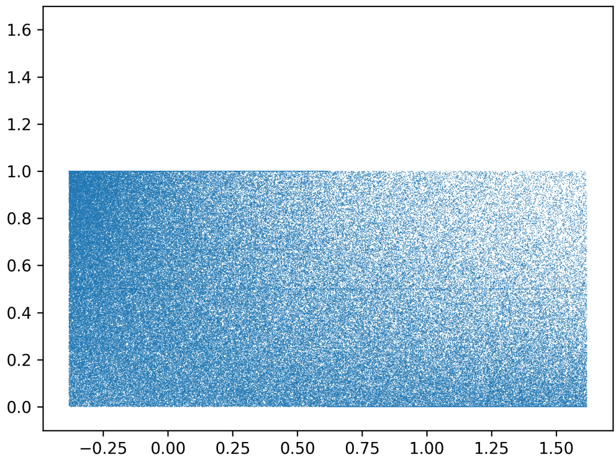
OCF Animation



First frame of the animiation.

Regular Continued Fractions

Way to represent $x > 0$ as

- $x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$

Regular Continued Fractions

Way to represent $x > 0$ as

- $$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$
- $$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \dots}}}}$$

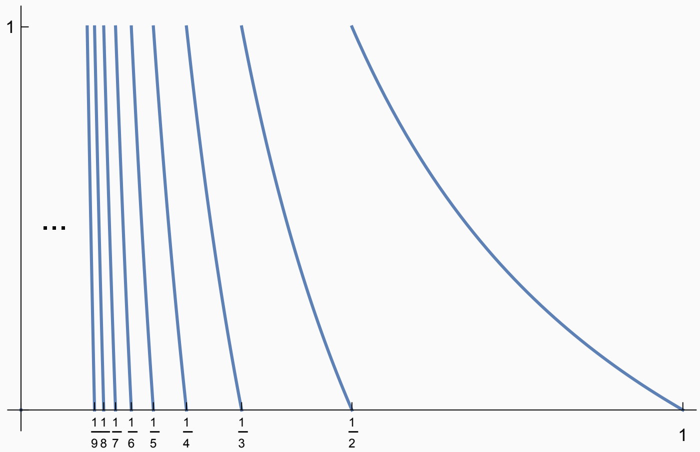
Dynamics

Define $T : [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k} \right] \\ 0 & \text{if } x = 0 \end{cases}.$$

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \mapsto \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

Gauss map



Natural Extension

Define $\bar{T} : [0,1)^2 \to [0,1)^2$ by

$$\bar{T}(x,y) = \begin{cases} \left(\frac{1}{x} - k, \frac{1}{y+k}\right) & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ (0,y) & \text{if } x = 0 \end{cases}.$$

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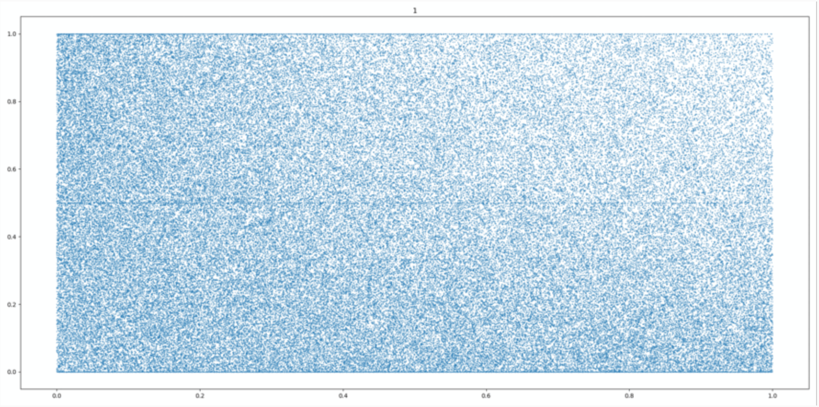
$$\left(\cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots}}, \cfrac{1}{a_{-1} + \cfrac{1}{a_2 + \dots}} \right) \mapsto \left(\cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}, \cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots}} \right)$$

Natural extension domain



Plot of $\bar{T}^n(x,0)$ for 1500 values of $x, 1 \leq n \leq 200$

Nakada α-continued fractions

Nakada (1981) introduced the α-continued fractions.

