Hadwiger Models: Low-Temperature Behavior in a Natural Extension of the Ising Model

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Abstract

All isometrically invariant Markov (strictly local) fields on binary assignments are induced by energy functions that can be represented as linear combinations of area, perimeter, and Euler characteristic. This class of model includes the Ising model, both ferro- and antiferro-magnetic, with and without a field, as well as the "triplet" Ising model We determine the low-temperature behavior for this class of model, and construct a phase diagram of that behavior. In particular, we identify regions with three geometric phases, regions with a single unique phase, and coexistence lines between them.

1 Introduction

Traditionally, the Ising Hamiltonian is formally defined as a map $H : \{-1, 1\}^V \to \mathbb{R}$ by $H(\sigma) = \sum_{(x,y)\in E} \sigma(x)\sigma(y) + h\sum_{x\in V} \sigma(x)$, where V, E are the vertex and edge sets of some graph G. However, the phase transition of the two-dimensional Ising model was first proved by Peierls [Pei36] using the equivalent formulation where $H : \{0, 1\}^{F^*} \to \mathbb{R}$ by $H = P(\sigma) + hA(\sigma)$ where F^* are the faces of the dual graph G^* of G, and P, A are respectively the perimeter and area of $\cup_{f\in\sigma} f$, as represented in figure 1. This formulation allowed Peierls to use a geometric argument to bound the perimeter and thus guarantee that one orientation dominates when the temperature is sufficiently low.

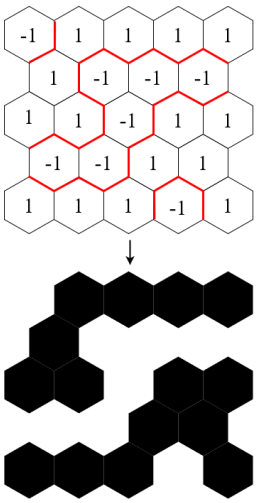


Figure 1: Converting binary assignments to polyconvex subsets

Perimeter and area are both invariant valuations, that is, given a union of convex sets (polyconvex) U, they satisfy $f(T(U)) = f(U)$ for any isometry T, $f(U \cup V) = f(U) + f(V) - f(U \cap V)$, and $f(\varnothing) = 0$. $f(U \cup V) = f(U) + f(V) - f(U \cap V)$ if and only if the energy depends only on immediately neighboring sites, as repeated application of the equation decomposes the energy function into a sum over faces, edges, and vertices. This decomposition property means the distribution within a given region depends only on its boundary; such a distribution is called a Markov field. For an infinite graph with finite fundamental domain, the dual faces are always compact and polygonal, so their union is polyconvex, and any isometrically invariant Markov Hamiltonian on binary assignments is a valuation, up to a constant term for the energy of the empty set. Hadwiger's theorem describes the set of all valuations explicitly:

Theorem (Hadwiger's Theorem[Had56])**.** *Any invariant valuation must be a linear combination of the perimeter, area, and Euler characteristic*

Hadwiger's theorem implies that any invariant Markov Hamiltonian must be a linear combination of perimeter term, area term, and Euler characteristic. In fact, all Markov fields are induced by Markov Hamiltonians [Gri73], so this class of function includes every possible Markov field on binary assignments. This paper is dedicated to the study of the distributions induced by members of this class of Hamiltonian on the hexagonal lattice. Note that the Euler characteristic can be expressed as a sum over particular subsets S of $\prod_{f\in S} \sigma_f$, so this class is a special case of the generalized Ising models. We choose to work on the

hexagonal lattice because each term behaves nicely under inversion: if a set of faces is replaced with its complement, the perimeter remains unchanged, whereas the area and the Euler characteristic are inverted, up to an additive constant.

Define a domain as a connected subset F of the faces of some lattice with connected complement, a configuration $\sigma \in \{0,1\}^F$ on a domain as a binary assignment to those faces, a boundary configuration ρ as a binary assignment to all the faces in the lattice, an occupied or filled hexagon f as one such that $\sigma(f) = 1$ if $f \in \Lambda$ or $\rho(f) = 1$ if $f \notin \Lambda$ and, $\partial\Lambda$ (the "boundary") as the set of faces neighboring Λ. We identify σ and ρ interchangeably with the associated set of occupied hexagons.

Definition (Hadwiger Model)**.** *Given a domain Λ, and a boundary configuration ρ, at a temperature T, define a probability distribution on the set of configurations Ω with parameters x, p, a by*

$$\mu(\sigma) \coloneqq \mu_{x,p,a,T}(\sigma) \propto e^{\mathcal{H}(\sigma)/T},\ \mathcal{H} = x\chi(\sigma) + pP(\sigma) + aA(\sigma)$$

where χ, P, A are Euler characteristic, perimeter, and area respectively.

