HOMOLOGICAL PERCOLATION ON A TORUS: PLAQUETTES AND PERMUTOHEDRA

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Abstract. We study higher-dimensional homological analogues of bond percolation on a square lattice and site percolation on a triangular lattice. By taking a quotient of certain infinite cell complexes by growing sub-lattices, we obtain finite cell complexes with a high degree of symmetry and with the topology of the torus $\mathbb{T}^d$. When random subcomplexes induce nontrivial $i$-dimensional cycles in the homology of the ambient torus, we call such cycles giant. We show that for every $i$ and $d$ there is a sharp transition from nonexistence of giant cycles to giant cycles spanning the homology of the torus.

We also prove convergence of the threshold function to a constant in certain cases. In particular, we prove that $p_c = 1/2$ in the case of middle dimension $i = d/2$ for both models. This gives finite-volume high-dimensional analogues of Kesten’s theorems that $p_c = 1/2$ for bond percolation on a square lattice and site percolation on a triangular lattice.

1. Introduction

Various models of percolation are fundamental in statistical mechanics; classically, they study the emergence of a giant component in random structures. From early in the mathematical study of percolation, geometry and topology have been at the heart of the subject. Indeed, Frisch and Hammersley wrote in 1963 [11] that, “Nearly all extant percolation theory deals with regular interconnecting structures, for lack of knowledge of how to define randomly irregular structures. Adventurous readers may care to rectify this deficiency by pioneering branches of mathematics that might be called stochastic geometry or statistical topology.”

The main geometric structure of interest in percolation theory thus far is often the existence of an infinite component or infinite path. A path which

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wraps around the torus can be seen as a finite volume analogue of this infinite component. We study events defined in terms of geometric structures that are higher-dimensional generalizations of such paths. The “homological percolation” property we consider is one that was recently studied by Bobrowski and Skraba [3, 4]. The possibility of studying homological percolation in a 2-dimensional torus or surface of genus $g$ was discussed earlier in [18], and [21], [20]. The more general setting of $i$-dimensional homological percolation in a $d$-dimensional torus was, to our knowledge, first carefully studied in [3] and [4]. The setup is as follows.

For a random shape $S$ in the torus $\mathbb{T}^d$, let $\phi: S \hookrightarrow \mathbb{T}^d$ denote the natural inclusion map. Bobrowski and Skraba suggest that nontrivial elements in the image of the induced map $\phi_*: H_i(P; \mathbb{Q}) \to H_i(\mathbb{T}^d; \mathbb{Q})$ can be considered “giant $i$-dimensional cycles”. In the case $i = 1$, these correspond to paths that loop around the torus as in Figure 1 whereas for the case $i = 2$ and $d = 3$ they are “sheets” going from one side of the torus to the other, as illustrated in Figure 2.

In [3], Bobrowski and Skraba provide experimental evidence that the appearance of giant cycles is closely correlated with the zeroes of the expected Euler characteristic curve, and moreover that this behavior seems universal across several models. In [4], they focus on continuum percolation and show that giant cycles appear in all dimensions within the so-called thermodynamic limit where $nr^d$ is bounded. One interesting suggestion is that there is a sharp, convergent transition from the map $\phi_*$ being trivial to being surjective. They prove that such a transition occurs when $i = 1$, for every $d \geq 2$.

The Harris–Kesten theorem [14, 16] (see also [6] for a short proof) establishes that for bond percolation in the square lattice, the critical probability for an infinite component to appear is $p = 1/2$. Kesten also showed that $p_c = 1/2$ for site percolation in the triangular lattice, using similar tools [17].

We seek higher-dimensional homological analogues of these classical theorems. We take a step in this direction by proving finitary versions on certain flat tori with a high degree of symmetry. We will discuss first an analogue of bond percolation on a square lattice, and then an analogue of site percolation on a triangular lattice.

A natural analogue of bond percolation in higher dimensions is given by plaquette percolation, which was first studied by Aizenman, Chayes, Chayes, Fröhlich, and Russo in [1]. They were motivated by notions of random surfaces in physics coming from lattice gauge theories and three-dimensional spin systems—see Sections 1 and 7 of that paper. The plaquette model starts with the entire 1-skeleton of $\mathbb{Z}^3$ and adds 2-dimensional square cells, or plaquettes,
with probability $p$ independently. The authors prove that the probability that a large planar loop is null-homologous undergoes a phase transition from an “area law” to a “perimeter law” that is dual to the phase transition for bond percolation in $\mathbb{Z}^3$. In particular, the critical probability for this threshold is at $p_c = 1 - \hat{p}_c$, where $\hat{p}_c$ is the threshold for bond percolation (this follows when their results are combined with those of [13]). They also show that infinite “surface sheets” appear at this threshold. At the end of their paper, the authors defined $i$-dimensional plaquette percolation on $\mathbb{Z}^d$ and suggested the study of analogous questions in higher dimensions.

Of particular interest are random $d/2$-cells in even dimension $d$. Clearly, if no intermediate phase exists in such a self-dual system, the transition point is $p = 1/2$. The most promising model for future study is random plaquettes in $d = 4$. [1]

One of the models we study is the following $i$-dimensional plaquette percolation model $P$ on the $d$-dimensional torus. Let $N \geq 1$ and consider the cubical complex $T^d_N = \mathbb{Z}^d/(N\mathbb{Z})^d$. Take the entire $(i-1)$-skeleton of $T^d_N$ and add each $i$-face independently with probability $p$.

We also study an analogue of site percolation in a triangular lattice. In general, site percolation is defined as a random induced subgraph of a graph where each vertex is open with probability $p$, independently. For the special case of site percolation on a triangular lattice, it is often convenient to consider a “dual” model where one tiles the plane by regular hexagons, and every closed hexagon is included independently with probability $p$. As observed by Bobrowski and Skraba in [3], this dual model generalizes naturally to higher dimensions—indeed, one can tile Euclidean space $\mathbb{R}^d$ by regular permutohedra, and then include each closed permutohedron independently with probability $p$. More details, including a topological justification for referring to this as site percolation, are given below.

1.1. Main results. Our first set of results is in the case of plaquette percolation on the torus. Let $d, N \in \mathbb{N}$, $d > 1$, $1 \leq i \leq d - 1$ and $p \in (0, 1)$. As above, denote by $T^d_N = \mathbb{Z}^d/(N\mathbb{Z})^d$ the regular cubical complex on the $d$-dimensional torus with $N^d$ cubes of width one. Let $H_i(X)$ be homology with coefficients in a fixed field $F$. The choice of $F$ only matters in the arguments of Section 3, which is reflected in the minor restrictions on the characteristic of $F$ (denoted $\text{char}(F)$) stated in the main theorems. Roughly speaking, our proofs rely on three symmetries: the symmetry of the plaquette system with its dual, the symmetry of individual plaquettes (that is, the existence of a transitive group action on the set of plaquettes), and the symmetry properties of the homology
groups of the torus. The hypotheses on the homology coefficient field are crucial for the third symmetry; they ensure that the homology groups of the torus are maximally symmetric in the sense that they are irreducible representations of the point symmetry group of the lattice.
Define the plaquette system $P = P(i, d, N, p)$ to be the random set obtained by taking the $(i-1)$-skeleton of $T^d_N$ and adding each $i$-face independently with probability $p$. Let $\phi : P \to T^d_N$ be the inclusion, and let $\phi_* : H_i(P) \to H_i(T^d_N)$ be the induced map on homology. Also, denote by $A^\square = A^\square(i, d, N, p)$ the event that $\phi_*$ is nontrivial, and denote by $S^\square = S^\square(i, d, N, p)$ the event that $\phi_*$ is surjective (we will use square and hexagon superscripts to indicate when we are working in the square and permutohedral lattices respectively). For example, in Figure 1 the two giant cycles shown are homologous with standard generators for $H_1(T^2)$, so we have the event $S^\square$.

Our main result for plaquette percolation is that if $d = 2i$, then $i$-dimensional percolation is self-dual and undergoes a sharp transition at $p = 1/2$.

**Theorem 1.** Suppose $\text{char} (F) \neq 2$. If $d = 2i$ then

$$
\begin{align*}
\mathbb{P}_p (A^\square) & \to 0 & \text{as } N \to \infty, \\
\mathbb{P}_p (S^\square) & \to 1
\end{align*}
$$

as $N \to \infty$.

Using results on bond percolation on $\mathbb{Z}^d$, we also prove dual sharp thresholds for 1-dimensional and $(d-1)$-dimensional percolation on the torus.

**Theorem 2.** Suppose $\text{char} (F) \neq 2$. Let $\hat{p}_c = \hat{p}_c(d)$ be the critical threshold for bond percolation on $\mathbb{Z}^d$. If $i = 1$ then

$$
\begin{align*}
\mathbb{P}_p (A^\square) & \to 0 & \text{as } N \to \infty, \\
\mathbb{P}_p (S^\square) & \to 1
\end{align*}
$$

Furthermore, if $i = d - 1$ then

$$
\begin{align*}
\mathbb{P}_p (A^\square) & \to 0 & \text{as } N \to \infty, \\
\mathbb{P}_p (S^\square) & \to 1
\end{align*}
$$

In the above, we also show that the decay of $\mathbb{P}_p (A^\square)$ below the threshold and $\mathbb{P}_p (S^\square)$ above the threshold is exponentially fast for both $i = 1$ and $i = d - 1$.

For other values of $i$ and $d$ we show the existence of a sharp threshold function as follows. For each $N \in \mathbb{N}$, let $\lambda^\square(N, i, d)$ satisfy

$$
\mathbb{P}_{\lambda^\square(N,i,d)} (A^\square) = \frac{1}{2}.
$$

(1)
Note that $\mathbb{P}_p (A^{\square})$ is continuous as a function of $p$, so such a $\lambda^{\square}(N, i, d)$ exists by the intermediate value theorem. Since the tori of different sizes do not embed nicely into each other, it is not obvious that $\lambda^{\square}(N, i, d)$ should be convergent a priori.

We should mention that this choice of $\lambda^{\square}(N, i, d)$ is somewhat arbitrary. We could replace $\frac{1}{2}$ in Equation 1 with any constant strictly between 0 and 1, for example, and the sharp threshold results we use would apply just as well. In several cases we show that $\lambda^{\square}(N, i, d)$ converges, and in those cases the limiting value could also be taken as a constant threshold function.

