Solutions for HW 11
STAT 346, Spring 2010

1) Since X and Y are independent and both have pmf p, for the joint pmf of X and Y we have $p_{X,Y}(x,y) = p(x)p(y)$. So

$$P(X+Y=3) = p_{X,Y}(1,2) + p_{X,Y}(2,1) = p(1)p(2) + p(2)p(1) = 2p(1)p(2) = 2(1/3)(2/9) = 4/27 \doteq 0.148.$$

3) Since the possible values that Y can take depends on the value that X assumes, X and Y can't be independent. (Note: If X assumes the value 1, then Y can take both 1 and 2 with positive probability, but if X assumes the value 2, Y can only assume the value 1.) More explicitly, we have that $p(2,2) = 0$, but $p_X(2)p_Y(2) > 0$. (We have $p_X(2)p_Y(2) = p(2,1)p(1,2) = (4/7)(2/7) = 8/49$.)

5) For $y \in (0,\, 1)$, the marginal pdf of Y is

$$f_Y(y) = \int_0^y 2\,dx = 2y.$$

So, for $y \in (0,\, 1)$, the conditional pdf of X given $Y = y$ is

$$\begin{aligned} f_{X|Y}(x|y) &= f(x,y)/f_Y(y) \\ &= [2/(2y)]\, I_{(0,\,y)}(x) \\ &= \frac{1}{y}\, I_{(0,\,y)}(x). \end{aligned}$$

6) Since we have a constant density within the cube, the desired probability is just the proportion of the volume of the cube which is contained within the sphere, which is

$$\left(\frac{4}{3}\pi a^3\right)/([2a]^3) = \pi/6 \doteq 0.524.$$

7) The desired probability is

$$P(X < 1/3, Y < 1/2, Z < 1/4)/P(Y < 1/2, Z < 1/4).$$

Since

$$P(X < 1/3, Y < 1/2, Z < 1/4) = \int_0^{1/3}\int_0^{1/2}\int_0^{1/4} (1/2)(x+y+2z)\,dx\,dy\,dz = 1/72$$

and

$$P(Y < 1/2, Z < 1/4) = \int_0^{1}\int_0^{1/2}\int_0^{1/4} (1/2)(x+y+2z)\,dx\,dy\,dz = 1/16,$$

it follows that the desired probability is $(1/72)/(1/16) = 16/72 = 2/9$.

8) We have

$$1 = \int_0^1\int_0^1\int_0^1 c(x+y+2z)\,dx\,dy\,dz = c\int_0^1\int_0^1\int_0^1 (x+y+2z)\,dx\,dy\,dz.$$

Since

$$\begin{aligned} &\int_0^1\int_0^1\int_0^1 c(x+y+2z)\,dx\,dy\,dz \\ &= \int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz + \int_0^1\int_0^1\int_0^1 y\,dy\,dx\,dz + \int_0^1\int_0^1\int_0^1 2z\,dz\,dx\,dy \\ &= 1/2 + 1/2 + 1 \\ &= 2, \end{aligned}$$

we must have $c = 1/2$.

9) For $y \in [0,\, e-1]$, the marginal pdf of Y is

$$f_Y(y) = \int_0^\infty e^{-x(y+1)}\, dx = -\frac{1}{y+1} e^{-x(y+1)} |_0^\infty = 1/(y+1).$$

So, for $y \in [0,\, e-1]$, the conditional pdf of X given $Y = y$ is

$$f_{X|Y}(x|y) = (y+1)e^{-(y+1)x}\, I_{(0,\,\infty)}(x),$$

which is the pdf of an exponential distribution having mean $1/(y+1)$. So the desired conditional mean is $1/(y+1)$.

10) Since "a random time between 400 and 1200 hours" means that we're dealing with a uniform $(400, 1200)$ distribution, the probability that a bulb burns out before 550 hours is $150/800 = 3/16$, the probability that a bulb burns out after 800 hours is $400/800 = 1/2$, and the probability that a bulb burns out after 550 hours but before 800 hours is $250/800 = 5/16$. Using the appropriate multinomial distribution pmf, the desired probability is

$$\frac{8!}{3!\,2!\,3!}(3/16)^3(1/2)^2(5/16)^3 \doteq 0.0282.$$