Solutions for HW 8
STAT 346, Spring 2010

1)
(a) $P(X < 5/4) = \int_1^{5/4} 2x^{-2}\,dx = -2x^{-1}|_1^{5/4} = 2/5$.
(b) $E(X) = \int_1^2 x\,2x^{-2}\,dx = \int_1^2 2x^{-1}\,dx = 2\log x|_1^2 = 2\log 2$.
(c) $E(1/X) = \int_1^2 x^{-1}\,2x^{-2}\,dx = \int_1^2 2x^{-3}\,dx = -x^{-2}|_1^2 = 3/4$.
2)
(a) Since the support of $Y = e^X$ is $[1, \infty)$, we have $f_Y(y) = 0$ for $y < 1$. For $y \geq 1$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(e^X \leq y) \\ &= P(X \leq \log y) \\ &= \int_0^{\log y} 3e^{-3x}\,dx \\ &= -e^{-3x}|_0^{\log y} \\ &= -e^{-3\log y} - (-e^0) \\ &= 1 - e^{\log y^{-3}} \\ &= 1 - y^{-3}. \end{aligned}$$

It follows that

$$f_Y(y) = \frac{3}{y^4}\, I_{[1,\,\infty)}(y).$$

Alternatively, using $h(x) = e^x$ and $h^{-1}(x) = \log x$, the transformation method gives us

$$f_Y(y) = f_X(\log y)\left|\frac{d}{dy}\log y\right| I_{[1,\,\infty)}(y) = 3e^{-3\log y}\,\frac{1}{y}\, I_{[1,\,\infty)}(y) = \frac{3}{y^4}\, I_{[1,\,\infty)}(y).$$

(b) $E(e^X) = \int_0^\infty e^x\,3e^{-3x}\,dx = \int_0^\infty 3e^{-2x}\,dx = -(3/2)e^{-2x}|_0^\infty = 3/2$. (Alternatively, one can use the pdf of $Y = e^X$ as follows: $E(e^X) = E(Y) = \int_1^\infty y\,3y^{-4}\,dy = \int_1^\infty 3y^{-3}\,dy = -(3/2)y^{-2}|_1^\infty = 3/2$.)
3) We have

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x\, f_X(x)\,dx \\ &= \int_0^1 x\,6x(1-x)\,dx \\ &= \int_0^1 (6x^2 - 6x^3)\,dx \\ &= [2x^3 - (3/2)x^4]\Big|_0^1 \\ &= 1/2 \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2\, f_X(x)\,dx \\ &= \int_0^1 x^2\,6x(1-x)\,dx \\ &= \int_0^1 (6x^3 - 6x^4)\,dx \\ &= [(3/2)x^4 - (6/5)x^5]\Big|_0^1 \\ &= 3/10, \end{aligned}$$

and so $\sigma_X^2 = E(X^2) - \mu_X^2 = 3/10 - (1/2)^2 = 1/20$, and $\sigma_X = 1/\sqrt{20} = 1/(2\sqrt{5})$. It follows that the desired probability is

$$\begin{aligned}
P(\mu_X - 2\sigma_X < X < \mu_X + 2\sigma_X) &= P(1/2 - 1/\sqrt{5} < X < 1/2 + 1/\sqrt{5}) \\
&= \int_{1/2-1/\sqrt{5}}^{1/2+1/\sqrt{5}} 6x(1-x)\,dx \\
&= (3x^2 - 2x^3)\Big|_{1/2-1/\sqrt{5}}^{1/2+1/\sqrt{5}} \\
&= [3(1/2 + 1/\sqrt{5})^2 - 2(1/2 + 1/\sqrt{5})^3] \\
&\qquad - [3(1/2 - 1/\sqrt{5})^2 - 2(1/2 - 1/\sqrt{5})^3] \\
&= 11/(5\sqrt{5}) \\
&\doteq 0.98387.
\end{aligned}$$