Define

\[ p_c^{\square}(i, d) = \inf \left\{ p : \lim_{N \to \infty} \mathbb{P}_p (A^{\square}) > 0 \right\} \]

and

\[ q_c^{\square}(i, d) = \sup \left\{ p : \lim_{N \to \infty} \mathbb{P}_p (S^{\square}) < 1 \right\}. \]

As we show below, we could alternatively define $p_c^{\square}(i, d)$ and $q_c^{\square}(i, d)$ as the limit supremum and limit infimum of the threshold function $\lambda^{\square}(N, i, d)$, respectively.

With the understanding that these depend on choice of $i$ and $d$, which are always understood in context, we sometimes abbreviate to simply $p_c^{\square}$, $q_c^{\square}$, and $\lambda^{\square}(N)$.

**Theorem 3.** Suppose $\text{char } (F) \neq 2$. For every $d \geq 2$, $1 \leq i \leq d - 1$, and $\epsilon > 0$

\[ \begin{cases} \mathbb{P}_{\lambda^{\square}(N)-\epsilon} (A^{\square}) \to 0 \\ \mathbb{P}_{\lambda^{\square}(N)+\epsilon} (S^{\square}) \to 1 \end{cases} \]

as $N \to \infty$.

Moreover, for every $d \geq 2$ and $1 \leq i \leq d - 1$ we have

\[ 0 < q_c^{\square} = \liminf_{N \to \infty} \lambda^{\square}(N) \leq \limsup_{N \to \infty} \lambda^{\square}(N) = p_c^{\square} < 1, \]

and $p_c^{\square}(i, d)$ has the following properties.

(a) (Duality) $p_c^{\square}(i, d) + q_c^{\square}(d - i, d) = 1$.

(b) (Monotonicity in $i$ and $d$) $p_c^{\square}(i, d) < p_c^{\square}(i, d - 1) < p_c^{\square}(i + 1, d)$ if $0 < i < d - 1$.

It follows that $p_c^{\square} = q_c^{\square}$ for $i = d/2$, $i = 1$, and $i = d - 1$, and we conjecture that this equality (and hence sharp threshold from a trivial map to a surjective
one at a constant value of $p$) holds for all $i$ and $d$. Bobrowksi and Skraba make analogous conjectures for the continuum percolation model in [4].

We also apply our methods to Bernoulli site percolation on the tiling of the torus by $d$-dimensional permutohedra, which was previously studied in [3]. The precise definitions are as follows. Let

$$\hat{\mathbb{R}}^d := \{(x_0, x_1, \ldots, x_d) : \sum_{k=0}^d x_k = 0\}.$$

Recall that the root lattice $\mathcal{A}_d$ is defined by

$$\mathcal{A}_d := \hat{\mathbb{R}}^d \cap \mathbb{Z}^{d+1}.$$

The dual lattice is then defined by

$$\mathcal{A}_d^* := \{x \in \hat{\mathbb{R}}^d : \forall y \in \mathcal{A}_d, x \cdot y \in \mathbb{Z}\}$$

which is generated by the basis

$$B := \{1 - de_k : 1 \leq k \leq d\}.$$

Let $\pi : \hat{\mathbb{R}}^d \to \mathbb{R}^d$ be the natural isometry. Then the closed Voronoi cells of $\pi(\mathcal{A}_d)$ are $d$-dimensional permutohedra. When $d = 2$, $\mathcal{A}_2^*$ is the triangular lattice and the permutohedra are hexagons. For the case $d = 3$, $\mathcal{A}_3^*$ is the body-centered cubic lattice and the permutohedra are truncated octahedra (see [2] for a detailed exposition).

Consider the torus $\mathbf{T}_N^d$ as the parallelepiped generated by $\{Nv : v \in B\}$ with opposite faces identified. Define $Q = Q(d, N, p)$ the be the random set obtained by adding each permutohedron independently with probability $p$. The topological justification for calling this site percolation is that the adjacency graph on the permutohedra of $Q$ is exactly site percolation on the lattice $\mathcal{A}_d^*$. In other words, site percolation is the one-skeleton of the nerve of the cover of $Q$ by the closed permutohedra. By the nerve theorem, $Q$ is homotopy equivalent to this nerve and as such has the same connected components as site percolation on $\mathcal{A}_d^*$.

The giant cycle events are defined as before, except that $i$-dimensional giant cycles exhibit interesting behavior for all $1 \leq i \leq d - 1$ (for the plaquette model $P(i, d, N, p)$, all giant cycles in homological dimensions less than $i$ are automatically present, and there can be no giant cycles in homological dimensions exceeding $i$.) More precisely, let $\varphi : Q \hookrightarrow \mathbf{T}_N^d$ be the inclusion, and let $\varphi_{is} : H_i(Q) \to H_i(\mathbf{T}_N^d)$ be the induced maps for homology in each dimension. For each $i$, Let $A_{i}^\bigcirc$ be the event that $\varphi_{is}$ is nonzero and let $S_{i}^{\bigcirc}$ be the event that $\varphi_{is}$ is surjective.
For each $N \in \mathbb{N}$, let $\lambda_i^O(N, d)$ satisfy
\[
\mathbb{P}_{\lambda_i^O(N, d)} \left( A_i^O \right) = \frac{1}{2}.
\]
Define
\[
p_i^O = p_i^O (d) = \inf \left\{ p : \liminf_{N \to \infty} \mathbb{P}_p \left( A_i^O \right) > 0 \right\}
\]
and
\[
q_i^O = q_i^O (d) = \sup \left\{ p : \limsup_{N \to \infty} \mathbb{P}_p \left( S_i^O \right) < 1 \right\}.
\]

**Theorem 4.** Suppose $\text{char} \ (F) \nmid d + 1$. For every $d \geq 2$, $1 \leq i \leq d - 1$, and $\epsilon > 0$
\[
\begin{align*}
\mathbb{P}_{\lambda_i^O(N-\epsilon)} \left( A_i^O \right) &\to 0 \\
\mathbb{P}_{\lambda_i^O(N+\epsilon)} \left( S_i^O \right) &\to 1
\end{align*}
\]
as $N \to \infty$. For every $d \geq 2$ and $1 \leq i \leq d - 1$ we have
\[
0 < q_i^O = \liminf_{N \to \infty} \lambda_i^O (N) \leq \limsup_{N \to \infty} \lambda_i^O (N) = p_i^O < 1.
\]
In some cases the threshold converges, namely
\[
\lim_{N \to \infty} \lambda_1^O (N) = p_1^O = q_1^O = p_c(A_d^*) ,
\]
\[
\lim_{N \to \infty} \lambda_{d-1}^O (N) = p_{d-1}^O = q_{d-1}^O = 1 - p_c(A_d^*) ,
\]
and if $d$ is even, then
\[
\lim_{N \to \infty} \lambda_{d/2}^O (N) = p_{d/2}^O = q_{d/2}^O = \frac{1}{2}.
\]
Moreover, $p_i^O (d)$ has the following properties.

(a) (Duality) $p_i^O (d) + q_{d-i}^O (d) = 1$.

(b) (Monotonicity in $i$ and $d$) $p_i^O (d) < p_i^O (d-1) < p_{i+1}^O (d)$ if $0 < i < d - 1$.

In particular, when $d = 4$, the random set $Q$ exhibits three qualitatively distinct phase transitions at $p_c(A_4^*), \frac{1}{2}$, and $1 - p_c(A_4^*)$, where $p_c(A_4^*)$ is the site percolation threshold for the lattice $A_4^*$. 
1.2. **Probabilistic tools.** Let $X$ be the probability space formed by the product of $n$ Bernoulli($p$) random variables, and let $\mu_p$ be the probability measure on the power set $\mathcal{P}(X)$ defined by taking each element of $X$ independently with probability $p$. That is, if $Y \subseteq X$, 
$$
\mu(Y) = p^{|Y|} (1-p)^{|Y|} .
$$

An event $B$ is increasing if 
$$
Y_0 \subset Y_1, Y_0 \in B \implies Y_1 \in B .
$$

We require Harris’s Inequality on increasing events [14], which is a special case of the FKG Inequality [9].

**Theorem 5** (Harris’s Inequality). If $B_1, \ldots, B_j$ are increasing events then 
$$
\mathbb{P} \left( \bigcap_{k=1}^{j} B_k \right) \geq \prod_{k=1}^{j} \mathbb{P} (B_k) .
$$

Another key tool for us is the following theorem of Friedgut and Kalai on sharpness of thresholds [10].

**Theorem 6** (Friedgut and Kalai). Let $B$ be an increasing event that is invariant under a transitive group action on $X$. There exists a constant $\rho > 0$ so that if $\mu_p(B) > \epsilon > 0$ and 
$$
q \geq p + \rho \frac{\log (1/(2\epsilon))}{\log (|X|)}
$$

then $\mu_q(B) > 1 - \epsilon$.

We use two more technical results on connection probabilities in the subcritical and supercritical phases in bond percolation in $\mathbb{Z}^d$ below. For clarity, we state them in Section 6 when they are needed.

1.3. **Definitions and notation.** In both models, we will need a notion of dual percolation. In the permutohedral case, we simply define 
$$
Q^\bullet = Q^\bullet (d, N, p) := Q^c,
$$
i.e. the union of the permutohedra that are not included in $Q$. Defining the dual system to plaquette percolation requires more work.

To define the dual system of plaquettes $P^\bullet = P^\bullet (i, d, N, p)$, let $(\mathbb{T}_N^d)^\bullet$ be the regular cubical complex obtained by shifting $\mathbb{T}_N^d$ by $\frac{1}{2}$ in each coordinate direction. Each $i$-face of $\mathbb{T}_N^d$ intersects a unique $(d-i)$-face of $(\mathbb{T}_N^d)^\bullet$ and they
Figure 3. Bond percolation at criticality (i.e. $p = 1/2$) on the torus $T_2^d$ in blue, with the corresponding dual system of bonds in orange. Giant cycles are shown in bold. Observe that while $\text{rank } \phi_* + \text{rank } \psi_* = 2$ (as required by duality), neither the bond system nor its dual has a giant cycle homologous to one of the standard basis elements of $H_1(T^2)$.

meet in a single point at their centers. For example, the faces $[0,1]^i \times \{0\}^{d-i}$ and $\{1/2\}^i \times [-1/2,1/2]^{d-i}$ intersect in the point $\{1/2\}^i \times \{0\}^{d-i}$.

Define the dual system $P^\square$ to be the subcomplex of $(T_N^d)^\square$ consisting of all faces for which the corresponding face in $T_N^d$ is not contained in $P$. See Figure 3. Observe that the distribution of $P^\square(i,d,N,p)$ is the same as that of $P(d-i,d,N,1-p)$. If $B^\square$ is an event defined in terms of $P^\square$ we will write $\mathbb{P}_p(B^\square)$ to mean the probability of $B^\square$ with respect to the parameter $p$ of $P$.

