Solutions for HW 6
STAT 346, Spring 2010

1) Letting W be the amount won, the expected value of W is

$$0p_W(0) + 5p_W(5) + 10p_W(10) = 0 + 5\frac{\binom{4}{3}\binom{6}{1}}{\binom{10}{4}} + 10\frac{\binom{4}{4}\binom{6}{0}}{\binom{10}{4}} = 13/21.$$

The expected gain is $E(W-1) = E(W) - 1 = 13/21 - 1 = -8/21$.

2) We have

$$E(X) = \sum_{x=1}^{10} xp_X(x) = \sum_{x=1}^{10} x(1/10) = (1/10)\sum_{x=1}^{10} x = (1/10)(10)(10+1)/2 = 11/2,$$

and

$$E(X^2) = \sum_{x=1}^{10} x^2 p_X(x) = \sum_{x=1}^{10} x^2(1/10) = (1/10)\sum_{x=1}^{10} x^2 = (1/10)(10)(10+1)(2(10)+1)/6 = 77/2.$$

So, for the desired expectation we have $E(X[11-X]) = 11E(X) - E(X^2) = 121/2 - 77/2 = 22$. (Alternatively, we have that the desired expectation is

$$\sum_{x=1}^{10} x(11-x)p_X(x) = [1(10)+2(9)+3(8)+4(7)+5(6)+6(5)+7(4)+8(3)+9(2)+10(1)]/10,$$

which equals 22.)

3) Letting X be the number of spades, we have that the desired mean is

$$E(X) = \sum_{x=0}^{5} x\frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}} = 5/4 = 1.25.$$

We have

$$E(X^2) = \sum_{x=0}^{5} x^2\frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}} = 105105/86632 \doteq 2.42647,$$

and so the desired variance is $E(X^2) - [E(X)]^2 = 235/272 \doteq 0.86397$. Upon taking the square root, we find that the desired standard deviation is about 0.9295. (*Note*: After learning the material in Ch. 5 one should recognize that X is a hypergeometric random variable, and that it's mean is $5(13/52) = 5/4 = 1.25$ and its variance is $5(13/52)(39/52)([52-5]/[52-1]) = 235/272 \doteq 0.86397$.)

4) The probability of getting at least one 6 when a pair of fair dice is rolled is 11/36. So letting X be a binomial(10, 11/36) random variable, the desired probability is

$$P(X=3) = \binom{10}{3}(11/36)^3(25/36)^7 \doteq 0.267.$$

5) The number of misprints in a 35 page chapter is a Poisson random variable having mean/parameter $(3/10)(35) = 10.5$. The probability that Ch. 1 has exactly 10 misprints is $P(X = 10)$, where X is a Poisson(10.5) random variable, which equals $(10.5)^{10}e^{-10.5}/10! \doteq 0.1236$. The probability that Ch. 5 has exactly 10 misprints is the same, and by the independence associated with the assumed Poisson process, the events that Ch. 1 has 10 misprints and Ch. 5 has 10 misprints are independent, giving us that the desired probability is

$$[(10.5)^{10}e^{-10.5}/10!][(10.5)^{10}e^{-10.5}/10!] = (10.5)^{20}e^{-21}/(10!)^2 \doteq 0.0153.$$

6) Letting X be the number of defectives in a random sample of size 10, X is a hypergeometric random variable and the desired probability is

$$P(X \leq 1) = \frac{\binom{10}{0}\binom{40}{10}}{\binom{50}{10}} + \frac{\binom{10}{1}\binom{40}{9}}{\binom{50}{10}} \doteq 0.3487.$$

7) Letting X be the number of defectives in a random sample of size 8, X is a hypergeometric random variable and the desired probability is

$$P(X = 1) = \frac{\binom{5}{1}\binom{45}{7}}{\binom{50}{8}} = \frac{3198}{7567} \doteq 0.423.$$

8) The desired probability is $P(X = 2)$, where X is a binomial(12, p) random variable, where p is the probability that a Poisson(5) random variable assumes the value 0, which is just e^{-5}. So the desired probability is

$$\binom{12}{2}(e^{-5})^2(1 - e^{-5})^{10} \doteq 0.00280.$$