Define T_α on $[\alpha - 1, \alpha]$:

$$\begin{aligned} T_\alpha(x) &= \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor \\ &= \frac{\epsilon}{x} - a_1 \text{ for } \epsilon x \in \left[\frac{1}{a_1 - 1 + \alpha}, \frac{1}{a_1 + \alpha}\right) \end{aligned}$$

$$x = \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \dots}}.$$

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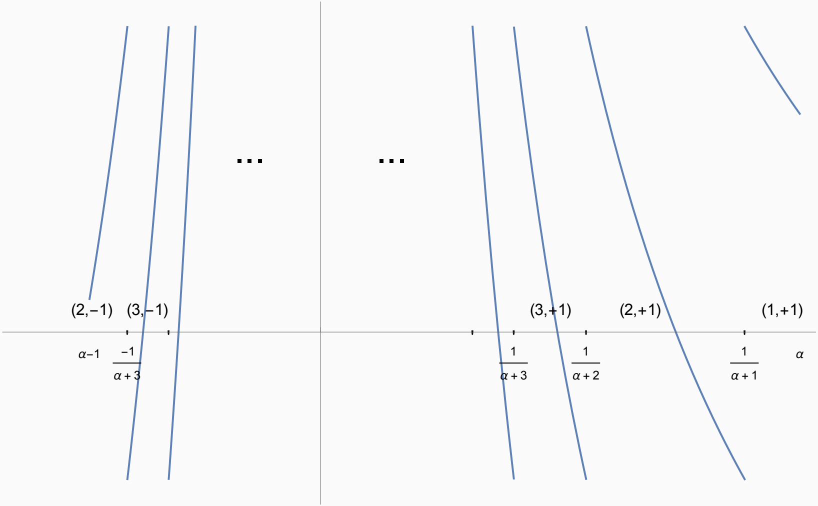
$$x = \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \dots}}.$$

When $\alpha = \frac{1}{2}$, $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{16 - \cfrac{1}{293 + \dots}}}$

Gauss map



(2,−1)
(3,−1)
(3,+1)
(2,+1)
(1,+1)
$\alpha-1$
$\frac{-1}{\alpha+3}$
$\frac{1}{\alpha+3}$
$\frac{1}{\alpha+2}$
$\frac{1}{\alpha+1}$
α

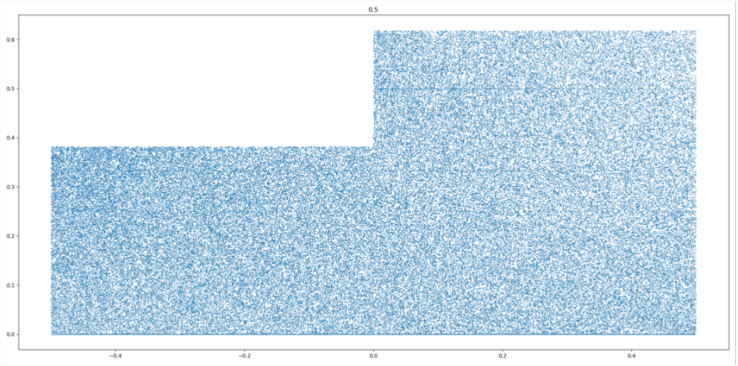
Natural extension

Natural extension defined on $[\alpha - 1, \alpha) \times R_\alpha$.

$$(x, y) \mapsto \left(\frac{\epsilon}{x} - a_1, \frac{1}{a_1 + \epsilon y}\right) \text{ for } \epsilon x \in \left[\frac{1}{a_1 - 1 + \alpha}, \frac{1}{a_1 + \alpha}\right)$$

$$\left(\frac{\epsilon_0}{a_0 + \frac{\epsilon_1}{a_1 + \dots}}, \frac{1}{a_{-1} + \frac{\epsilon_{-1}}{a_2 + \dots}}\right) \mapsto \left(\frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_3 + \dots}}, \frac{1}{a_0 + \frac{\epsilon_0}{a_1 + \dots}}\right)$$

