Solutions for HW 10
STAT 346, Spring 2010

1) We have

$$P(T > t) = \int_t^\infty f_T(s)\,ds = \int_t^\infty 2s^{-3}\,ds = -s^{-2}|_t^\infty = 1/t^2.$$

So, for the hazard function we have

$$\lambda(t) = f_T(t)/P(T > t) = \frac{2/t^3}{1/t^2} = 2/t.$$

2)
(a) We have

$$\begin{aligned} 1 &= p(1,1) + p(1,2) + p(2,1) + p(2,2) + p(3,1) + p(3,2) \\ &= c(1+1) + c(1+2) + c(2+1) + c(2+2) + c(3+1) + c(3+2) \\ &= 21c, \end{aligned}$$

which implies that $c = 1/21$.

(b) X can only assume the values 1, 2, and 3. For $x = 1$ we have $p_X(1) = p(1,1) + p(1,2) = 2/21 + 3/21 = 5/21$. For $x = 2$ we have $p_X(2) = p(2,1) + p(2,2) = 3/21 + 4/21 = 7/21$. For $x = 3$ we have $p_X(3) = p(3,1) + p(3,2) = 4/21 + 5/21 = 9/21$. Altogether, we have

$$p_X(x) = \begin{cases} 9/21, & x = 3, \\ 7/21, & x = 2, \\ 5/21, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Y can only assume the values 1 and 2. For $y = 1$ we have $p_Y(1) = p(1,1) + p(2,1) + p(3,1) = 2/21 + 3/21 + 4/21 = 9/21$. For $y = 2$ we have $p_Y(2) = p(1,2) + p(2,2) + p(3,2) = 3/21 + 4/21 + 5/21 = 12/21$. Altogether, we have

$$p_Y(y) = \begin{cases} 12/21, & y = 2, \\ 9/21, & y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We have

$$\begin{aligned} P(X \geq 2 | Y = 1) &= P(\{X \geq 2\} \cap \{Y = 1\})/P(Y = 1) \\ &= [p(2,1) + p(3,1)]/p_Y(1) \\ &= [3/21 + 4/21]/(9/21) \\ &= 7/9. \end{aligned}$$

(d) We have

$$E(X) = 1p_X(1) + 2p_X(2) + 3p_X(3) = 1(5/21) + 2(7/21) + 3(9/21) = 46/21 \doteq 2.19$$

and

$$E(Y) = 1p_Y(1) + 2p_Y(2) = 1(9/21) + 2(12/21) = 33/21 = 11/7 \doteq 1.57.$$

3)
(a) We have

$$1 = p(1,1) + p(1,3) + p(2,3) = k(1^2 + 1^2) + k(1^2 + 3^2) + k(2^2 + 3^2) = 25k,$$

which implies that $k = 1/25$.

(b) X can only assume the values 1 and 2. For $x = 1$ we have $p_X(1) = p(1,1) + p(1,3) = 2/25 + 10/25 = 12/25$. For $x = 2$ we have $p_X(2) = p(2,3) = 13/25$. Altogether, we have

$$p_X(x) = \begin{cases} 13/25, & x = 2, \\ 12/25, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Y can only assume the values 1 and 3. For $y = 1$ we have $p_Y(1) = p(1,1) = 2/25$. For $y = 3$ we have $p_Y(3) = p(1,3) + p(2,3) = 10/25 + 13/25 = 23/25$. Altogether, we have

$$p_Y(y) = \begin{cases} 23/25, & y = 3, \\ 2/25, & y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We have

$$E(X) = 1p_X(1) + 2p_X(2) = 1(12/25) + 2(13/25) = 38/25 = 1.52$$

and

$$E(Y) = 1p_Y(1) + 3p_Y(3) = 1(2/25) + 3(23/25) = 71/25 = 2.84.$$

4) The desired probability is

$$p(1,1) + p(2,0) = 2/25 + 4/25 = 6/25 = 0.24.$$

5) In class I showed that the marginal pdf of Y is

$$f_Y(y) = 4y(1 - y^2)\, I_{(0,\,1)}(y).$$

So for the desired expectation we have

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = 4\int_0^1 (y^2 - y^4)\, dy = 4\left[\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^1\right] = 8/15.$$

8) The desired probability is

$$p(1,0) + p(1,1) + p(2,0) = 1/25 + 2/25 + 4/25 = 7/25 = 0.28.$$

9) For $y \notin [0,\,1]$, $f_X(x) = 0$. For $y \in [0,\,1]$,

$$f_X(x) = \int_0^x 8xy\, dy = 4xy^2|_0^x = 4x^3.$$

Altogether we have

$$f_X(x) = 4x^3\, I_{(0,\,1]}(x).$$

So for the desired expectation we have

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^1 x\, 4x^3\, dx = \int_0^1 4x^4\, dx = [(4/5)x^5|_0^1] = 4/5.$$