We always use the notation $\phi : P \hookrightarrow T^d$ and $\psi : P^\square \hookrightarrow T^d$ for the respective inclusion maps, and $\phi_* : H_i(P) \to H_i(T_N^d)$ and $\psi_* : H_{d-i}(P^\square) \to H_{d-i}(T^d)$ for the induced maps on homology. Also, we consistently use notation $A^\square = A^\square(i,d,N,p)$ for the event that $\text{im } \phi_* \neq 0$, $S^\square = S^\square(i,d,N,p)$ for the event that $\phi_*$ is surjective, and $Z^\square = Z^\square(i,d,N,p)$ for the event that $\phi_*$ is zero. Denote by $A^\square, S^\square, Z^\square$ the corresponding events for $\psi_*$.

Throughout this paper, we always write $H_i(X;F)$ for the $i$-dimensional singular homology of a topological space $X$ with coefficients in a field $F$, and we
often write \( H_i(X) \) to suppress the dependence on the field \( F \). For a review of singular homology, please refer to a reference such as [15].

1.4. **Proof sketch.** We provide an overview of our main argument. Much of it is the same, mutatis mutandis, whether we are working with plaquettes or permutohedra. Throughout the paper, we shall make a note at points of substantial difference, but otherwise we only include proofs with plaquettes for brevity. In the section on topological results (Section 2), we show that duality holds in the sense that \( \text{rank} \, \phi_* + \text{rank} \, \psi_* = \text{rank} \, H_i(T^d) \) (Lemma 10). This is similar in spirit to results of [3] and [4] for other models of percolation on the torus including permutohedral site percolation. In particular, at least one of the events \( A^\square \) and \( A^\blacksquare \) occurs, \( S^\square \) occurs if and only if \( Z^\blacksquare \) occurs, and \( S^\blacksquare \) occurs if and only if \( Z^\square \) occurs.

Our strategy is to exploit the duality between the events \( S^\square \) and \( Z^\blacksquare = (A^\blacksquare)^c \). Toward that end, we show that a threshold for \( A^\square \) is also a threshold for \( S^\square \) in Section 3. First, we use the action of the point symmetry group of the torus on the homology to show that there are constants \( b_0 \) and \( b_1 \) so that \( \mathbb{P}_p(S^\square) \geq b_0 \mathbb{P}_p(A^\square)^{b_1} \). This follows from a more general result for events defined in terms of an irreducible representation of the point symmetry group (Lemma 11) and the fact that \( H_i(T^d; F) \) is an irreducible representation of the point symmetry group of \( \mathbb{Z}^d \) assuming the characteristic of \( F \) does not equal 2 (Proposition 13). This is one point at which the argument differs for site percolation on the permutohedral lattice; to account for the symmetries of that lattice we include a different argument that assumes that the characteristic of \( F \) is not divisible by \( d + 1 \) (Proposition 12).

Recall that \( \lambda^\square \) was chosen such that

\[
\mathbb{P}_{\lambda^\square(N,i,d)}(A^\square) = \frac{1}{2}.
\]

By the above, it follows that \( \mathbb{P}_{\lambda^\square(N)}(S^\square) > b_0 (\frac{1}{2})^{b_1} \). \( S^\square \) is increasing and invariant under the symmetry group of \( T^d \) so Friedgut and Kalai’s theorem on sharpness of thresholds (Theorem 6) implies that for any \( \epsilon > 0 \),

\[
\mathbb{P}_{\lambda^\square(N)+\epsilon}(S^\square) \to 1 \text{ as } N \to \infty.
\]

The proof of Theorem 1 is then straightforward (Section 4). By duality

\[
\mathbb{P}_{1/2}(A^\square) = \mathbb{P}_{1/2}(A^\blacksquare) \quad \text{and} \quad \mathbb{P}_{1/2}(A^\square) + \mathbb{P}_{1/2}(A^\blacksquare) \geq 1,
\]

so \( \mathbb{P}_{1/2}(A^\square) \geq \frac{1}{2} \) for all \( N \). It follows from the previous argument that \( \mathbb{P}_p(S^\square) \to 1 \) for \( p > 1/2 \). On the other hand, if \( p < 1/2 \) duality implies that

\[
\mathbb{P}_p(A^\square) = 1 - \mathbb{P}_p(S^\blacksquare) = 1 - \mathbb{P}_{1-p}(S^\square) \to 0.
\]
Next, in Section 5 we study the relationship between duality and convergence. Recall that

\[ q^\square_c (i, d) = \sup \left\{ p : \limsup_{N \to \infty} \mathbb{P}_p (S^\square) < 1 \right\}. \]

We show that \( p^\square_c (i, d) + q^\square_c (d - i, d) = 1 \) by using Lemma 10 and applying Theorem 6 to \( A^\square \) above and below \( q^\square_c \) (Proposition 18). It follows that the threshold for \( A^\square \) converges if and only if \( p^\square_c (d - i, d) + p^\square_c (i, d) = 1 \) (Corollary 19).

In Section 6 we show that \( p^\square_c (1, d) \) and \( q^\square_c (1, d) \) coincide and equal the critical threshold for bond percolation on \( \mathbb{Z}^d \) by applying classical results on connection probabilities in the subcritical and supercritical regimes (in the proofs of Propositions 23 and 21). This together with Corollary 19 demonstrates Theorem 2.

Finally, in Section 7, we complete the proof of Theorem 3 by showing the monotonicity property \( p^\square_c (i, d) < p^\square_c (i, d - 1) < p^\square_c (i + 1, d) \) if \( 0 < i < d - 1 \), and corresponding result for the thresholds \( q^\square_c \) (Proposition 24). This is done by comparing percolation on \( \mathbb{T}^d_N \) with percolation on a subset homotopy equivalent to \( \mathbb{T}^{d-1} \). The proof of the corresponding result for permutohedral site percolation is different, but the overall idea is the similar (Proposition 26).

2. Topological Results

In this section, we discuss duality lemmas which will be useful in many of our arguments.

In [3], Bobrowski and Skraba prove a duality lemma for the permutohedral lattice. We will use their notation which, for a subcomplex \( X \subset \mathbb{T}^d_N \), defines

\[ B_k (X) := \text{rank} \varphi_* , \]

where \( \varphi_* : H_k (X) \to H_k (\mathbb{T}^d_N) \) is the map on homology induced by inclusion.

**Lemma 7** (Bobrowski and Skraba). For \( 0 \leq k \leq d \),

\[ B_k (Q) + B_{d-k} (Q^c) = \text{rank} \ H_k (\mathbb{T}^d) . \]

This is a point at which one must consider permutohedra and plaquettes separately. We use the previous lemma in the permutohedral case, but we must prove an analogue in order to work with plaquettes. First, we show a preliminary result demonstrating a relationship between the complement of \( P \) and \( P^\square \).
Lemma 8. $\mathbb{T}^d \setminus P$ deformation retracts to $P^\square$.

Proof. Let $T^{(j)}$ and $T^{(i)}$ denote the $j$-skeleta of $\mathbb{T}^d_N$ and $(\mathbb{T}^d_N)^\square$, respectively, and let

$$S_j = T^{(d-j)} \setminus T^{(j)}.$$

Observe that $S_j$ is obtained from $T^{(d-j)}$ by removing the central point of each $(d-j)$-cell. Also, let

$$\hat{S}_j = T^{(d-j)} \setminus P.$$

We construct a deformation retraction from $T^d \setminus P = \hat{S}_0$ to $P^\square$ by iteratively collapsing $\hat{S}_j$ to $\hat{S}_{j+1}$ for $j < i$, then collapsing $\hat{S}_i$ to $P^\square$.

For an $j$-cell $\sigma$ of $T^{(j)}$ with center point $q$ let

$$f_\sigma : \sigma \setminus \{q\} \times [0, 1] \to \sigma \setminus \{q\}$$

be the deformation retraction from the punctured $j$-dimensional cube to its boundary along straight lines radiating from the center. Observe that the restriction of $f_\sigma$ to $(\sigma \setminus P) \times [0, 1]$ defines a deformation retraction from $\sigma \setminus P$ to $\partial \sigma \setminus P$ (for $j > d - i$); this is because $\sigma$ intersects $P$ in hyperplanes spanned by the coordinate vectors based at $q$. When projecting radially from $q$, points inside $\sigma \cap P$ remain inside $\sigma \cap P$ and points outside of $\sigma \cap P$ remain outside of $\sigma \cap P$.

For $x \in \mathbb{T}^d$, let $\sigma (x)$ be the unique $(d-j)$-cell of $(\mathbb{T}^d_N)^\square$ that contains $x$ in its interior. Define $G_j : S_j \times [0, 1] \to S_j$ by

$$G_j (x, t) = \begin{cases} f_{\sigma (x)} (x, t) & x \in S_j \setminus T^{d-j-1} \\ x & \text{otherwise.} \end{cases}$$

$G_j$ collapses $S_j$ to $T^{(d-j-1)}$ by retracting the punctured $(d-j)$-cells to their boundaries. It follows from the discussion in the previous paragraph that the restriction of $G_j$ to $\hat{S}_j \times [0, 1]$ defines a deformation retraction from $\hat{S}_j$ to $\hat{S}_{j+1}$.

Similarly, define $H : \hat{S}_i \times [0, 1] \to \hat{S}_i$ by

$$H (x, t) = \begin{cases} f_{\sigma (x)} (x, t) & x \in \hat{S}_i \setminus P^\square \\ x & \text{otherwise.} \end{cases}$$

That is, $H$ collapses the $(d-i)$-faces of $T^{(d-i)}$ that are punctured by $i$-faces of $P$ to deformation retract $\hat{S}_i$ to $P^\square$. 
Figure 4. of the deformation retraction for the case $N = 3$, $d = 2$, $i = 1$. $P$ is shown in blue and $P^\bullet$ in orange. $T^d \setminus P$ is first retracted to $T^d_1 \setminus P$ via the dashed gray arrows radiating from each vertex of $P$, then to $P^\bullet$ by the solid black arrows radiating from the midpoints of the edges of $P$.

In summary, we can deformation retract $T^d \setminus P$ to $P^\bullet$ via the function $F : T^d \setminus P \times [0, i] \to T^d \setminus P$ defined by

$$F(x, t) = \begin{cases} 
G_0(x, t) & t \in [0, 1] \\
G_j(F(x, j), t - j) & t \in (j, j + 1), 0 < j < i \\
H(F(x, i), t - i) & t \in (i, i + 1) 
\end{cases}.$$ 

In fact, the same deformation retraction works when $P$ is slightly thickened, which will be useful for the next Lemma. Let $P_\epsilon$ denote the $\epsilon$-neighborhood $P_\epsilon = \{ x \in T^d_N : d(x, P) \leq \epsilon \}$.