RCF Animation



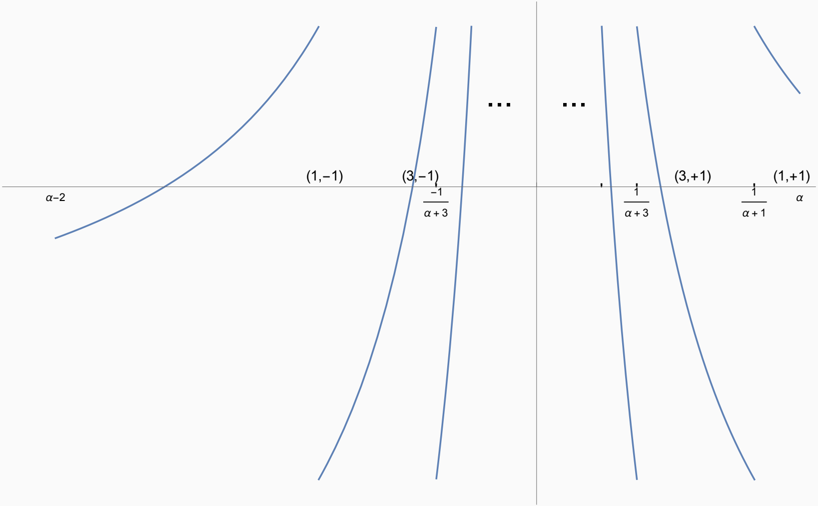
Frame of the animation of the natural extension domain where $\alpha = .5$.

α-odd continued fractions

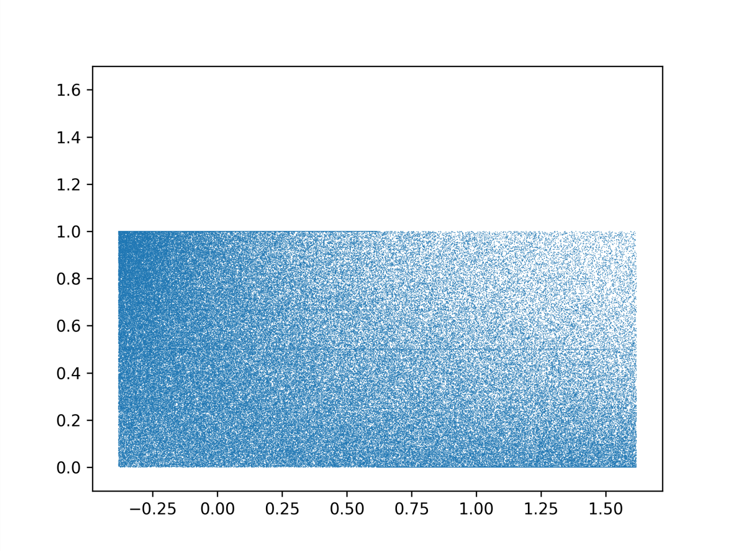
Boca-M (2019) introduced the α-odd continued fractions. Define φ_α on $[\alpha - 2, \alpha]$

$$\varphi_\alpha(x) = \frac{\epsilon}{x} - 2a_1 + 1 \text{ for } \epsilon x \in \left[\frac{1}{2a_1 + 1 + \alpha}, \frac{1}{2a_1 - 1 + \alpha}\right)$$

