Solutions for HW 9
STAT 346, Spring 2010

1)
(a) We have

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \\
&= \int_1^2 x\, \frac{4}{15} x^3\, dx \\
&= \int_1^2 \frac{4}{15} x^4\, dx \\
&= \frac{4}{75} x^5 \Big|_1^2 \\
&= \frac{4}{75}(32-1) \\
&= \frac{124}{75}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx \\
&= \int_1^2 x^2\, \frac{4}{15} x^3\, dx \\
&= \int_1^2 \frac{4}{15} x^5\, dx \\
&= \frac{4}{90} x^6 \Big|_1^2 \\
&= \frac{4}{90}(64-1) \\
&= \frac{14}{5}.
\end{aligned}$$

Alternatively, using the pdf of $Y = X^2$ obtained for HW 7,

$$\begin{aligned}
E(X^2) &= E(Y) \\
&= \int_{-\infty}^{\infty} y\, f_Y(x)\, dy \\
&= \int_1^4 y\, \frac{2}{15} y\, dy \\
&= \int_1^4 \frac{2}{15} y^2\, dy \\
&= \frac{2}{45} y^3 \Big|_1^4 \\
&= \frac{2}{45}(64-1) \\
&= \frac{14}{5}.
\end{aligned}$$

2) From the pdf, one can obtain that the cdf is

$$F(x) = \begin{cases} 1 - 1/x, & x > 1, \\ 0, & x \leq 1. \end{cases}$$

Then, from $x = F(F^{-1}(x)) = 1 - 1/[F^{-1}(x)]$, one can solve to obtain that $F^{-1}(x) = 1/(1-x)$. So the desired function of U is $1/(1-U)$. (Note: Since U has the same distribution as $1-U$, we could also use $1/U$.)

3) From $x = F(F^{-1}(x)) = (F^{-1}(x)-3)/(F^{-1}(x)-2)$, one can solve to obtain that $F^{-1}(x) = (2x-3)/(x-1)$. So if U is a uniform $(0, 1)$ random variable, $(2U-3)/(U-1)$ will have cdf F (the same distribution as X).

4) Letting X be the length of a randomly selected sheet, we have that $X \sim \mathrm{N}\,(75,\, 1)$. The desired probability is

$$\begin{aligned} P(74.5 < X < 75.8) &= P(X < 75.8) - P(X \le 74.5) \\ &= \Phi([75.8 - 75]/1) - \Phi([74.5 - 75]/1) \\ &= \Phi(0.8) - \Phi(-0.5) \\ &\doteq 0.7881 - 0.3085 \\ &\doteq 0.480. \end{aligned}$$

5) Letting X be the lifetime of a randomly selected bulb, we have that $X \sim \mathrm{N}\,(1000,\, 100^2)$. We have

$$\begin{aligned} P(X \ge 900) &= 1 - P(X < 900) \\ &= 1 - \Phi([900 - 1000]/100) \\ &= 1 - \Phi(-1) \\ &= \Phi(1) \\ &\doteq 0.8413. \end{aligned}$$

This probability indicates that only about 84.13% of the bulbs last at least 900 hours, and so the company's claim is not correct.

6) Letting X be the number of 5s that appear when 1000 random digits are generated, we have that $X \sim \text{binomial}\,(1000,\, 0.1)$. Using the normal approximation, the desired probability is

$$\begin{aligned} P(X \le 93) &= \Phi([93 + 1/2 - 1000(0.1)]/\sqrt{1000(0.1)(0.9)}) \\ &= \Phi(-6.5/\sqrt{90}) \\ &\doteq \Phi(-0.68516) \\ &\doteq 0.484\Phi(-0.68) + 0.516\Phi(-0.69) \\ &\doteq 0.484(0.2483) + 0.516(0.2451) \\ &\doteq 0.247. \end{aligned}$$