**Corollary 9.** For any $0 < \epsilon < 1/2$, the closure $\overline{(T^d \setminus P_\epsilon)}$ deformation retracts to $P^\bullet$.

**Proof.** Consider the deformation retraction as in Lemma 8 restricted to $\overline{(T^d \setminus P_\epsilon)}$. When a punctured $j$-cell $\sigma$ is retracted via $f_\sigma$, the property that points outside of $\sigma \cap P_\epsilon$ remain outside of $\sigma \cap P_\epsilon$ is preserved even though $\sigma \cap P_\epsilon$ now is a union of thickened hyperplanes. The deformation retractions $G_j$ and $H$ are defined
by collapsing different cells via the functions \( f_\sigma \), so the restricted retraction does not pass through \( P_\epsilon \).

The next result is a key topological tool we use in many of our arguments. It is very similar to results of [3] and [4] for other models of percolation on the torus including Lemma 7 above. For convenience, let

\[
D = \text{rank } H_i \left( \mathbb{T}^d \right) = \binom{d}{i}.
\]

**Lemma 10 (Duality Lemma).** \( \text{rank } \phi_* + \text{rank } \psi_* = D \). In particular, at least one of the events \( A^\square \) and \( A^\blacksquare \) occurs, \( S^\blacksquare \iff Z^\square \), and \( Z^\blacksquare \iff S^\square \).

**Proof.** We proceed similarly to Bobrowski and Skraba’s proof of Lemma 7. Let \( \epsilon = 1/4 \) and define \( P^\epsilon : = \left( \mathbb{T}^d_N \setminus P_\epsilon \right) \). Consider the diagram

\[
\begin{array}{c}
H_i \left( P_\epsilon \right) \xrightarrow{i_*} H_i \left( \mathbb{T}^d_N \right) \xrightarrow{\delta_i} H_{i-1} \left( P_\epsilon \right) \\
\cong \quad \cong \quad \cong \\
H^{d-i} \left( \mathbb{T}^d_N, P^\epsilon \right) \xrightarrow{j^*} H^{d-i} \left( \mathbb{T}^d_N \right) \xrightarrow{j} H^{d-i} \left( P^\epsilon \right) \xrightarrow{\delta^{d-i}} H^{d-i+1} \left( \mathbb{T}^d_N, P^\epsilon \right)
\end{array}
\]

Here \( i \) and \( j \) are the inclusions of \( P_\epsilon \) and \( P^\epsilon \) respectively into \( \mathbb{T}^d_N \). The first isomorphism from the left is from Lefschetz Duality, the second is from Poincaré Duality, and the third is from the five lemma. (A similar diagram is used in the proof of Alexander duality). In particular note that by exactness and a diagram chase,

\[
H_i \left( \mathbb{T}^d_N \right) \cong \text{im } i_* \oplus \text{im } j^*.
\]

Furthermore, since we are considering homology with field coefficients, \( \text{rank } j^* = \text{rank } j_* \). Now by Corollary 9, \( \left( \mathbb{T}^d_N \setminus P_\epsilon \right) \) retracts to \( P^\blacksquare \), and \( P_\epsilon \) clearly retracts to \( P \), so \( \text{rank } \phi_* = \text{rank } i_* \) and \( \text{rank } \psi_* = \text{rank } j^* \). Putting these together gives \( \text{rank } \phi_* + \text{rank } \psi_* = D \). \( \square \)

3. **Surjectivity**

The goal of this section is to show that if \( p > p^\square_c \) then \( \mathbb{P}_p \left( S^\square \right) \to 1 \) as \( N \to \infty \), where

\[
p^\square_c = p^\square_c \left( i, d \right) = \inf \left\{ p : \liminf_{N \to \infty} \mathbb{P}_p \left( A^\square \right) > 0 \right\}.
\]

First, we will prove that \( \mathbb{P}_p \left( S^\square \right) \geq b_0 p \mathbb{P}_p \left( A^\square \right)^{b_1} \) for some \( b_0, b_1 > 0 \) that do not depend on \( N \). This argument is another point of distinction between our argument in the permutohedral lattice and the cubical lattice because the symmetries of the lattices become relevant. However, we start with a general lemma that we use in both cases.
Remark that a vector space \( V \) that is acted on by a group \( G \) is called an \textbf{irreducible representation} of \( G \) if it has no proper, non-trivial \( G \)-invariant subspaces. That is, the only subspaces \( W \) of \( V \) so that \( \{ gw : w \in W \} = W \) are \( \{0\} \) and \( V \).

**Lemma 11.** Let \( V \) be a finite dimensional vector space and \( Y \) be a set. Let \( A \) be the lattice of subspaces of \( V \). Suppose \( f : \mathcal{P}(Y) \rightarrow A \) is an increasing function, i.e. if \( A \subset B \) then \( f(A) \subset f(B) \). Let \( G \) be a finite group which acts on both \( Y \) and \( V \) whose action is compatible with \( f \). That is, for each \( g \in G \) and \( v \in V \) \( g \) \( f(v) = f(gv) \). Let \( X \) be a \( \mathcal{P}(Y) \)-valued random variable with a \( G \)-invariant distribution that satisfies the conclusion of Harris’ Lemma, meaning that increasing events with respect to \( X \) are non-negatively correlated. Then if \( V \) is an irreducible representation of \( G \), there are positive constants \( C_0, C_1 \) so that

\[
P_p(f(X) = V) \geq C_0 P_p(f(X) \neq 0)^{C_1},
\]

where \( C_0 \) only depends on \( G \) and \( C_1 \) only depends on \( \dim V \).

**Proof.** Let \( A_k = \{ X \in \mathcal{P}(Y) : \text{rank } f(X) \geq k \} \) and \( \mathcal{W}_k = f(A_k) \). For a subspace \( W \) of \( V \) let \( \text{Stab}(W) \) denote the stabilizer of \( W \), \( \{ g \in G : gW = W \} \), and for \( H \leq G \) let

\[
S_k(H) = \{ X : \text{Stab}(f(X)) = H \} \cap A_k.
\]

Then in particular, since \( A_k = \bigsqcup_{H \leq G} S_k(H) \), there is a subgroup \( H' \) of \( G \) so that

\[
(3) \quad P_p(S_k(H')) \geq \frac{1}{c_G} P_p(A_k),
\]

where \( c_G \) is the number of subgroups of \( G \).

If \( H' = G \) then

\[
S_k(H') = S_k(G) = \{ X : f(X) = V \}
\]

because \( V \) is an irreducible representation of \( G \), and it follows that

\[
(4) \quad P_p(f(X) = V) = P_p(S_k(H')) \geq \frac{1}{c_G} P_p(A_k).
\]

Otherwise, if \( \text{Stab}(W) = H' \) then the orbit \( \{ gw : g \in G \} \) contains \( |G| / |H'| \) elements, where the elements of each coset of \( H' \) in \( G \) have the same action on \( W \). Let \( \mathcal{B} \) be a collection of subspaces of \( V \) that contains one element from each orbit of \( \{ W \in \mathcal{W}_k : \text{Stab}(W) = H' \} \) so

\[
f(S_k(H')) = \bigbigsqcup_{g \in G/H'} g\mathcal{B}.
\]
Taking $B := \{ X : f(X) \in \mathcal{B} \}$, we have that

$$S_k(H') = \bigsqcup_{g \in G/H'} gB.$$ 

Let $g \in G \setminus H'$. The events $B$ and $gB$ are symmetric so

$$\mathbb{P}_p(B) = \mathbb{P}_p(gB) = \frac{|H'|}{|G|} \mathbb{P}_p(S_k(H')) \geq \frac{1}{c_G |G|} \mathbb{P}_p(A_k)$$

using Equation 3. By construction, $gB \cap B \subseteq A_{k+1}$ and the Harris’ Lemma-like property of $X$ yields

$$\mathbb{P}_p(A_{k+1}) \geq \mathbb{P}_p(B \cap gB) \geq \mathbb{P}_p(B)^2 \geq \left( \frac{1}{c_G |G|} \mathbb{P}_p(A_k) \right)^2.$$ 

Since either the preceding equation or Equation 4 holds for all $k$, we can conclude that there are positive constants $C_0(G,V)$ and $C_1(V)$ so that

$$\mathbb{P}_p(f(X) = V) = \mathbb{P}_p(A_{\dim V}) \geq C_0 \mathbb{P}_p(A_1)^{2 \dim V - 2} = C_0 \mathbb{P}_p(f(X) \neq 0)^{C_1}.$$ 

□

Now it suffices to check the irreducibility of the homology of the torus as a representation of the point symmetry group of each lattice, which we do separately. We begin with the case of the permutohedral lattice $A^*_d$ whose point symmetry group is the symmetric group $S_{d+1}$.

**Proposition 12.** Let $F$ be a field, $d > 0$, and $1 \leq k \leq d - 1$. $H_k(\mathbf{T}_N^d; F)$ is an irreducible representation of $S_{d+1}$ if and only if char $(F) \nmid d + 1$.

**Proof.** First, we review the action of $S_{d+1}$ on $A^*_d$. The lattice $A^*_d \subset F^{d+1}$ has a basis

$$B := \{ 1 - d e_k : 1 \leq k \leq d \}$$

where $1$ is the vector whose entries all equal $1$. $S_{d+1}$ acts on $F^{d+1}$ by permuting the coordinates, and this restricts to an action on $A^*_d$ which permutes the elements of $B \cup \{ 1 - d e_{d+1} \}$. The $F$-vector space generated by $A^*_d$ is called the standard representation of $S_{d+1}$. Denote it by $\hat{F}^d$. $\hat{F}^d$ is an irreducible representation of $S_{d+1}$ if and only if char $(F) \nmid d + 1$. This can be shown directly or deduced from [8], for example.

The exterior powers of the standard representation $\bigwedge^k \hat{F}^d$ give other representations of $S_{d+1}$. $S_{d+1}$ acts on $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \hat{F}^d$ by $g(v_1 \wedge \ldots \wedge v_k) =$
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Consider $T_N^d$ as the parallelepiped generated by \( \{ Nv : v \in B \} \) with opposite faces identified. The homology group $H_1(T_N^d; F)$ is generated by the circles in the coordinate directions corresponding to the elements of $B$. This correspondence induces an isomorphism of $S_{d+1}$-representations $H_1(T_N^d; F) \simeq \hat{F}_d$. By the Künneth formula for homology, $H_k(T_N^d; F) \simeq \bigwedge^k \hat{F}_d$. This is easily seen to be an isomorphism of $S_{d+1}$-representations by comparing the action of $S_{d+1}$ on the homology generators with the definition of the $k^{th}$ exterior power of a representation. As such, the proposition follows from the previous paragraph.