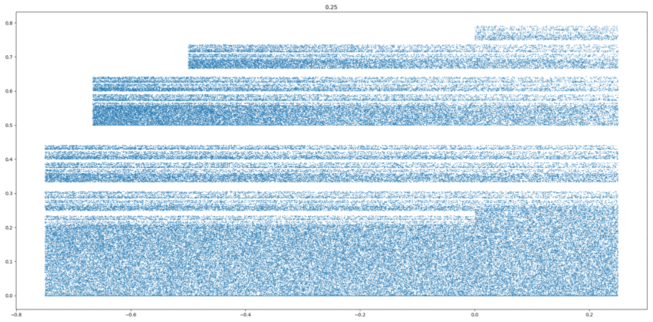
α-odd continued fractions

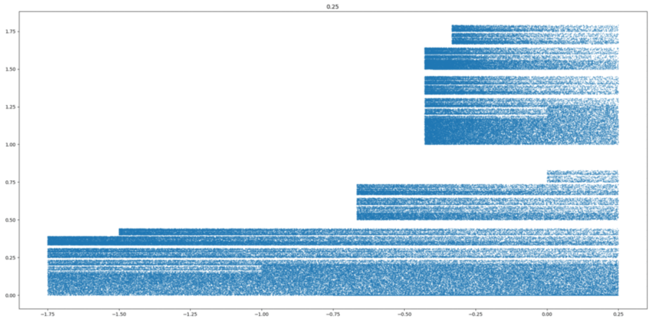


OCF Animation



First frame of the animation of the natural extension domain with $\frac{1+\sqrt{5}}{}$



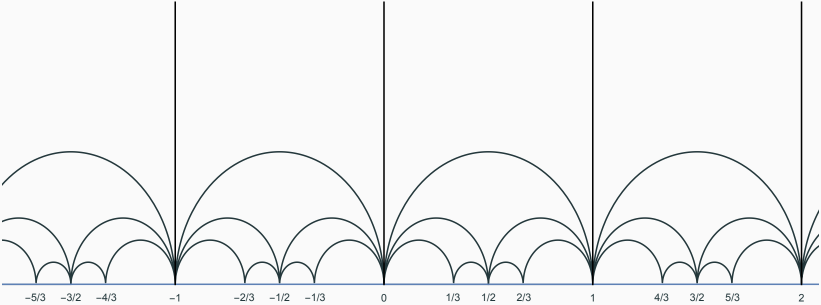


Two stills from the animationof the natural extension domain,

Farey Tessellation

$\mathbb{H} := \{x + iy \mid y > 0\}$

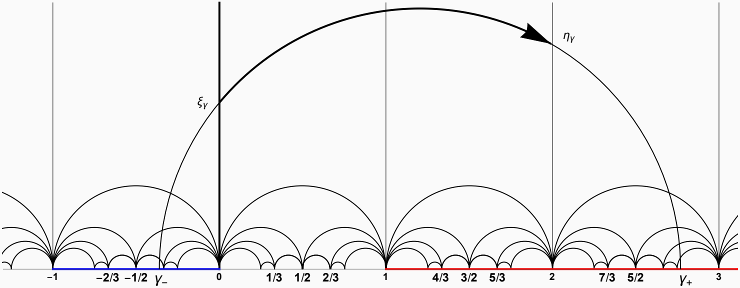
Connect two rational numbers $\frac{p}{q}, \frac{p'}{q'}$ iff $pq' - p'q = \pm 1$.

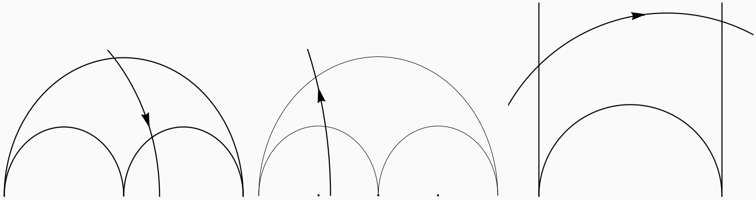


Geodesics

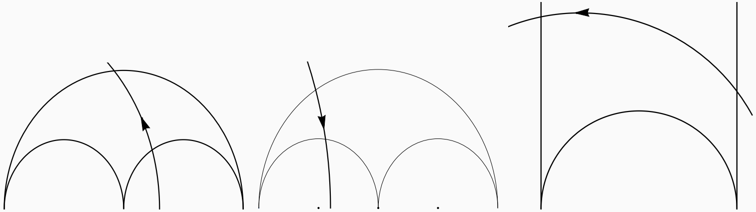
Let $\mathcal{S}$ be the set of geodesics γ with endpoints

- $\gamma_{-\infty} \in (-1, 0), \gamma_{\infty} \geq 1$
- $\gamma_{-\infty} \in (0, 1), \gamma_{\infty} \leq -1$