While we define the Hadwiger models in terms of the Ising, area, and Euler functions, the phase diagram is naturally parametrized by an equivalent formulation in terms of vertex energies. Consider the energy as a sum over vertices, and evaluate each term in the Hamiltonian on each of the four possible vertex states. The associated energy assignments are represented in figure 2. Note that only number and not position of adjacent hexagons matters, because each model is rotation invariant, and that every model assigns the empty state energy 0. Refer to the vertex state with 0,1,2, and 3 incident filled hexagons as E, C, H, F respectively. We can notate the energy assigned by the Hamiltonian to vertex state X with e_X.

Proposition. *Area, perimeter, and Euler characteristic span the full space of possible energy assignments to vertex states with $e_E = 0$, so the set of Hadwiger functions is*

$$\{g(\sigma) = \sum_{v \in V} f(\sigma_v) | f(\sigma_v) = e_C \cdot \delta_{C,\sigma_v} + e_H \cdot \delta_{H,\sigma_v} + e_F \cdot \partial_{F,\sigma_v}\}$$

where σ_v is the vertex state induced by σ on v, and δ is the Kronecker delta.

Coordinates of e_C, e_H, e_F are the most natural parametrization to consider when evaluating low temperature behavior, as they determine the ground configuration. In particular, the ground configurations are those which consist only of the lowest-energy vertex states for that model. The full space of Hamiltonians is 3-dimensional, but two Hamiltonians related by a scaling factor are considered to represent the same model at different temperatures. Therefore, the space of all "models" is a sphere. If we unwrap this sphere to the plane, we have a 2 dimensional diagram of the full set of models, represented in Figure 4.

Each region in the diagram is characterized by the vertex state with the lowest energy, with the transition lines representing points where two vertex states have equal and minimal energy. Continuing along transition

			
Area	1	2	3
Perimeter	1	1	0
Euler	1	-1	0

Figure 2: Energy assigned by each term to the three non-empty vertex states

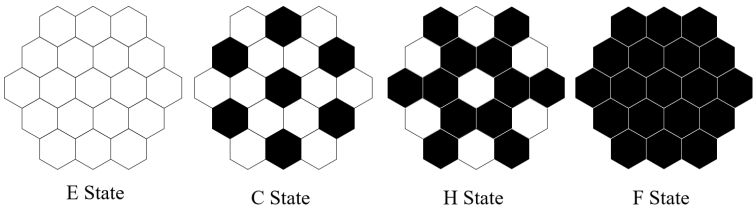


Figure 3: Representative subsets of each of the configurations with exactly 1 vertex state

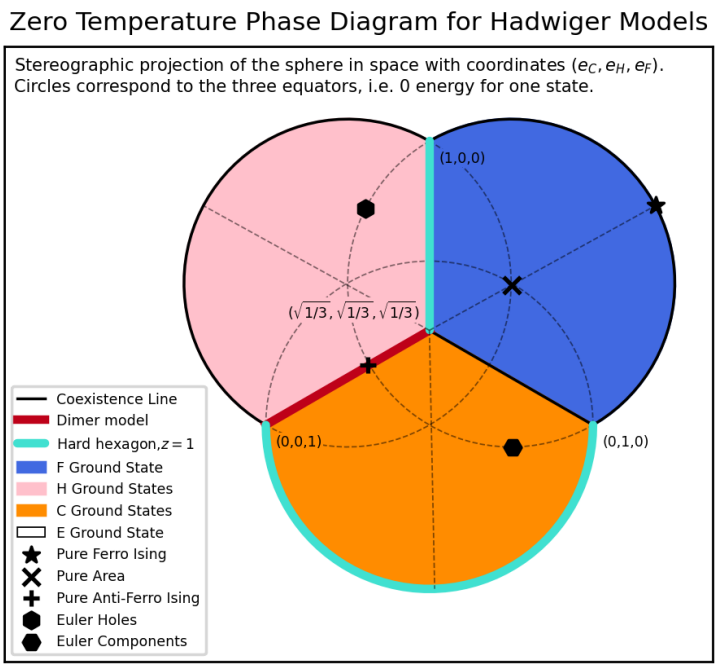


Figure 4: At $T = 0$, the behavior is determined only by states with minimal energy. Along three of the transition lines, where multiple vertex states have minimal energy, there is nonzero entropy, and so these represent "models" in a fuller sense than other points, where only finitely many configurations are permitted. When both C and H vertex states are minimal, the allowed configurations are those of the 0 temperature antiferromagnetic Ising model, which are the same as those of the dimer model [BH82]. When both E and C vertex states are minimal, the allowed configurations are those of the hard hexagon model, with the even distribution on allowed configurations [Bax89]. The F and H line is symmetric to the E and C under inversion. "Landmark points" are marked, including the pure Euler, Perimeter, and Area terms.

lines maintains the equality of the energies even when they are no longer minimal, so their intersections represent particularly symmetrical points. The H region has three ground configurations, each having every vertex incident to two hexagons, and possessing maximally many holes. Each ground configuration corresponds to one of the 3 sublattices in the hexagonal lattice. The C region is the inverse of the H region, having three ground configurations with maximally many components. The F and E regions have one ground configuration each, such that the value on every hexagon is 1 and 0 respectively. You can see the structure of each in Figure 3