Next, we consider the case of the square lattice $Z^d$. The point symmetry group of $Z^d$ is the hyperoctahedral group $W = S_2 \wr S_d$, where $\wr$ denotes the wreath product. It is generated by permutations of the coordinate directions and reflections which reverse a coordinate direction.

**Proposition 13.** Let $F$ be a field, $d > 0$, and $1 \leq i \leq d - 1$. $H_i(T_N^d; F)$ is an irreducible representation of $W_d$ if and only if $\text{char}(F) \neq 2$.

**Proof.** Consider $T_N^d$ as the cube generated by \( \{ N\text{e}_k \}_{1 \leq k \leq d} \) with opposite sides identified. For similar reasons as in the proof of Proposition 12, $H_i(T_N^d; F)$ is isomorphic to the $W_d$-representation $\bigwedge^i F^d$. We will give a direct proof that this is an irreducible representation of $W_d$ by showing that if $w \in \bigwedge^i F^d \setminus \{ 0 \}$ then $\langle W_dw \rangle = \bigwedge^i F^d$.

Let $w$ be an arbitrary non-zero element of $\bigwedge^i F^d \setminus \{ 0 \}$ by dividing by the leading coefficient if necessary we may write

$$w = e_{i_1,1} \wedge \ldots \wedge e_{i_1,i} + \ldots + c_m e_{i_m,1} \wedge \ldots \wedge e_{i_m,i}.$$ 

Let $\sigma_v \in S_d \leq W_d$ be a permutation so that

$$\sigma_w (e_{i_1,1} \wedge \ldots \wedge e_{i_1,i}) = e_1 \wedge e_2 \wedge \ldots \wedge e_i.$$ 

For each $1 \leq k \leq d$, let $\rho_k \in W_d$ be the reflection about the hyperplane orthogonal to $e_k$, and let $f_k(v) := v + \rho_k(v)$ for $v \in \bigwedge^i F^d$. Then

$$f_{i+1} (f_{i+2} (\ldots f_d (\sigma_w (w)))) = 2^{d-i} e_1 \wedge e_2 \wedge \ldots \wedge e_i.$$ 

$\text{char}(F) \neq 2$ so $2^{d-i} \neq 0$ and thus $e_1 \wedge e_2 \wedge \ldots \wedge e_i \in \langle W_dw \rangle$. But then using the action of $S_d \leq W_d$, we can obtain a basis for $\bigwedge^i F^d$, so $\langle W_dw \rangle = \bigwedge^i F^d$ for any non-zero $w$ and the action of $W_d$ is irreducible. 

$\square$
We can combine Lemma 11 with the preceding propositions to obtain the following corollary.

**Corollary 14.** Then there are constants $C_0, C_1 > 0$ not depending on $N, i$ such that

\[
P_p \left( S_i^\square \right) \geq C_0 P_p \left( A_i^\square \right)^{C_1}
\]

and

\[
P_p \left( S_i^\Box \right) \geq C_0 P_p \left( A_i^\Box \right)^{C_1}.
\]

It is worth noting that Lemma 11 is more general than some of our other tools. For example, in the case of continuum percolation studied in [4], this Lemma can be used to show the analogue of Corollary 14, even in the absence of stronger duality results.

**Proposition 15.** Let $\{Y_N\}_{N \in \mathbb{N}}$ be a sequence of finite sets with $|Y_N| \to \infty$, each of which has a transitive action by a symmetry group $H_N$. Also, let $R(N, p)$ be the random set obtained by including each element of $Y_N$ independently with probability $p$, and suppose there are functions $f_N : \mathcal{P}(Y_N) \to V$ which satisfy the hypotheses of Lemma 11 for some fixed symmetry group $G$. Assume that $G$ is a subgroup of $H_N$ for all $N$ and that the action of $H_N/G$ on $V$ is trivial. If $f_N(\emptyset) = 0$ and $f_N(Y_N) \neq 0$ for all sufficiently large $N$ then there exists a threshold function $\lambda(N)$ so that for any $\epsilon > 0$

\[
P \left( f_N \left( R(N, \lambda(N) - \epsilon) \right) = 0 \right) \to 1
\]

\[
P \left( f_N \left( R(N, \lambda(N) + \epsilon) \right) = V \right) \to 1
\]

as $N \to \infty$.

**Proof.** For a fixed value of $N$, $P \left( R(N, p) \right)$ is an increasing, continuous function of $p$ with $P \left( R(N, 0) \right) = 0$ and $P \left( R(N, 1) \right) = 1$ for all sufficiently large $N$. By the intermediate value theorem we can choose $\lambda(N)$ so that for all sufficiently large $N$,

\[
P \left( f_N \left( R(N, \lambda(N)) \right) \neq 0 \right) = 1/2.
\]

Then by Lemma 11, there exist $C_0, C_1 > 0$ such that

\[
P \left( f_N \left( R(N, \lambda(N)) \right) = V \right) \geq C_0 P \left( f_N \left( R(N, \lambda(N)) \right) \neq 0 \right)^{C_1} = \frac{C_0}{2^{C_1}} > 0.
\]

Choose an $\epsilon_0$ between 0 and $\frac{C_0}{2^{C_1}}$. Note that the event $\{f_N \left( R(N, p) \right) = V \}$ is increasing in $p$ and invariant under the action of $H_N$. By assumption, $H_N$ acts
transitively on $X$, so the hypotheses of Theorem 6 are met. Let $\epsilon > 0$. Rearranging Equation 2 gives that $\mathbb{P}(f_N(R(N, \lambda(N) + \epsilon)) = V) > 1 - \delta$ when
\[
\log (|Y_N|) > \frac{\rho \log(1/(2\delta))}{\epsilon}.
\]
On the other hand, the event $\{f(R(N, p)^c) = 0\}$ is also increasing, so by a similar argument, $\mathbb{P}(f_N(R(N, \lambda(N) - \epsilon)) = 0) \to 1$. □

In our models of interest, this tells us that $\lambda^\square(N)$ and $\lambda^\bigcirc(N)$ are sharp threshold functions of $N$ for the appearance of all giant cycles. From the definitions of $p_c$ and $q_c$, we also obtain the inequalities
\[
q_c^\square = \liminf_{N \to \infty} \lambda^\square(N) \leq \limsup_{N \to \infty} \lambda^\square(N) = p_c^\square
\]
and
\[
q_c^\bigcirc = \liminf_{N \to \infty} \lambda^\bigcirc(N) \leq \limsup_{N \to \infty} \lambda^\bigcirc(N) = p_c^\bigcirc.
\]
We can then describe the behavior of both models below $q_c$ and above $p_c$.

**Corollary 16.** If $p_1 > p_c^\square(i,d)$ and $p_2 > p_c^\bigcirc(i,d)$ then
\[
\mathbb{P}_{p_1}(S^\square) \to 1
\]
and
\[
\mathbb{P}_{p_2}(S^\bigcirc) \to 1
\]
as $N \to \infty$. Conversely, if $p_1 < q_c^\square(i,d)$ and $p_2 < q_c^\bigcirc(i,d)$ then
\[
\mathbb{P}_{p_1}(A^\square) \to 0
\]
and
\[
\mathbb{P}_{p_2}(A^\bigcirc) \to 0
\]
as $N \to \infty$.

4. **The Case $d = 2i$**

We now prove Theorem 1, that $p_c^\square(i,2i) = 1/2$ is a sharp threshold for $A^\square$ when $d = 2i$. The proof of the corresponding result for the site percolation model is nearly identical.

**Proof of Theorem 1.** Half-dimensional plaquette percolation is self-dual so $\mathbb{P}_{1/2}(A^\square) = \mathbb{P}_{1/2}(A^\bullet)$. By Lemma 10 at least one of the events $A^\square$ and $A^\bullet$ must occur. Therefore,
\[
2\mathbb{P}_{1/2}(A^\square) = \mathbb{P}_{1/2}(A^\square) + \mathbb{P}_{1/2}(A^\bullet) \geq 1
\]
and
\[ \mathbb{P}_{1/2} (A^□) \geq 1/2 \]
for all \( N \). It follows that \( p_c^\square \leq 1/2 \). Thus, if \( p > 1/2 \) then
\[ \mathbb{P}_p (S^\square) \to 1 \]
as \( N \to \infty \), and if \( p < 1/2 \) then
\[ \mathbb{P}_p (A^\square) \to 0 \]
as \( N \to \infty \) by Corollary 16. \( \square \)

5. Sharpness and Duality

In this section, we combine the Duality Lemma (Lemma 10) with Corollary 16 to examine the behavior of \( \mathbb{P}_p (A^\square) \) below \( q_c^\square (i,d) \) and above \( p_c^\square (i,d) \). We also relate these thresholds to \( p_c^\square (d-i,d) \) and \( q_c^\square (d-i,d) \). Direct analogues of these statements hold for site percolation model hold by very similar arguments, and we do not state them separately here.

We remind the reader that
\[ q_c^\square (i,d) = \sup \left\{ p : \limsup_{N \to \infty} \mathbb{P}_p (S^\square) < 1 \right\} . \]

First, Corollary 16 above has the following corollary.

**Corollary 17.** \( q_c^\square (i,d) \leq p_c^\square (i,d) \).

Now we show a partial duality result for any \( i \) and \( d \).

**Proposition 18.**
\[ p_c^\square (i,d) + q_c^\square (d-i,d) = 1 . \]

**Proof.** Let \( p > p_c^\square (i,d) \). Then
\[ \mathbb{P}_p (A^\square) = 1 - \mathbb{P}_p (Z^\square) \quad \text{by definition} \]
\[ = 1 - \mathbb{P}_p (S^\square) \quad \text{by Lemma 10} \]
\[ \to 0 \quad \text{by Corollary 16} \]
as \( N \to \infty \). Therefore, \( 1 - p \leq q_c^\square (d-i,d) \) for all \( p > p_c^\square (i,d) \) and
\[ p_c^\square (i,d) + q_c^\square (d-i,d) \geq 1 . \] (5)

Until now, we have suppressed the dependence of probabilities of events on \( N \). To work with subsequences in this argument, denote the probability of an event \( B \) for \( P(i,d,N,p) \) by \( \mathbb{P}_{p,N} (B) \).
Let \( p < p_c^\square(i,d) \). Then there is a subsequence \( \{n_1,n_2,\ldots\} \) of \( \mathbb{N} \) for which

\[
\mathbb{P}_{p,n_k}(A^\square) \to 0 .
\]

By Lemma 10,

\[
\mathbb{P}_{p,n_k}(S^\square) \to 1
\]

so

\[
\limsup_{N \to \infty} \mathbb{P}_p(S^\square) = 1
\]

and \( 1 - p \geq q_c^\square(i,d) \) for all \( p < p_c^\square(i,d) \). Therefore,

\[
p_c^\square(i,d) + q_c^\square(d-i,d) \leq 1
\]

which holds with equality by Equation 5. \( \square \)

Propositions 15 and 18 show that duality between \( p_c^\square(i,d) \) and \( p_c^\square(d-i,d) \) is equivalent to the convergence of the threshold function \( \lambda^\square(N) \).