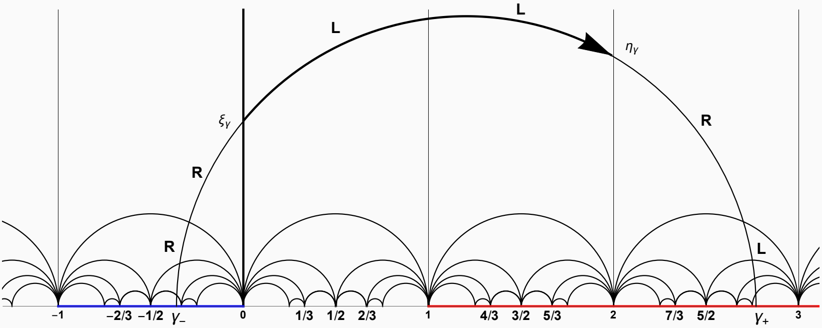


Some segments of type L



Some segments of type R

Example



Cutting sequence $\ldots RR\xi_\gamma L^2 R^1 L^3 \ldots$

Theorem (Series, '85)

A geodesic from $\gamma_{-\infty}$ to γ_∞ has two options:

- $\gamma_{-\infty} \in (-1,0),\ \gamma_\infty \in (1,\infty)$. *This geodesic has the coding* $\dots L^{n_{-2}} R^{n_{-1}} \xi_\gamma L^{n_0} R^{n_1} L^{n_2} \dots$

$$\gamma_{-\infty} = -[n_{-1}, n_{-2}, \dots] \text{ and } \gamma_\infty = n_0 + [n_1, n_2, \dots]$$

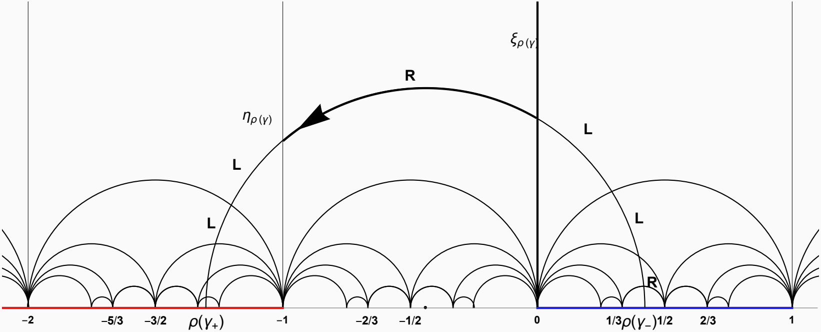
- $\gamma_{-\infty} \in (0,1),\ \gamma_\infty \in (-\infty,-1)$. *This geodesic has the coding* $\dots L^{n_{-2}} L^{n_{-1}} \xi_\gamma R^{n_0} L^{n_1} R^{n_2} \dots$

$$\gamma_{-\infty} = [n_{-1}, n_{-2}, \dots] \text{ and } \gamma_\infty = -\left(n_0 + [n_1, n_2, \dots]\right).$$

Action on Upper Half Plane

Case 1, $\gamma_\infty > 1$.

Define ρ on $\mathcal{S}$ by $(x, y) \mapsto (\frac{1}{a_1 - x}, \frac{1}{a_1 - y})$.



$$\dots L^1 R^2 \xi_\gamma L^2 \eta_\gamma R^1 L^3 \dots \mapsto L^1 R^2 L^2 \xi_{\rho(\gamma)} R^1 \eta_{\rho(\gamma)} L^3 \dots$$

Lehner expansions

Lehner (1994) defined continued fractions $x \in [1, 2]$

$$x = a_0 + \cfrac{e_o}{a_1 + \cfrac{e_1}{a_2 + \dots}}$$

$(a_i, e_i) = (1, +1), (2, -1)$.