2 Main Result

First, we will recall some terms, so we can use them freely in the following sections. A Gibbs state of a model with Hamiltonian $\mathcal{H}$ is a mapping from bounded measurable functions f on the set of configurations Ω to expectations $\langle f \rangle$ of those functions, such that for any finite region, the expectations conditional on a particular boundary configuration ρ are those generated in the above way by the Hamiltonian, given ρ. Note that this implicitly defines probability distributions on configurations for any finite regions via indicator functions. Such a Gibbs state is "extremal" if it cannot be represented as a linear combination of other Gibbs states with positive coefficients. A configuration or distribution is "translation-invariant" if it is invariant under all translations that preserve the lattice, while it is "periodic" if it's invariant spanning sublattice. These terms are technically distinct, but any periodic model can be rendered translation-invariant by considering larger configuration spaces on tiles formed from the fundamental domain of the period lattice [RL 85]. The Peierls condition states that there exists a finite collection of translation-invariant minimal-energy configurations and a constant c such that, for any other configuration ω that differs from a ground configuration η only on a finite set Γ, $\mathcal{H}(\omega) - \mathcal{H}(\eta) \geq c|\Gamma|$. A transition line refers to a curve such that multiple configurations, not related by a symmetry, have minimal energy. Conversely, a coexistence line refers to a curve such that there is more than one Gibbs state, which are not related by a symmetry.

With the space of models laid out, we now consider the low temperature behavior of the Hadwiger models. There is a chain of statements we must string together to restrict our Gibbs states. Dobrushin tells us that in 2-dimensions and at low temperatures, all Gibbs states of a translation-invariant (resp. periodic) Peierls

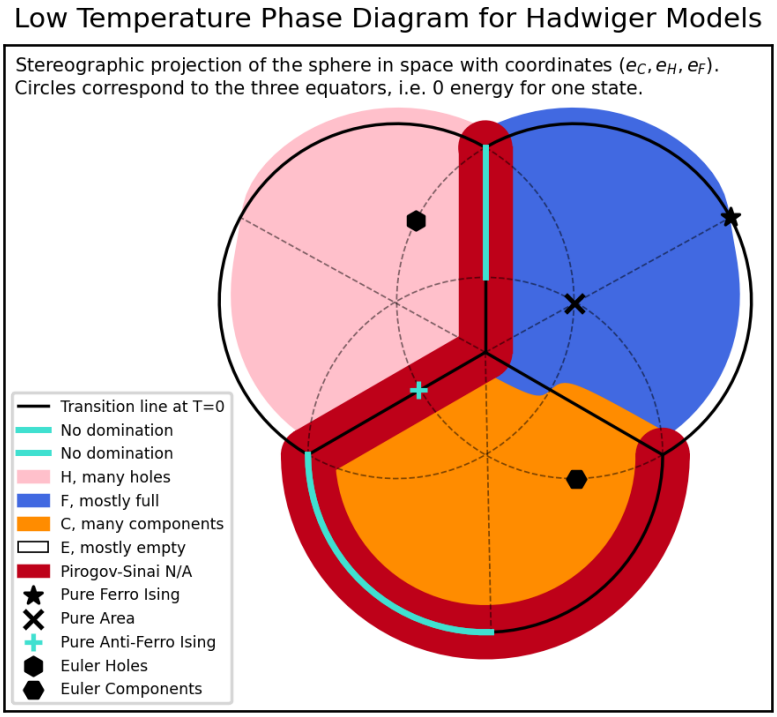


Figure 5: Phase diagram showing the regions where given configurations dominate. Each color, including white, represents a region where a particular set of configurations dominates. On the boundaries between the regions, multiple Gibbs states not related by a symmetry dominate. Note that the dark red region surrounding the non-Peierls transition lines indicates points where Pirogov–Sinai techniques are not applicable. This region shrinks with decreasing temperature, but is never empty. On some of the non-Peierls lines, we use disagreement percolation to prove no domination at any temperature.

model are translation-invariant (periodic)[RL 85]. Zahradnik, extending Pirogov–Sinai, tells us that, given the Peierls condition, all translation invariant (or equivalently, periodic) Gibbs states are linear combinations of extremal "pure phases", each of which is dominated by a particular ground configuration of the model [Zah84]. Slawny tells us the position of the transition curves between areas such that different ground configurations have an extremal Gibbs state corresponding to them for a particular model . [Sla87]