**Corollary 19.** The following are equivalent.

(a) \( \lim_{N \to \infty} \lambda^\square(N) \) exists.

(b) \( p_c^\square(i,d) = q_c^\square(i,d) \).

(c) \( p_c^\square(i,d) + p_c^\square(d-i,d) = 1 \).

In the next section, we demonstrate that the statements of Corollary 19 hold in the cases \( i = 1 \) and \( i = d - 1 \).

### 6. The Cases \( i = 1 \) and \( i = d - 1 \)

We show that \( p_c^\square(1,d) \) and \( q_c^\square(1,d) \) coincide and equal the critical threshold for bond percolation on \( \mathbb{Z}^d \), denoted here by \( \hat{p}_c = \hat{p}_c(d) \). As in the previous sections, the proofs for the corresponding results for the site percolation model are nearly identical and we do not state them here.

We rely on two results from the classical theory of this system in the subcritical and supercritical regimes. In the former regime, we use Menshikov’s Theorem [19], also proven independently by Aizenman and Barsky, showing an exponential decay in the radius of the cluster at the origin.

For a vertex \( x \) and a subset \( S \) of \( \mathbb{T}^d_N \), denote the event that \( x \) is connected to a vertex in \( S \) by a path of edges in \( P \) by \( x \leftrightarrow S \).
**Theorem 20** (Menshikov/Aizenman-Barsky). Consider bond percolation on $\mathbb{Z}^d$. If $p < \hat{p}_c$ then there exists a $\kappa(p) > 0$ so that

$$\mathbb{P}_p (0 \leftrightarrow \partial[-M, M]^d) \leq e^{-\kappa(p)M}$$

for all $M > 0$.

We apply Menshikov’s Theorem to show that the probability of a giant one-cycle limits to zero as $N \to \infty$ when $p < \hat{p}_c$.

**Proposition 21.** $q_c(1, d) \geq \hat{p}_c$

**Proof.** Let $p < \hat{p}_c$ and let $M = \lfloor N/2 \rfloor$. For a vertex $x$ of $\mathbb{T}_N^d$, denote by $A_x$ the event that there is a connected, giant 1-cycle containing an edge adjacent to $x$. If $A_0$ occurs then $0 \leftrightarrow \partial[-M, M]^d$ because a 1-cycle contained in $[-M, M]^d$ is null-homologous in $\mathbb{T}_N^d$. Therefore,

$$\mathbb{P}_p (A_x) \leq e^{-\kappa(p)M}$$

for all vertices $x$ of $\mathbb{T}_N^d$, using translation invariance and Theorem 20.

Let $X$ be the number of vertices in $\mathbb{T}_N^d$ that are contained in a connected, giant 1-cycle. $A^\Box = \{X \geq 1\}$ so

$$\mathbb{P}_p (A^\Box) = \mathbb{P}_p (X \geq 1) \leq \mathbb{E}_p (X)$$

by Markov’s Inequality

$$= \sum_{x \in \mathbb{T}_N^d} \mathbb{P}_p (A_x) \leq N^D e^{-\kappa(p)M}$$

using Equation 6

$$= N^d e^{-\kappa(p)\lfloor N/2 \rfloor}$$

which goes 0 as $N \to \infty$. \hfill \Box

In the supercritical regime, we use the following lemma on crossing probabilities inside a rectangle (which is Lemma 7.78 in [12]).

**Lemma 22.** Let $p > \hat{p}_c$. Then there is an $L > 0$ and a $\delta > 0$ so that if $N > 0$ and $x \in [0, N - 1]^{d-1} \times [0, L]$, then probability that 0 is connected to $x$ inside $P \cap \left( [0, N - 1]^{d-1} \times [0, L] \right)$ is at least $\delta$.

**Proposition 23.** $p_c(1, d) \leq \hat{p}_c$
Proof. Let $p > \hat{p}_c$, and let $B$ be the event that there is a path of edges of $P$ connecting $0$ to $(N - 1)e_1 = (N - 1, 0, \ldots, 0)$ inside of $[0, N - 1]^d$. By the previous lemma, there is a $\delta > 0$ so that $P_p(B) \geq \delta$.

If $B$ occurs, then the path obtained by adding the edge between $(N - 1)e_1$ and $Ne_1 = 0$ is a giant 1-cycle. It follows that $P_p(A) \geq pP_p(B) \geq p\delta$ for any choice of $N$.

Therefore, $p \leq \hat{p}_c$ for all $p \geq \hat{p}_c$ and $p \leq \hat{p}_c$.

The proof of Theorem 2 is completed by combining Propositions 21 and 23 with Corollary 19.

Proof of Theorem 2. Propositions 21 and 23 show that $p \leq \hat{p}_c \leq q \leq p$ for all $p \geq \hat{p}_c$ and $p \leq \hat{p}_c$. Therefore, $p \leq \hat{p}_c$ and $1 - \hat{p}_c$ are sharp thresholds for 1-dimensional and $(d - 1)$-dimensional percolation on the $\mathbb{T}^d$, respectively.

7. Monotonicity

Next, we prove that the critical probabilities $p^c(i, d)$ are strictly increasing in $i$ and strictly decreasing in $d$. This will complete the proof of Theorem 3. Here again we will need to differentiate between the cubical and permutohedral lattices.

First we consider the cubical case, in which we compare percolation on $\mathbb{T}_N^d$ with the thickened $d - 1$-dimensional slice $\mathbb{T}_N^d \cap \{0 \leq x_1 \leq 1\}$. Compare the first part of the proof to that of Lemma 4.9 of [4].

**Proposition 24.** For $0 < i < d - 1$,

$$p^c(i, d) < p^c(i, d - 1) < p^c(i + 1, d).$$

Proof. First, we will show that

(7) $$p^c(i, d) \leq p^c(i, d - 1) \leq p^c(i + 1, d).$$

Let $T = \mathbb{T}_N^d \cap \{x_1 = 0\}$. $T$ is a torus of dimension $d - 1$ and, by a standard argument, the map on homology $\alpha_\ast : H_j(T) \to H_j(\mathbb{T}^d)$ induced by the inclusion $T \hookrightarrow \mathbb{T}^d$ is injective for all $j$. $P \cap T$ is distributed as $P(i, d - 1, N, p)$. 
Define $A_{d-1}^\square$ to be the event that $\gamma_s : H_i(\mathbb{T} \cap T) \to H_i(T)$ is non-zero, where $\gamma_s$ is induced by the inclusion $\mathbb{T} \cap T \hookrightarrow T$. If $A_{d-1}^\square$ holds then $\alpha_s \circ \phi_s$ is also non-zero, as $\alpha_s$ is injective. But $\alpha_s \circ \gamma_s = \phi_s \circ \beta_s$, where $\beta_s$ is the map on homology $\beta_s : H_i(\mathbb{T} \cap T) \to H_i(P)$ induced by the inclusion $\mathbb{T} \cap T \hookrightarrow P$, so $\phi_s$ is also non-zero. It follows that $A_{d-1}^\square \implies A^\square$. Therefore $p_{c}^\square(i, d - 1) \geq p_{c}^\square(i, d)$ by the definition of that threshold.

Observe that $H_i(\mathbb{T}^d)$ is generated by the images of the maps on homology $H_i(\mathbb{T}^d \cap \{x_j = 0\}) \to H_i(\mathbb{T}^d)$ induced by the inclusions $\mathbb{T}^d \cap \{x_j = 0\} \hookrightarrow \mathbb{T}^d$ as $j$ ranges from 1 to $d$. Denote by $S_j$ the event that the map $H_i(P \cap \{x_j = 0\}) \to H_i(\mathbb{T}^d \cap \{x_j = 0\})$ induced by inclusion is surjective and let $q > q_{c}^\square(i, d - 1)$. Then there is a subsequence $(n_1, n_2, \ldots)$ of $\mathbb{N}$ so that

$$\mathbb{P}_{p, n_k}(S_j) \to 1$$

as $k \to \infty$ for $j = 1, \ldots, d$. As $S \subset \bigcap S_j$, Harris’s Inequality implies that $\mathbb{P}_{p, n_k}(S) \to 1$ also. Therefore, $p > q_{c}^\square(i, d)$ and $q_{c}^\square(i, d - 1) \geq q_{c}^\square(i, d)$. For all $i$ and $d$. Combining this inequality (for a different choice of $i$ and $d$) with Proposition 18 we obtain

$$p_{c}^\square(i, d - 1) = 1 - q_{c}^\square(d - i - 1, d - 1) \leq 1 - q_{c}^\square(d - i - 1, d) = p_{c}^\square(i + 1, d),$$

which shows Equation 7.

It will be useful later in the argument to observe that these inequalities, together with Theorem 2 and known lower bounds on $\hat{p}_c$ (see [5], for example), imply that

$$0 < p_{c}^\square(1, d) \leq p_{c}^\square(i, d) \leq p_{c}^\square(i, i + 1) < 1.$$ 

Furthermore, we can show $p_{c}^\square(i, d) < p_{c}^\square(i, d - 1)$ using the thicker cross-section $T' = \mathbb{T}^d_{\mathbb{N}} \cap \{0 \leq x_1 \leq 1\}$. Note that an $i$-face $v$ of $T'$ is in the boundary of a unique $(i + 1)$-face $w(v)$ of $T'$ that is not contained in $T$ (for example, if $v = \{0\} \times [0, 1]^i \times \{0\}^{d-i-1}$, then $w(v) = [0, 1]^{i+1} \times \{0\}^{d-i-1}$). The idea is to sometimes add $v$ to $T'$ when the other $i$-faces of $w(v)$ are present, effectively increasing the percolation probability in $T$ by a small amount. However, we must be careful to do so in a way so that the $i$-faces remain independent.

