Solutions for HW 7 STAT 346, Spring 2010

1) (Note: Unless you've had a decent course in statistics, I don't expect you to have done this one the same way I did it.) Letting X be the number of defectives in a sample of 24, it makes sense to reject the manufacturer's claim if the observed value of X is sufficiently large. To reject with the observed value, x = 2, the rejection region would have to be $\{2, 3, 3, \ldots, 24\}$; i.e., we'd have to consider 2 to be sufficiently large. To determine whether it's sensible to reject for x = 2, we should consider the probability of getting two or more defectives if the manufacturer's claim is true. If this probability is small, then the fact that X took a value of 2 or larger is not compatible with the manufacturer's claim, and we should reject the claim. But if the probability is not too small, we can say the observed number of defectives is not highly inconsistent with the claim and we don't have strong evidence to reject the claim. (Note: What is meant by small and large is flexible, but often we say that we have significant evidence to reject if the probability is less than or equal to 0.05.) So the key probability to consider is $P(X \ge 2)$ where X is a binomial (24, 0.03) random variable. (*Note*: One might think that if the sample was drawn without replacement then using a hypergeometric random variable would be appropriate. But to do that we'd have to know the overall number of nails and that's not given. Assuming the sample size of 24 is rather small compared to the overall population size, it won't matter much if we use the binomial distribution even if the sample was drawn without replacement, resulting in a lack of independence.) Since

$$P(X \ge 2) = 1 - p_X(0) - p_X(1) = 1 - (0.97)^{24} - 24(0.03)(0.97)^{23} \doteq 0.161$$

isn't real small, we don't have strong evidence with which to reject the manufacturer's claim.

2) We can consider iid Bernoulli(1/12) trials, so that if V is the number of tries needed to open the door, V is a geometric(1/12) random variable.

(a) The desired expectation is E(V) = 1/(1/12) = 12.

(b) The desired probability is $P(V=3) = (11/12)^2(1/12) = 121/1728 \doteq 0.0700$.

3) Since the cards are drawn with replacement, we can consider independent Bernoulli trials with the probability of success (a \blacklozenge) being 13/52 = 1/4 on each trial. Letting X be a binomial(8, 1/4) random variable, the desired probability is

$$P(X \ge 4) = 1 - P(X \le 3) = 1 - \sum_{x=0}^{3} \binom{8}{x} (1/4)^x (3/4)^{8-x} = 0.114$$

4) Letting N(2) be the number of earthquakes in the next two weeks, N(2) is a Poisson random variable having mean $\lambda t = (3)(2) = 6$, and the desired probability is $P(N(2) = 0) = e^{-6} \doteq 0.00248$.

5) Using a geometric distribution pmf the desired probability is $(5/6)^5(1/6) = 3125/46656 \doteq 0.0670$.

6) Using the formula for the expected value of a hypergeometric random variable we have that the desired expectation is 5(6/16) = 15/8 = 1.875.

7)

(a) $P(X > 8) = 1 - P(X \le 8) = 1 - F_X(8) = 1 - (1 - 16/8^2) = 16/64 = 1/4.$

(b) Differentiating the cdf we have

$$f_X(x) = \frac{32}{x^3} I_{(4,\infty)}(x)$$

(but we can note that since the cdf isn't differentiable at x = 4, how $f_X(4)$ is defined is arbitrary).

8)

(a) Since the support of X is (1, 2), we have $F_X(x) = 0$ for $x \le 1$ and $F_X(x) = 1$ for $x \ge 2$. For $x \in (1, 2)$, $F_X(x) = \int_1^x \frac{4}{15} t^3 dt = \frac{t^4}{15} |_1^x = (x^4 - 1)/15$. Altogether, we have

$$F_X(x) = \begin{cases} 1, & x \ge 2, \\ (x^4 - 1)/15, & 1 < x < 2, \\ 0, & x \le 0. \end{cases}$$

(b) $P(X > 3/2) = 1 - F_X(3/2) = 1 - [(3/2)^4 - 1]/15 = 35/48$ (making use of the answer for part (a)). (c) Since the support of $Y = X^2$ is $(1, 4), f_Y(y) = 0$ for $y \notin (1, 4)$. For $y \in (1, 4), F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) = (y^2 - 1)/15$, and (upon differentiating) $f_Y(y) = 2y/15$. Altogether, we have

$$f_Y(y) = \frac{2y}{15} I_{(1,4)}(y).$$

9) Letting Y be the number of radios lasting more than 15 years, Y is a binomial (8, p) random variable, where

$$p = P(X > 15) = \int_{15}^{\infty} \frac{1}{15} e^{-x/15} \, dx = -e^{-x/15} \big|_{15}^{\infty} = e^{-1}.$$

The desired probability is

$$P(Y \ge 4) = 1 - P(Y \le 3) = 1 - \sum_{y=0}^{3} \binom{8}{y} (e^{-1})^{y} (1 - e^{-1})^{8-y} \doteq 0.3327.$$

10) The support of Y is $(1, \infty)$. So for $y \leq 1$, $F_Y(y) = 0$. For y > 1,

$$F_Y(y) = P(Y \le y) = P(1/X \le y) = P(1/y \le X) = 1 - P(X < 1/y) = 1 - 1/y,$$

where the last equality is due to X being a random number from (0, 1). So, altogether,

$$F_Y(y) = \begin{cases} 1 - 1/y, & y > 1, \\ 0, & y \le 1. \end{cases}$$

Differentiating, it follows that

$$f_Y(y) = \frac{1}{y^2} I_{(1,\infty)}(y)$$

Alternatively, since we have a monotone function (and hence the method of transformations may be used), if

$$Y = h(X) = \frac{1}{X},$$

then the support of Y is $(1, \infty)$ and $h^{-1}(x) = x^{-1}$. So

$$f_Y(y) = f_X(y^{-1}) \left| \frac{d}{dy} y^{-1} \right| I_{(1,\infty)}(y)$$

= $1 \left| -\frac{1}{y^2} \right| I_{(1,\infty)}(y)$
= $\frac{1}{y^2} I_{(1,\infty)}(y).$