Definition. *Given a ground configuration G, notate by $B(\omega)$ the set of vertices not agreeing with G We refer to a Gibbs state as* ***"dominated"*** *by a certain global configuration G, or equivalently as a "pure phase" if the portion of vertices in $B(\omega)$ goes to 0 as $T \to 0$, and every connected component of $B(\omega)$ is finite. [PS75]. These pure phases are exactly those induced by choosing boundary conditions agreeing with that configuration, although they may be induced by other boundary conditions. To respect the symmetry of the model, if a set of n configurations is symmetrically related under a given model, we instead mean that the model has a $1/n$ chance of being dominated by any of the n pure Gibbs states given general boundary conditions.*

In the interior of any region such that a single configuration (or symmetric set of configurations) has minimal energy density, there will always be a temperature low enough that the model is dominated by the lowest energy configurations. However, at any fixed low temperature, the lowest energy configuration may not dominate in the entire region. The asymptotic transition curves between areas where different phases dominate, which also determine the behavior at any nonzero low temperature of models with more than one distinct asymmetric ground configuration, can be calculated based on the technique of Slawny [Sla87]. Using these techniques, we generate a phase diagram representing the low-temperature behavior of the Hadwiger models, depicted in Figure 5

3 Proving the Phase Diagram

Before we make any specific arguments, first we should note the symmetry of the model. Inverting the state of all hexagons interchanges E vertices for F vertices, C vertices for H vertices, and vice versa. Thus,

inverting the hexagons while swapping the energies assigned to the respective vertex states maintains the assignment of probabilities. With this in mind, any statement that concerns particular vertex states applies just as well to its "twin" under inversion symmetry. We will frequently make an argument for one state, then apply the same to its twin by inversion symmetry.

Using techniques from Pirogov–Sinai, we address the large regions dominated by a particular vertex state first, in subsection 3.1. Next, with Slawny's technique we handle the position of the coexistence lines between these regions in subsection 3.2. Finally, with disagreement percolation and reflection positivity we determine the behavior along the transition lines for which Pirogov–Sinai techniques are inapplicable, in subsection 3.3.

3.1 Applying Pirogov–Sinai

We begin by describing the ground states of the model, as depicted in Figure 4. Given two vertex states A, B, we will refer to the points where $e_A = e_B$ and both achieve the minimal vertex energy as the $A - B$ transition lines.

Lemma. *Along the $H-F$, $E-C$, or $H-C$ transition lines, there are infinitely many ground configurations. At all other points, there are finitely many ground configurations, all of which are periodic.*

Proof. First, we consider the case where exactly one vertex state has minimal energy. If a configuration with only that vertex state is possible, it will be minimal. For the F and E vertex states, minimal energy is achieved by the full and empty configurations respectively; any other configuration must have other vertex states, and so the ground configuration is unique and translation invariant. For the H and C vertex states, we start with a single vertex. This could have three distinct assignments, as the C state is realizable by three different three-hexagon configurations. However, given a trio of three hexagons and state C, there is only one configuration on its neighbors such that every vertex is in state C. By induction, this uniquely determines a periodic ground configuration given one of our original choices, so there are three periodic ground configurations. Along the $E - H$, $E - F$, and $C - F$ transition lines, the minimal vertex states cannot coexist, i.e. neighboring vertices cannot have distinct minimal vertex states, there must be a third state between them. Therefore, all minimal configurations must have only one vertex state, so the set of minimal configurations is the union of the set on the two neighboring regions, and as such is finite and periodic.

Along the $H - F$, $E - C$, and $H - C$ boundary lines, minimal vertex states can coexist. Thus there are infinitely many energy-preserving modifications to particular ground configurations (for example, given the empty configuration, along the $E - C$ line, one can fill in any single hexagon), so the set of ground configurations is infinite. These ground configurations are also not in general periodic: the lower half-plane may have one vertex state, while the upper may have the other.

□

For Pirogov–Sinai arguments to apply, we need the model to satisfy the Peierls condition.

Lemma. *All Hadwiger models satisfy the Peierls condition, except on the $H-F$, $E-C$, or $H-C$ transition lines.*

Proof. An m-potential is a function on $\omega \in \{0,1\}^S$ that can be written as $\sum_{U \subset S} f_U(\omega)$, where each f_U, called a "potential", depends only on elements in U, and there exists a configuration ω such that f_U achieves its minimal value for all U. As described above, any Hadwiger Hamiltonian can be represented as the sum over the vertices of terms that depend only on the neighboring hexagons, and may have every vertex in a minimum-energy state, so the Hadwiger models are m-potentials. If an m-potential's set of minimal configurations is finite, it satisfies the Peierls condition [FV17]. As just demonstrated, outside the $H - F$, $E - C$, or $H - C$ transition lines the set of minimal configurations is finite, so at those points the Peierls conditions is satisfied

Along the $H-F$, $E-C$, and $H-C$ transition lines, there are infinitely many ground configurations; these are achieved by starting from one ground configuration and flipping any of the infinitely many hexagons that do not increase the energy. Thus, they do not satisfy the Peierls condition and have nonzero entropy at 0

temperature. In particular, the ground states of $H-F$ and $E-C$ are the allowed states of the hard hexagon model, while the ground states of the $H-C$ model are ground states of the frustrated antiferromagnetic Ising model, or equivalently, random lozenge tilings [Gor21]. □

The low temperature behavior in the large two-dimensional regions of the phase diagram is a straightforward consequence of Pirogov–Sinai Theory.