Lehner expansions

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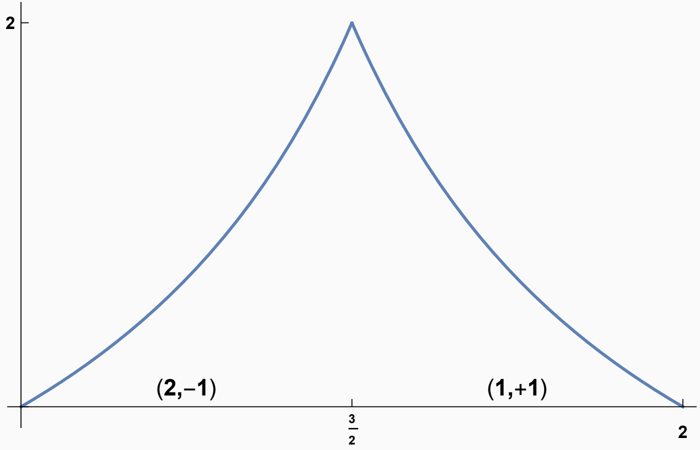
$$x = a_0 + \cfrac{e_o}{a_1 + \cfrac{e_1}{a_2 + \dots}}$$

$(a_i, e_i) = (1, +1), (2, -1)$.

Define $L : [1,2] \to [1,2]$ by

$$L(x) = \begin{cases} \frac{1}{2-x} & \text{if } x \in \left[1, \frac{3}{2}\right] \\ \frac{1}{x-1} & \text{if } x \in \left(\frac{3}{2}, 2\right] \end{cases}$$

Tent map



Dajani and Kraaikamp (2000) introduced the Farey expansions for $y \in [-1, \infty)$

$$y = \cfrac{f_0}{b_0 + \cfrac{f_1}{b_1 + \dots}} = \langle\!\langle (f_0/b_0)(f_1/b_1)(f_2/b_2)\dots \rangle\!\rangle$$

$(f_i/b_i) = (+1/1), (-1/2).$

$$\pi = \langle\!\langle (1/1)(-1/2)^3(1/1)(-1/2)^6(1/1)(-1/2)^{14}\dots \rangle\!\rangle$$

Natural extension

$\mathcal{L} : [1,2) \times [-1,\infty) \to [1,2) \times [-1,\infty)$ by

$$\left(\frac{e_0}{x - a_0}, \frac{e_0}{y + a_0}\right) = \begin{cases} \left(\frac{-1}{x-2}, \frac{-1}{y+2}\right) & x \in \left[1, \frac{3}{2}\right) \\ \left(\frac{1}{x-1}, \frac{1}{y+1}\right) & x \in \left[\frac{3}{2}, 2\right) \end{cases}.$$

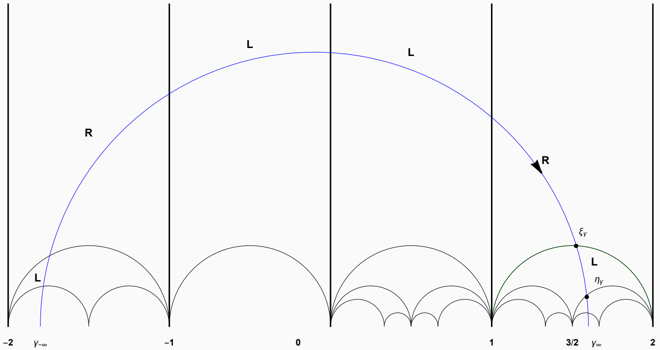
$$\left(a_0 + \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \ldots}}, \cfrac{1}{a_{-1} + \cfrac{\epsilon_{-1}}{a_2 + \ldots}} \right) \mapsto \left(a_1 + \cfrac{\epsilon_2}{a_3 + \ldots}, \cfrac{1}{a_0 + \cfrac{\epsilon_0}{a_1 + \ldots}} \right)$$

