Solutions for HW 12
STAT 346, Spring 2010

1) The possible values X can assume are $1, 2, \ldots, 6$. So the desired cdf is a step function with jump discontinuities at these values. For $x < 1$, $F_X(x) = 0$, and for $x \geq 6$, $F_X(x) = 1$. Letting Y_1 and Y_2 be the two individual die outcomes, for $k \in \{1, 2, 3, 4, 5\}$, we have that for $k \leq x < k + 1$,

$$\begin{aligned} F_X(x) &= P(Y_{(1)} \leq x) \\ &= 1 - P(Y_{(1)} > x) \\ &= 1 - P(Y_1 > x, Y_2 > x) \\ &= 1 - P(Y_1 > x)P(Y_2 > x) \\ &= 1 - [1 - P(Y_1 \leq x)][1 - P(Y_2 \leq x)] \\ &= 1 - [1 - k/6][1 - k/6] \\ &= 1 - [(6 - k)/6]^2 \\ &= k(12 - k)/36. \end{aligned}$$

Altogether, we have

$$F_X(x) = \begin{cases} 1, & x \geq 6, \\ 35/36, & 5 \leq x < 6, \\ 32/36, & 4 \leq x < 5, \\ 27/36, & 3 \leq x < 4, \\ 20/36, & 2 \leq x < 3, \\ 11/36, & 1 \leq x < 2, \\ 0, & x < 1. \end{cases}$$

2) For $0 < x < y$, the desired marginal pdf equals

$$\int_y^\infty 6e^{-x-y-z}\, dz = 6e^{-x-y} \int_y^\infty e^{-z}\, dz = 6e^{-x-y}(-e^{-z}|_y^\infty) = 6e^{-x-2y}.$$

Elsewhere the desired marginal pdf equals 0. So, altogether, the desired marginal pdf is

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-x-2y}, & 0 < x < y, \\ 0, & \text{elsewhere.} \end{cases}$$

3) I'll integrate the joint pdf found in the problem immediately above to obtain the marginal pdf of X. The support of X is $(0, \infty)$, and for $x > 0$ we have

$$f_X(x) = \int_x^\infty 6e^{-x-2y}\, dy = 3e^{-x} \int_x^\infty 2e^{-2y}\, dy = 3e^{-x}(-e^{-2y}|_x^\infty) = 3e^{-3x}.$$

Altogether, we have

$$f_X(x) = 3e^{-3x}\, I_{(0,\infty)}(x),$$

which is the pdf of an exponential random variable having mean 1/3, and so the desired expectation is 1/3. (*Note*: If I didn't recognize the distribution of the marginal pdf of X I could have just multiplied it by x and integrated to obtain $E(X)$. Alternatively, I could have multiplied the joint marginal pdf of X and Y by x and have done a double integral to obtain the desired mean. But I generally like to obtain the marginal pdf of X in these cases, since checking that it is a valid pdf provides an overall check of my work.)

4) For the desired marginal pmf we have

$$p_X(x) = \sum_{y=1}^{3} \sum_{z=1}^{2} p(x, y, z).$$

For $x \notin \{4, 5\}$, $p_X(x) = 0$. For $x \in \{4, 5\}$,

$$p_X(x) = \sum_{y=1}^{3} \sum_{z=1}^{2} xyz/162 = \sum_{y=1}^{3} xy/54 = x/9.$$

Altogether, we have

$$p_X(x) = \frac{x}{9}\, I_{\{4,5\}}(x).$$

5) I'll integrate the joint pdf to obtain the marginal pdf of Y. The support of Y is $[0,\, 1]$, and for $y \in [0,\, 1]$ we have

$$f_Y(y) = \int_0^1 (x + y)\, dx = (x^2/2 + yx)|_0^1 = (1/2 + y)$$

(which is nonnegative and integrates to 1 over $[0,\, 1]$). So for the desired conditional pdf we have

$$f_{X|Y}(x|y) = \frac{x + y}{1/2 + y}\, I_{[0,\, 1]}(x).$$

6) Since the X_i are iid continuous random variables, we can use the general result for the sample maximum:

$$f_{X_{(n)}}(x) = n[F_X(x)]^{n-1} f_X(x).$$

For the case of $n = 2$ and the exponential random variables being considered we have

$$f_{X_{(2)}}(x) = 2(1 - e^{-x})e^{-x}\, I_{(0,\,\infty)}(x) = (2e^{-x} - 2e^{-2x})\, I_{(0,\,\infty)}(x).$$

For the desired expectation we have

$$2\int_0^\infty x\, e^{-x}\, dx - \int_0^\infty x\, 2e^{-2x}\, dx = 2(1) - 1/2 = 3/2.$$

(*Note*: The integrals are easily evaluated by noting that they are expressions for the means of exponential random variables having means 1 and 1/2, respectively.)

7) Letting

$$I_j = \begin{cases} 1, & \text{if } j\text{th box remains empty,} \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$E(I_j) = P(I_j = 1) = P(\textit{none of the 80 balls go into the jth box}) = (39/40)^{80}.$$

The desired expectation is

$$E(I_1 + I_2 \cdots + I_{40}) = \sum_{j=1}^{40} E(X_j) = 40(39/40)^{80} \doteq 5.2775.$$

8) Using the X_i described in the hint, for $i = 1, 2, \ldots, 25$, we have

$$E(X_i) = P(X_i = 1) = P(\textit{none of the other students have same birthday as ith student}) = (364/365)^{24}.$$

It follows that the desired expectation is

$$25(364/365)^{24} \doteq 23.407.$$