Theorem 1. *At any point such that the unique lowest energy vertex configuration is H or C, there exists T_0 such that for $T < T_0$ there are exactly 3 distinct extremal Gibbs states, dominated respectively by the three ground configurations, and every Gibbs state is a linear combination of these.*

Proof. We can introduce two periodic but not rotationally invariant "dummy fields" f_i that distinguish between the three forms of the C vertex state, $\{C_i\}$, by $f_i(C_j) = \delta(i,j)$. Note that we only need two fields, as up to a constant the third can be represented implicitly by negative energy on the other two states. When these fields have nonzero coefficient, they lift the degeneracy of the three configurations, so by Pirogov–Sinai [PS75], $\exists T_0$ such that $\forall T < T_0$, for each form there is a choice of parameters where this forms dominates, and that at their coexistence point there is an equal chance of any dominating. By the symmetry of three ground configurations, we know this coexistence point must be at the point where both dummy fields are 0, i.e. the original model.

Zahradnik [Zah84] states for any periodic Peierls model with finite ground configurations $\exists T_0$ such that $\forall T < T_0$, every periodic Gibbs state is a linear combination of extremal Gibbs states dominated by a periodic ground configuration. Dobrushin [RL 85] states that for 2-dimensional models, under the same conditions, every Gibbs state is periodic. Therefore, for any 2-dimensional periodic Peierls model with finite ground configurations $\exists T_0$ such that $\forall T < T_0$, every Gibbs state is a linear combination of those dominated by the ground states. The model always satisfies the other conditions, so at every Peierls point at sufficiently low temperatures, the space of Gibbs state is exactly those dominated by the ground states.

□

3.2 Coexistence Lines

At fixed low temperature, there are three coexistence lines to consider, the latter two of which are equivalent under inversion symmetry: $E-F$, $E-H$, and $C-F$. Zahradnik tells us that coexistence lines approach the zero-temperature transition lines, but does not tell us from which direction they approach it, or whether they achieve it at some nonzero temperature. There is a technique by Slawny [Sla87] to calculate the asymptotic low-temperature positions of these coexistence lines explicitly, by calculating at each point which of the ground configurations has the lowest-energy or highest-multiplicity minimal excitation. We use this technique in a slightly unconventional way. Normally — in two dimensions — Slawny's technique is used to determine the location of a low-temperature triple point, where three coexistence lines meet, relative to a Peierls zero-temperature triple point. No Peierls triple point exists here, but we can still determine the relative location of the coexistence line by calculating 1-dimensional "slices" perpendicular to each point along the zero-temperature transition line. In each slice, there is a nonzero T such that the transient terms are smaller than the asymptotic term in the expansion we describe below. From Slawny we know the terms vary continuously, so on a closed region there will be a nonzero minimum, such that the overall curve (which by implicit function theorem must be continuous) approximates the asymptotic curve.

Theorem 2. *There exists T_0 such that $\forall 0 < T < T_0$, along the $E-H$ transition line, if $e_F > e_C$, the E state dominates and if $e_F < e_C$ the H state dominates.*

Proof. Along this line, the E and H vertex states have energy 0, and moving along the line changes the energy of F and C. The space of models lies on a sphere, but we're interested only in the asymptotic shape and position relative to the transition lines, so we can change coordinates at will as long as it does not affect relative position. Project to a plane to simplify coordinates, so along the transition line $e_F = f \in [\epsilon, 1-\epsilon]$ and $e_C = 1-f$. Moving perpendicular to this line, $e_H = h$. Following Slawny, we can treat the three symmetric ground configurations as a single ground configuration for the purposes of the calculation.

Slawny's technique consists of solving the equation

$$\dot{P}^G(\beta\mathcal{H}_0 + \mathcal{H}) - e_G(\mathcal{H}') = \dot{P}^{G'}(\beta\mathcal{H}_0 + \mathcal{H}') - e_{G'}(\mathcal{H}')$$

where $\dot{P}^G$ is the pressure of the gas of excitations from the ground configuration G with respect to a given Hamiltonian, $\mathcal{H}_0$ is the Hamiltonian you're expanding around, $\mathcal{H}'$ is the perturbation to the Hamiltonian, and e_G is the energy density of G with respect to a given Hamiltonian, equivalent to the energy the Hamiltonian assigns to the associated vertex state. By "pressure of the gas of excitations", we mean $\log(\sum_{m\in M} e^{-\mathcal{H}(M)})$, where M is the set of configurations that disagree with your ground configuration at only finitely many points, and have energy below your chosen threshold. Such a solution identifies the points where the two modified pressure of the two ground states are equal, and so both have corresponding Gibbs states: the coexistence line. This equation is difficult to solve explicitly, so we instead solve up to the leading term in the expansion in $e^{-\varepsilon_i}$, where $\varepsilon_i \in \mathcal{E}$ and $\mathcal{E}$ is the set of possible excitation energies. Notate equation up to leading term by $\approx$. We find the coefficients using the cluster expansion. Because we only consider the leading order term, which consist of single polymers, the coefficient in $\dot{P}^G$ simplifies to the density of minimal excitations. In the following calculations, we will use β for $1/T$, to simplify the notation.