Geodesics

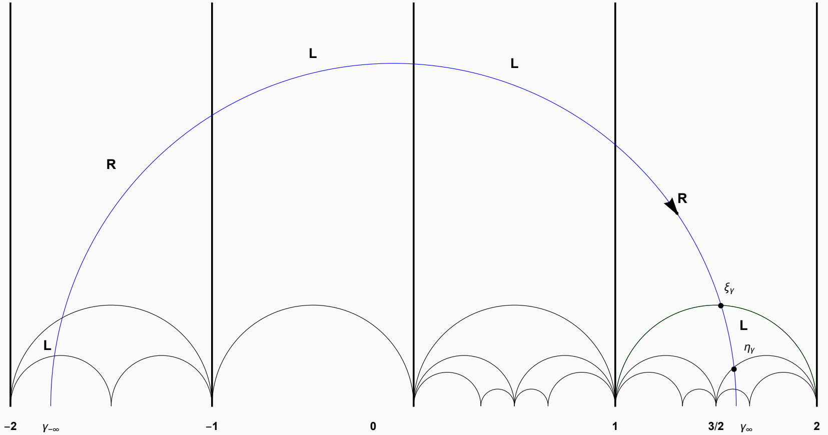
Connect backwards endpoint $\gamma_{-\infty}$ to forward endpoint γ_{∞} with γ

Either

- $\gamma_{-\infty} < 1, 1 < \gamma_{\infty} < 2$
- $\gamma_{-\infty}\ -1, -2 < \gamma_{\infty} < -1$



Example



Cutting sequence $\ldots LRL^2 R \xi_\gamma L \eta_\gamma R \ldots$

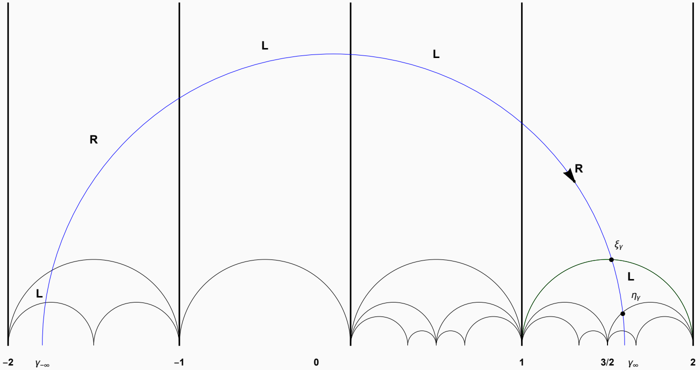
Converting to Lehner and Farey expansions

Read Lehner expansion of γ_∞ starting at ξ_γ.

Farey expansion of $\gamma_{-\infty}$ from right to left starting at ξ_γ.

If the letter is the same as the previous (letter to the left), the digit is $(2, -1)$, if it is different than the previous letter, the digit is $(1, +1)$.

Example



Cutting sequence $\ldots RRL^2R\xi_\gamma L\eta_\gamma R\ldots$

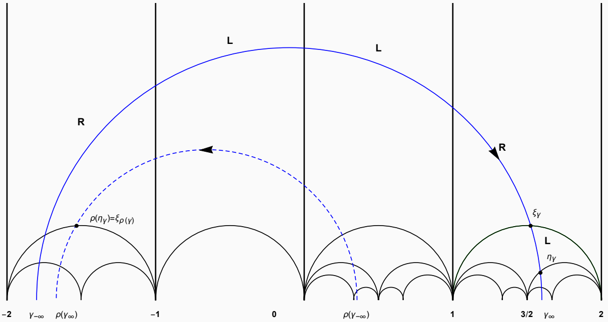
$$R\xi_\gamma LR\ldots \rightsquigarrow [\![(1,+1),(1,+1),\ldots]\!]$$

$$\ldots LRLLR\xi_\gamma \rightsquigarrow \langle\!\langle(+1/1)(-1/2)(+1/1)(-1/2)\ldots\rangle\!\rangle$$

Action on Upper Half Plane

Case 1, $1 < \gamma_\infty < 2$.

Define ρ on $\pm((1,2) \times (-\infty, 1))$ by $(x,y) \mapsto \left(\frac{1}{a_1 - x}, \frac{1}{a_1 - y}\right)$.



$$\ldots LRL^2R\xi_\gamma L\eta_\gamma R \cdots \mapsto \ldots LRL^2RL\xi_{\rho(\gamma)} R\eta_{\rho(\gamma)} \ldots$$

Thank You

Questions?