7) Letting X be an exponential random variable having mean 1700, and $Y \sim \text{binomial}\,(20,\, p)$, where

$$p = P(X < 1700) = 1 - \exp(-1700/1700) = 1 - e^{-1},$$

the desired probability is

$$\begin{aligned} P(Y \ge 2) &= 1 - P(Y \le 1) \\ &= 1 - P(Y = 0) - P(Y = 1) \\ &= 1 - (1-p)^{20} - 20p(1-p)^{19} \\ &= 1 - (1 - [1 - e^{-1}])^{20} - 20(1 - e^{-1})(1 - [1 - e^{-1}])^{19} \\ &= 1 - (e^{-1}])^{20} - 20(1 - e^{-1})(e^{-1})^{19} \\ &= 1 - e^{-20} - 20(1 - e^{-1})e^{-19} \\ &= 1 - 20e^{-19} + 19e^{-20} \\ &\doteq 0.999999927. \end{aligned}$$

(*Note*: When a probability is real close to being 1, instead of rounding it to 1, it's better to give enough digits to give some indication of how much the probability differs from 1.)

8) Letting X be the number of the 100 patients who can be cured with the drug, $X \sim \text{binomial}\,(100,\, 0.9)$. Using the normal approximation, the desired probability is

$$\begin{aligned}
P(X \geq 85) &= 1 - P(X \leq 84) \\
&\doteq 1 - \Phi([84 + 1/2 - 100(0.9)]/\sqrt{100(0.1)(0.9)}) \\
&= 1 - \Phi(-5.5/\sqrt{9}) \\
&\doteq 1 - \Phi(-1.83333) \\
&\doteq \Phi(1.83333) \\
&\doteq 0.667\Phi(1.83) + 0.333(1.84) \\
&\doteq 0.967.
\end{aligned}$$

9) Letting X_1 be the lifetime of a randomly selected bulb from company 1, we have that $X_1 \sim \mathrm{N}\,(1000,\, 100^2)$. Letting X_2 be the lifetime of a randomly selected bulb from company 2, we have that $X_2 \sim \mathrm{N}\,(900,\, 150^2)$. Letting $p_i = P(X_i \geq 980)$ $(i = 1, 2)$, we have

$$\begin{aligned}
P(X_1 \geq 980) &= 1 - P(X_1 < 980) \\
&= 1 - \Phi([980 - 1000]/100) \\
&= 1 - \Phi(-0.2) \\
&= \Phi(0.2) \\
&\doteq 0.5793
\end{aligned}$$

and

$$\begin{aligned}
P(X_2 \geq 980) &= 1 - P(X_2 < 980) \\
&= 1 - \Phi([980 - 900]/150) \\
&\doteq 1 - \Phi(0.5333) \\
&\doteq \Phi(-0.5333) \\
&\doteq 0.2969.
\end{aligned}$$

Assuming independence, and using a result from Ch. 1 for the probability of the union of two events, the desired probability is

$$p_1 + p_2 - p_1 p_2 \doteq 0.704.$$

10) Due to the memoryless property of exponential random variables, it doesn't matter that the person has already gone 5 years since his/her previous heart attack. There is a 0.5 probability that another heart attack will occur within 5 years from now, and a 0.5 probability that another heart attack won't occur within 5 years from now. So the desired probability is 0.5.

11) If she finishes documents according to a Poisson process having a rate of 1 per hour, the number she can finish in 8 hours will be a Poisson random variable, $N(8)$, having a mean of 8. The desired probability is $P(N(8) \geq 4) = 1 - P(N(8) \leq 3) = 1 - e^{-8} - 8e^{-8} - (8^2/2)e^{-8} - (8^3/6)e^{-8} = 1 - (379/3)e^{-8} \doteq 0.958$.

12) Since X is an exponential random variable having mean $1/\lambda$, for the desired probability we have

$$\begin{aligned}
P(X > E(X)) &= P(X > 1/\lambda) \\
&= \int_{1/\lambda}^{\infty} \lambda e^{-\lambda x}\, dx \\
&= -e^{-\lambda x}\big|_{1/\lambda}^{\infty} \\
&= e^{-1} \\
&= 0.368.
\end{aligned}$$