The E configuration only has one kind of minimal excitation, flipping a single hexagon. This changes 6 vertices in state E to 6 vertices in state C, changing the energy by $1-f$. The perturbation of the Hamiltonian assigns E 0 energy, so that term disappears.

The H configuration has one of two possible minimal excitations, depending on the choice of parameters. Flipping one of the empty hexagons which form 1/3 of the configuration replaces 6 H vertices with F vertices, changing the energy by $6(f-h)$. Flipping a full hexagon replaces 6 H vertices with C vertices, changing the energy by $6(1-f-h)$. Combining this information, we solve

$$e^{6(1-f)\beta} \approx \frac{1}{3}e^{6f\beta-6h} + \frac{2}{3}e^{6(1-f)\beta-6h} + h$$

We set $h = 0$ in the exponent, because when $h << A$, $e^{A+h} - e^A$ is lower order than e^A. Thus, we have

$$h \approx \frac{1}{3}e^{6(1-f)\beta} - \frac{1}{3}e^{6f\beta}$$

Therefore $h > 0$ when $f < 1/2$ and $h > 0$ when $f > 1/2$.

At $f = 1/2$, there is a symmetry between the E and H excitation expansions. For any minimal excitation, flipping the value of hexagons in two sublattices replaces vertices in state E with H and vice versa, while vertices in state F are replaced with C, again vice versa. Therefore if $e_H = e_E$ and $e_f = e_C$, the multiplicities and energies of excitations around E and H will be the same, the pressures will be the same, and so by symmetry the line of the phase transition must pass through $f = 1/2$ at $h = 0$.

The behavior along $F - C$ curve is the same, by inversion symmetry.

□

Theorem 3. *There exists T_0 such that $\forall T < T_0$, along the $E - F$ transition line, if $e_H > e_C$, the E state dominates and if $e_H < e_C$ the F state dominates.*

Proof. Again we project to a plane, F and E both have energy 0, and $e_h = h \in [-1+\epsilon, 1-\epsilon]$, and $e_C = 1-h$ in $[-1+\epsilon, 1-\epsilon]$. Moving perpendicular to this line, $e_F = f$. In the E configuration, flipping an empty hexagon replaces 6 E vertices with C vertices, changing the energy by $1-h$, while in the F configuration, flipping a full hexagon replaces 6 F vertices with H vertices, changing the energy by $h-f$. Thus, we solve

$$e^{(1-h)\beta} \approx e^{h\beta-f} + f$$

Again setting the varying element to 0 in the exponent, we get

$$f \approx e^{(1-h)\beta} - e^{h\beta}$$

so $f > 0$ when $h < 1/2$ and $f > 0$ when $h > 1/2$. By the symmetry of the two ground states, we know this curve must be odd around the equality point, and thus passes exactly through 0 there. □

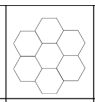
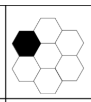
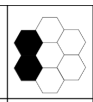
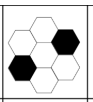
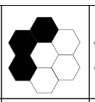
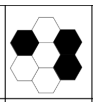
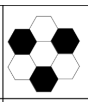
Boundary Conditions													
 Energy	6E	2C+4E	H+2C+3E	4C+2E	4C+2E	2H+2C+2E	4C+E+H	6C	E+2C+3H	4C+2H	4C+2H	4H+2C	6H
 Energy	6C	4C+2H	F+2H+3C	4H+2C	4H+2C	2F+2H+2C	F+4H+C	6H	C+2H+3E	4H+2E	4H+2E	4F+2H	6F

Figure 6: Quantities of the vertex states given either face assignment for each boundary condition, up to rotation and reflection. With this, the relative probability of the face assignments can be determined for any boundary condition and energy assignment, to determine the variational distance between induced probability distributions for disagreement percolation.