The $i$-faces of $T'$ are divided into three subsets: those included in $T$, those which are perpendicular to $T$ (that is, $i$-faces not included in $T$ which intersect $T$ in their boundary), and those parallel to $T$ (that is, $i$-faces of the form $v + e_1$, where $v$ is an $i$-face of $T$). For an $i$-face $v$ of $T$, let $J(v)$ be the set of all perpendicular $i$-faces that meet $v$ at an $i - 1$ face. $v$, $v + e_1$, and $J(v)$ are the $i$-faces of the $(i + 1)$-face $w(v)$. Also, for a perpendicular $i$-face $u$ of $T'$, let
Figure 5. The setup in the proof of Proposition 24 for the case $d = 2, i = 1$. On the left, $P'$ is shown in black and the remaining faces of $P$ are depicted in gray. On the right, $P \cap T$ is in black and the additional faces of $R$ are shown in blue. Note that giant cycles exist in $P, P', \text{and } R$, but not in $P \cap T$.

$K(u) = \{v : u \in J(v)\}$. Note that for any $u$ and $v$

$$|J(v)| = 2i \quad \text{and} \quad |K(u)| = 2(d - i).$$

We define a coupling between $i$-dimensional plaquette percolation $P'$ on $T'$ with probability $p$ and $i$-dimensional percolation $R$ with probability $p + p(1 - p)q^{2i}$ on $T$, where $q = q(p)$ is chosen to satisfy $p = 1 - (1 - q)^{2(d - i)}$. For all pairs $(v, u)$ where $v$ is an $i$-face of $T$ and $u \in J(v)$, define independent Bernoulli random variables $\kappa(u, v)$ to be 1 with probability $q$ and 0 with probability $1 - q$. Let $P' \subset T'$ be the subcomplex containing the $i - 1$-skeleton of $T'$ where each $i$-face in $T$ or parallel to $T$ is included independently with probability $p$, and the other $i$-faces $u$ of $T'$ are included if $\kappa(u, v) = 1$ for at least one $v \in K(u)$. Observe that

$$\mathbb{P}(u \in P') = 1 - \mathbb{P} \left( \cap_{v \in K(u)} \{ \kappa(u, v) = 0 \} \right) = 1 - (1 - q)^{2(d - i)} = p$$

(using Equation 10), and that the faces $u$ are included independently. That is, $P'$ is percolation with probability $p$ on $T'$. On the other hand, define $R \subset T$ by starting with all faces of $P' \cap T$ and adding an $i$-face $v \notin P'$ if $v + e_1 \in P'$ and $\kappa(v, u) = 1$ for all $u \in J(v)$. Then $R$ is percolation on $T$ with probability $p + p(1 - p)q^{2i} > p$. See Figure 5.

As $p + p(1 - p)q^{2i}$ is a continuous function of $p$ and $0 < p_c^{\square}(i, d - 1) < 1$ (Equation 9), we can choose $p$ to satisfy

$$0 < p < p_c^{\square}(i, d - 1) < p + p(1 - p)q^{2i} < 1.$$

Then

$$\mathbb{P}_{p + p(1 - p)q^{2i}} (\xi_* \text{ is non-trivial}) \to 1$$
HOMOLOGICAL PERCOLATION ON A TORUS

Figure 6. Percolation on $T''$ (left) mapping to percolation on $T'$ (right) for $i = 1, d = 2$. The blue edges are in $R \setminus P$.

As $N \to \infty$ by the definition of $p^\square_c(i, d - 1)$, where $\xi_* : H_i(R) \to H_i(T)$ is the map on homology induced by the inclusion $R \hookrightarrow T$.

Extend $P'$ to plaquette percolation $P$ on all of $\mathbb{T}^d_N$ by including the $i$-faces in $\mathbb{T}^d_N \setminus T'$ independently with probability $p$. If $\sigma$ is an $i$-cycle of $R$ we can write

$$\sigma = \sum_j a_j u_j + \sum_k b_k v_k$$

where $u_j \notin P$ and $v_k \in P$ for all $j$ and $k$. Then, by construction, we can form a corresponding $i$-cycle $\sigma'$ of $P$ by setting

$$\sigma' = \sigma + \sum_j a_j \partial w(u_j) .$$

$\sigma$ and $\sigma'$ are homologous in $\mathbb{T}^d$, so $\alpha_* \circ \xi_* ([\sigma]) = \phi_* ([\sigma'])$ In particular, if $\xi_*$ is non-trivial then $\phi_*$ is non-trivial as well. Using Equation 11, it follows that

$$\mathbb{P}_p(A^\square) \geq \mathbb{P}_{p+p(1-p)q^2i} (\xi_* \text{ is non-trivial}) \to 1 ,$$

as $N \to \infty$. Therefore,

$$p^\square_c(i, d) \leq p < p^\square_c(i, d - 1) .$$

We can define similar couplings between percolations on $\mathbb{T}^d_N \cap \{x_j = 0\}$ and on $\mathbb{T}^d_N \cap \{0 \leq x_j \leq 1\}$ for $j = 1 \ldots d$. Combining these couplings with the argument leading to Equation 8 yields $q^\square_c(i, d) < q^\square_c(i, d - 1)$. Then from Proposition 18 we obtain

$$p^\square_c(i, d - 1) = 1 - q^\square_c(d - i - 1, d - 1) < 1 - q^\square_c(d - i - 1, d) = p^\square_c(i + 1, d) .$$

If an alternative approach to the proof of Proposition 24 is to construct a third space $T''$ by attaching a new $i + 1$-cube to each $i$-face of $T$ along one of the cube’s $i$-faces. We can define inhomogeneous percolation $P''$ on $T''$ by starting with the $i - 1$-skeleton of $T''$, adding each $i$-face of $T$ and each $i$-face parallel to $T$ independently with probability $p$, and adding the perpendicular $i$-faces independently with probability $q$ (these faces play the same role as the random variables $\kappa(u, v)$ above). Giant cycles in $P''$ are ones that are mapped
non-trivially to $H_i(T'')$ by the map on homology induced by the inclusion $P'' \hookrightarrow T''$, and they appear at a lower value of $p$ than $p_{c}(i,d-1)$ (precisely when they appear in $R$ as defined above). The proof is finished by observing that the quotient map $\pi : T'' \to T'$ identifying the corresponding perpendicular faces of neighboring cubes induces an injective map on homology, and therefore the existence of giant cycles in $P''$ implies the existence of giant cycles in $P$. This idea is illustrated in Figure 6. Note that our definition of giant cycles in $T''$ can be adapted to give a more general notion of homological percolation in the $i$-skeleton of a cubical or simplicial complex whose $i$-dimensional homology is nontrivial.

Now we consider the permutohedral lattice. The idea of the proof is again to find a copy of $T^{d-1}$ within $T^d$, but here the correct embedding is slightly less obvious.

Before beginning the proof, it will be useful to discuss the combinatorial structure of the permutohedron. Recall that $\hat{R}^d \subset \mathbb{R}^{d+1}$ is the subspace

$$\hat{R}^d := \left\{ (x_0, x_1, \ldots, x_d) : \sum_{k=0}^{d} x_k = 0 \right\}.$$

We review the expositions of [7] and [22]. The $d$-permutohedron centered at the origin in $\hat{R}^d$ has vertices obtained from permuting the coordinates of

$$\left(\frac{1}{2d+1}(d,d-2,d-4,\ldots,-d+2,-d)\right).$$

Let $\sigma$ be this permutohedron centered at 0. It is enough to understand the geometry of $\sigma$ because the others permutohedrons of the lattice are translates of $\sigma$. The $k$-faces of $\sigma$ correspond to ordered partitions of the coordinate indices into $d-k+1$ subsets $M_1, \ldots, M_{d-k+1}$, where every coordinate in places $M_j$ is smaller than every coordinate in $M_l$ for $j < l$. We will abuse notation and identify these partitions with the faces associated to them. For example, the 4-permutohedron has a 2-face $\{\{1,3\},\{4\},\{2,5\}\}$ containing the vertices, and by extension all points, satisfying the property that the two smallest coordinates are in positions 1 and 3, and the next smallest coordinate is in position 4. This face has 1-subfaces corresponding to refinements of its ordered partition, namely $\{\{1\},\{3\},\{4\},\{2,5\}\}$, $\{\{3\},\{1\},\{4\},\{2,5\}\}$, $\{\{1,3\},\{4\},\{2\},\{5\}\}$, and $\{\{1,3\},\{4\},\{5\},\{2\}\}$. Since antipodal vertices have reversed coordinate order, it is not hard to check that opposite $k$-faces correspond to the same partition with reversed blocks.

We will be also interested in the combinatorial structure of the permutohedral lattice as it relates to translates of a fixed $(d-1)$-face. Let $f$ be the $(d-1)$-face corresponding to the partition $\{\{1,2,\ldots,d\},\{d+1\}\}$. Then $f$ is itself a
(d − 1)-permutohedron with subfaces corresponding to ordered partitions of \{1, 2, \ldots, d\}. Let \( f' \) be the opposite \((d − 1)\)-face, i.e. the one corresponding to \{\{d + 1\}, \{1, 2, \ldots, d\}\}. Consider \((d − 1)\)-face paths between \( f \) and \( f' \), meaning sequences \((f_1, f_2, \ldots, f_m)\) of \((d − 1)\)-faces such that \( f_1 = f \), \( f_m = f' \), and \( f_k \cap f_{k+1} \) is a \((d − 2)\)-face for each \( 1 \leq k \leq m − 1 \). Then there is no path of length 3, but for a \((d − 2)\)-face \( h \subset f \) corresponding to an ordered partition \( \{A, B, \{d + 1\}\}, \{A \cup \{d + 1\}, B\}\) of length 4 passing through \( h \). Thus, the other \((d − 1)\)-faces can be decomposed into one pair for each \((d − 2)\)-face of \( f \), each pair consisting of a neighbor of \( f \) and a neighbor of \( f' \).

We start with an easy lemma.

**Lemma 25.** Let \( \sigma_1 \) and \( \sigma_2 \) be the two other permutohedra adjacent to \( \sigma \) along \( \{A, B \cup \{d + 1\}\} \) and \( \{A \cup \{d + 1\}, B\}\) respectively. Then \( \sigma_1 \cap \sigma_2 \) is a translate of \( f \). In fact, letting \( v \) be such that \( \sigma + v = \sigma_1 \), we have \( \sigma_1 \cap \sigma_2 = f' + v \).

**Proof.** By Lemma 3.4 of [7], the centers of the permutohedra are in general position, so a \((d − k)\)-cell of the Voronoi complex is an intersection of exactly \((k + 1)\) top dimensional cells. In particular, there are exactly 3 \((d − 1)\)-faces among all permutohedra of the lattice that contain \( \{A, \{d + 1\}, B\} \). We will show that the face not contained in \( \sigma \) is \( \sigma_1 \cap \sigma_2 = f' + v \).