3.3 Non-Peierls Lines

Now we handle two of the three non-Peierls infinitely degenerate transition lines. We can't use Pirogov–Sinai here, so we will instead use disagreement percolation to prove uniqueness, then apply reflection positivity to understand the behavior. Our argument relies on the following uniqueness condition [BM94]:

Theorem (Disagreement Percolation)**.** *Fix a particular Markov field, and notate N_i as the set of neighbors of a single site, $d(\cdot,\cdot)$ as the variational distance, and $Y_i(\cdot,\eta)$ as the one-site distribution given its neighbors. If $p_i < p_c$, where $p_i = \max_{\eta,\eta' \in \{0,1\}^{N_i}} d(Y_i(\cdot,\eta), Y_i(\cdot,\eta')$ and p_c is the critical site percolation threshold for the given lattice, then there is exactly one Gibbs measure induced by the Markov field.*

Theorem 4. *Along the $E-C$ transition line, if $e_F \geq e_H$, there is a unique Gibbs state at all temperatures. By inversion symmetry, the same is true along the $F-H$ line when $e_C \geq e_E$.*

Proof. When $e_F \geq e_H$, filling the central hexagon never increases the energy, so it always has probability greater or equal to $1/2$ regardless of the boundary conditions. The probability of a filled hexagon is always less than 1, so the variational distance between any two boundary conditions is less than $1/2$, the critical site percolation threshold for the triangular lattice [Kes82], and so we have uniqueness for all T. When $e_F < e_H$, then the maximum variational distance approaches 1: for entirely filled boundary conditions an empty central hexagon is preferred, but for boundary conditions with a single filled face, a filled central hexagon is preferred.

□

Theorem 5. *Along the $E-C$ transition line, if $e_F \geq e_H$ then no configuration dominates at any temperature.*

Proof. If a Gibbs state is unique, it must also be translation-invariant: any translation of a Gibbs state is still a Gibbs state, so if there's exactly 1, it must be equal to all it's translations. Therefore, any dominating configuration must be invariant for any translation preserving the lattice. However, the only translation-invariant configurations are E or F. The F configuration has maximal energy, and so is minimally likely among all configurations. If we assume the E configuration dominates, then the probability of any face being in state E must go to unity as $T \to 0$. However, any face whose neighbors are all in state E can have its value flipped without increasing the energy, so that configuration must be equally likely as the E configuration, conditional on empty neighbors. Thus, the probability of a filled hexagon cannot go to 0: it must be no less than $1/2$ the probability that all its neighbors are empty. If the probability that all its neighbors are empty goes to 0, the probability of a filled neighbor cannot go to 0, and we have a contradiction by translation invariance.

□

Although we lack domination, at low temperatures the set of allowed configurations are still constrained with high probability. However, instead of approximating a particular configuration, they are constrained to avoid particular vertex states. Because the Gibbs measure is unique and the space of measures is sequentially compact, the unique infinite Gibbs measure is achieved by any sequence of local measures with infinite

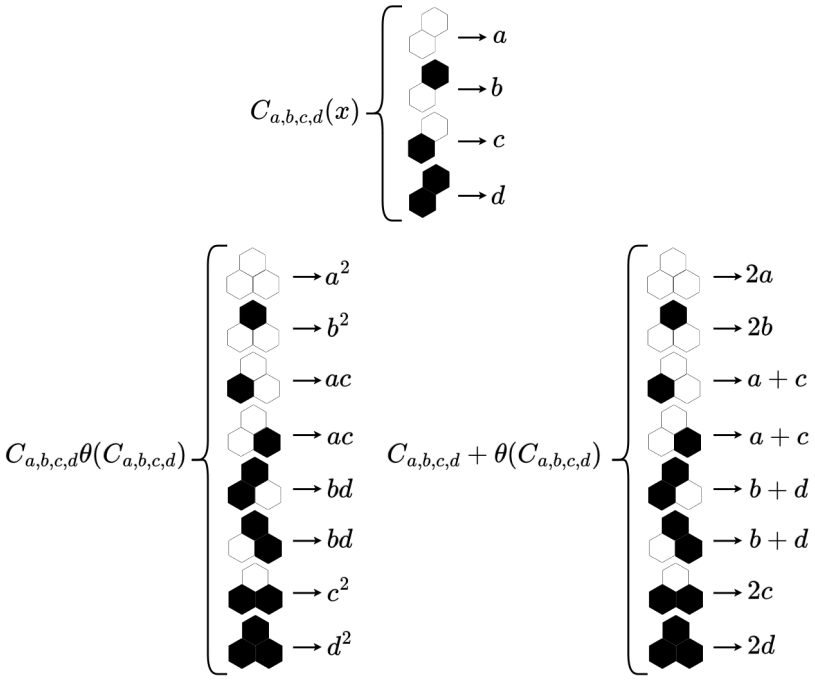


Figure 7: Local energy functions considering two boundary hexagons, and the induced functions on three boundary hexagons, two in each half space, with the top hexagon in both closed half-spaces. With these we construct indicator functions for each vertex state

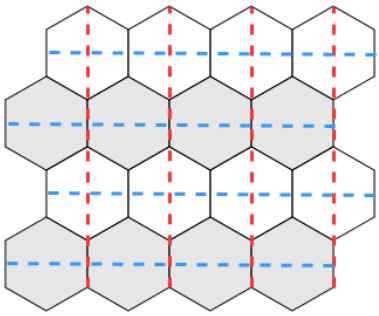


Figure 8: Horizontal reflections (blue) and vertical reflections (red) define blocks containing one full vertex

limiting radius. Thus, we can restrict our attention to measures defined on the rectangular torus. A reflection is defined by a partition of the torus into two halves, by a pair of horizontal or vertical lines, such that reflecting across the lines preserves the lattice. We will use the "closed half-space" to refer to the set of faces intersecting of one of these two halves. The particular reflections we're considering here are represented in Figure 8.