Let \( G = \sigma \cap \sigma_1 = \{A, B \cup \{d + 1\}\} \subset \sigma \) and \( G' = \{B \cup \{d + 1\}, A\} \subset \sigma \), and let \( T_v : \mathbb{R}^d \to \mathbb{R}^d \) be translation by \( v \). Also, denote the reflections about the line spanned by \( v \) and the hyperplane orthogonal to \( v \) by \( \rho_1 \) and \( \rho_2 \), respectively. Then \( G = T_v(G') \) since opposite \((d − 1)\)-faces of \( \sigma \) have reversed ordered partitions. Moreover, \( G \) and \( G' \) are orthogonal to \( v \) so \( T_v|_{G'} = \rho_2|_{G'} \).

Now, consider the action of \( \rho_1 \) on the \((d − 2)\)-faces of \( G' \). \( \rho_1 \) sends a \((d − 2)\) face to the face opposite to it in \( G' \), which can be obtained by reversing the subpartitions within each of \( B \cup \{d + 1\} \) and \( A \). This can be seen from the fact that opposite vertices of \( G' \) are maximally far apart via edge paths within \( G' \), so they must have reversed coordinates within \( B \cup \{d + 1\} \) and \( A \).

Then \( \rho_2 \circ \rho_1 \) is reflection about the origin (the antipodal map), so since antipodal \((d − 2)\)-faces have reversed ordered partitions,
\[
\{A, \{d + 1\}, B\} = \rho_2 \circ \rho_1 (\{B, \{d + 1\}, A\}) = \rho_2 (\{\{d + 1\}, B, A\}) = T_v (\{\{d + 1\}, B, A\}) \subset f' + v.
\]

But \( f' + v \) is not contained in \( \sigma \), so it must be the third \((d − 1)\)-face adjacent to \( \{A, \{d + 1\}, B\} \) (in addition to \( \sigma \cap \sigma_1 \) and \( \sigma \cap \sigma_2 \), which are both contained in \( \sigma \)) and therefore \( \sigma_1 \cap \sigma_2 = f' + v \). \( \square \)
Now we are ready to prove the monotonicity of \( p_i^O(d) \) in \( i \) and \( d \).

**Proposition 26.** \( p_i^O(d) < p_i^O(d - 1) < p_{i+1}^O(d) \)

**Proof.** First we prove \( p_i^O(d) \leq p_i^O(d - 1) \leq p_{i+1}^O(d) \). Unlike the case of plaquettes in \( \mathbb{Z}^d \), there is no obvious isometric embedding of the \((d - 1)\)-dimensional permutohedral lattice into the \(d\)-dimensional permutohedral lattice. We will instead find a set of \(d\)-permutohedra that is combinatorially and homotopy equivalent to the \((d - 1)\)-lattice. The idea is to associate each \(d\)-permutohedron with a fixed \((d - 1)\)-face and then project a thickened \((d - 1)\)-surface of permutohedra orthogonally to that face.

Now we can begin constructing the sublattice. Let \( w \) be the vector pointing to the center of \( f \). Let \( \mathbb{L}_{d-1} \cong A^*_d \) be the sublattice of \( A_d^* \) generated by the vectors of \( A_d^* \) orthogonal to \( w \) and let \( \mathbb{L}_d \) be the sublattice generated by \( \mathbb{L}_{d-1} \cup \{2w\} \). Call an equivalence class of the \(d\)-permutohedra under the action of translation by \(2\mathbb{Z}w\) a pile of permutohedra. For any permutohedron \( \theta \), define

\[
B_\theta = \{ \theta' : \theta' \cap \theta \neq \emptyset \},
\]

the union of the permutohedra that intersect \( \theta \). Identifying permutohedra with their centers, let

\[
S := \bigcup_{\theta \in \mathbb{L}_{d-1}} B_\theta.
\]

Then take \( S' \) be the set of permutohedra intersecting the upper envelope of \( S \) with respect to \( w \). In other words, \( S' \) is the union of one permutohedron of \( S \) from each pile such that for each pile \( \Pi \) we have

\[
(S' \cap \Pi) \cdot w = \sup_{\theta \in S' \cap \Pi} \theta \cdot w.
\]

Alternatively, one can construct \( S' \) explicitly. For a permutohedron \( \theta \), let \( v_\theta \) be such that \( \sigma + v_\theta = \theta \). Define

\[
U_\theta := \{ \theta' \in A_d^* : \theta' \cap (f + v_\theta) \neq \emptyset \} \setminus \theta.
\]

Then we can write

\[
S' = \bigcup_{\theta \in \mathbb{L}_{d-1}} U_\theta.
\]

Now we will show the homotopy equivalence respecting the cell structure via the nerve theorem, seen in Corollary 4G.3 of [15]. Let \( \mathcal{U} \) be the open cover of \( S' \) induced by the permutohedra it contains. Since the permutohedra are convex, it is a good cover and therefore \( S' \) is homotopy equivalent to \( \mathcal{N} \mathcal{U} \). Then we compare this to the cover \( \mathcal{V} \) of the \((d - 1)\)-dimensional permutohedral lattice induced by its \((d - 1)\)-permutohedra and the corresponding nerve \( \mathcal{N} \mathcal{V} \).
Let \( \theta \) be a permutohedron of \( S' \). Let \( f_\theta = f + v_\theta \) and let \( f'_\theta = f' + v_\theta \). Lastly, let \( h \) be an arbitrary \((d-2)\)-dimensional face of \( f_\theta \) and let \( h' \) be the corresponding \((d-2)\)-dimensional of \( f'_\theta \) obtained via translation by \(-2w\). By the definition of \( S' \), there is at most one other permutohedron of \( S' \) containing \( h \), since the one adjacent to \( \theta \) along \( f_\theta \) would be in the same pile. The same is true for \( h' \), and by Lemma 25 we cannot have both because this would again be two elements of the same pile. However, one or the other must be present because adjacent piles of \( S \) are connected by \((d-1)\)-faces and taking the upper envelope preserves this property. Thus, the permutohedra of \( S' \) adjacent to \( \theta \) are in bijection with adjacent \((d-1)\)-permutohedra to a fixed permutohedron \( \theta^{d-1} \in A_{d-1}^* \). Then it only remains to check that the intersections are the same in each case.

Let \( \alpha_1, \alpha_2, \ldots, \alpha_k \in A_{d-1}^* \) be \( d-1 \)-permutohedra adjacent to \( \theta^{d-1} \). For each \( j \), take \( \gamma_j \in S' \) to be the \( d \)-permutohedron in the pile corresponding to \( \alpha_j \). Again using Lemma 3.4 of [7], any \( k \) pairwise adjacent permutohedra intersect at a \((d - k + 1)\)-face and any set that is not pairwise adjacent has empty
intersection. From the construction of $S'$, $\gamma_j \cap \gamma_l = \emptyset$ if and only if $\alpha_j \cap \alpha_l = \emptyset$. Thus, \{\gamma_1, \ldots, \gamma_k, \theta\} are pairwise adjacent if and only if \{\alpha_1, \ldots, \alpha_k, \theta^{d-1}\} are, and so we have

$$\left( \bigcap_{j \leq k} \gamma_j \right) \cap \theta \neq \emptyset \iff \left( \bigcap_{j \leq k} \alpha_j \right) \cap \theta^{d-1} \neq \emptyset.$$ 

We have then shown that the $NU \simeq NV$. Furthermore, we have shown that there is a bijection between the permutohedra of $S'$ and those of $A_{d-1}^*$ such that for any $U' \subset U$, the corresponding $V' \subset V$ satisfies $NU' \simeq NV'$.

The strict inequality $p^O_i (d) < p^O_i (d - 1)$ can be obtained by a similar proof to the plaquette case. For a permutohedron $\theta \in S'$, it is easy to check that if there is a giant cycle in $Q \cup \theta$, there is also a giant cycle in $Q \cup (B_G (\theta, 1) \setminus S')$. There is overlap between the added permutohedra, but we deal with this in the same way as the overlap in the plaquette construction. This also gives the strict inequality $p^O_i (d - 1) < p^O_{i+1} (d)$ by duality as before. \hfill $\square$

**Proof of Theorem 3.** By Equation 9, $q^c_i (i, d), p^c_i (i, d) \in (0, 1)$. The remaining statements follow from Corollary 16 and Propositions 18 and 24. \hfill $\square$

Note that we could alternatively show that $p^c_i (i, d), q^c_i (i, d) \in (0, 1)$ by modifying the proof of Proposition 21 to work for the lattice of $i$-plaquettes in $\mathbb{Z}^d$ and using a Peierls-type argument to obtain the bound

$$\frac{1}{2d - i + 1} \leq q^c_i (i, d) \leq p^c_i (i, d) \leq 1 - \frac{1}{d + i + 1}.$$ 

**Proof of Theorem 4.** Theorem 4 follows from the proofs of Theorems 1, 2, and 3 with the adjustments for the permutohedral lattice noted throughout the paper. In particular, Lemma 10 and Propositions 13 and 24 are replaced by Lemma 7 and Propositions 12 and 26 respectively. \hfill $\square$

8. **Future directions**

It seems that not much is known about percolation with higher-dimensional cells or homological analogues of bond or site percolation.

- Do $\lambda^c (N, i, d)$ and $\lambda^O_i (N, d)$ converge as $N \to \infty$ for all $i, d$?
- Can the minor restrictions on the characteristic of the coefficient field of the homology be removed?
- Are there scaling limits for plaquette percolation? For bond percolation in the plane at criticality, conjecturally we get SLE. This could be a reasonable question to approach experimentally.
• Is there a limiting distribution for rank $\phi_*$, as $N \to \infty$? When $d = 2i$, our results imply that the distribution is symmetric and the expectation satisfies $\mathbb{E}[\text{rank } \phi_*] = \left(\frac{d}{d/2}\right)/2$, but at the moment we do not know anything else.

• One of the most interesting possibilities we can imagine would be a generalization of the Harris–Kesten theorem when $d = 2i$, on the whole lattice $\mathbb{Z}^d$ rather than on the torus $T_N^d$. One possibility might be to compactify $\mathbb{R}^d$ to a torus $T^d$. In various proofs of the Harris–Kesten theorem, a key step is to go from crossing squares to crossing long, skinny rectangles—see, for example, Chapter 3 of [5]. One difficulty is that we do not currently have a high-dimensional version of the Russo–Seymour–Welsh method, passing from homological “crossings” of high-dimensional cubes to long, skinny boxes.

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References


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