Theorem 6. *All Hadwiger models on the torus satisfy reflection positivity, i.e if θ is a horizontal reflection through faces or a vertical reflection through faces and vertices, and $f, g \in A^+$, the set of functions on the closed half-space of the reflection, $\langle f\theta(g)\rangle = \langle g\theta(f)$ and $\langle f\theta(f)\rangle \geq 0$*

Proof. The first equation follows immediately from θ invariance of the Hamiltonian under both kinds of reflections. For horizontal reflections through the faces, the locality of the model immediately implies the second relation holds; see the argument in [HP22]. For vertical reflections through sites and bonds, we use the following statement: if the Hamiltonian can be realized as $A + \theta(A) + \sum_i C_i\theta(C_i)$, with $A, C_i \in A^+$, the induced measure is reflection positive [FV17]. Each vertex internal to one of the half-spaces has an associated potential $A \in A^+$, and a corresponding vertex in the other half with the potential in $\theta(A)$. However, the vertices along the border do not have potentials in A^+, so representing the associated potential in these terms is nontrivial. To achieve this, we parametrize all functions $C_{a,b,c,d} \in A^+$ that depend only on a pair of boundary hexagons by the energy they assign to each of the four possible configurations. Then, we combine them to look at the associated functions on triplets in $A^+\theta(A^+)$ and $A^+ + \theta(A^+)$. These functions are represented visually in Figure 7.

We can construct an indicator function for the E and F vertex states with $C_{1,0,0,0}\theta(C_{1,0,0,0})$ and $C_{0,0,0,1}\theta(C_{0,0,0,1})$ respectively. For the H vertex state we can construct an indicator functions with $C_{1,1,1,0}\theta(C_{1,1,1,0}) + C_{-1/2,0,-1/2,0} + \theta(C_{-1/2,0,-1/2,0})$, and similarly for the F vertex state. Thus any Hadwiger energy function, represented in terms of relative energy values of vertex states, leads to a reflection positive measure on the torus. □

With reflection positivity we are able to use the chessboard estimate. Define a "block" as the fundamental domain of the set of reflections including every horizontal reflection and every other vertical reflection, displayed in Figure 8. This will include one vertex in its center, and two along its edge. For a local event A on a single block B, we can define an event A_i on any other block B_i by applying the necessary reflections to map that block to B. Notate by α the global event that A_i occurs simultaneously on all blocks, indexed by S. The chessboard estimate states [FV17]:

$$\mathbb{P}(A) \leq \mathbb{P}(\alpha)^{1/|S|}.$$

Theorem 7. *Along the $E-C$ transition line, if $e_F \geq e_H$, the probability a vertex is in the H or F states decays exponentially in the inverse temperature*

Proof. We decompose the event into $\sigma = F$ and $\sigma = H$. First, consider $\sigma = F$. In this case, if $\sigma_i = F \forall i \in S$, then every hexagon is filled, so $\sigma_i = F, \forall i$, which corresponds to exactly 1 configuration. The total number of vertices is twice the number of blocks, so the weight of this configuration is $e^{-2e_F|S|/T}$ and the partition function of the whole system is bounded below by 1, so $\mathbb{P}(\sigma = F) < (e^{-2e_F|S|/T})^{1/|S|} = e^{-2e_F/T}$. Now we consider $\sigma = H$. There are no more than $3^{|S|}$ configurations such that $\sigma_i = H$, $\forall i \in S$, because each block can take no more than 3 independent configurations with state S. The non-central vertices may be in a lower energy state, but the energy of the total configuration must be at least $e_H|S|$, so $\mathbb{P}(\sigma = H) < (3^{|S|}e^{-e_F|S|/T})^{1/|S|} = 3e^{-e_F/T}$. Thus $\mathbb{P}(\sigma = H \text{ or } F) < e^{-2e_F/T} + 3e^{-e_F/T}$. The same statement holds along the $H-F$ transition line, when $e_E \geq e_C$. □

With this, we can make the connection to the hard hexagon model explicit at sufficiently low temperatures. For a fixed infinite configuration η, notate the probability of an event A for the subcritical (i.e. no domination) Hadwiger model at temperature T on the finite region R with boundary conditions induced by η as $\mathbb{P}^R_{S,\eta,T}(A)$ and the distribution of the Hard Hexagon model with $z = 1$ (i.e. even distribution) on the same region with the same boundary conditions by $\mathbb{P}^R_{H,\eta}$. Notate their respective infinite volume limits by $\mathbb{P}_{S,\eta,T}(A)$ and $\mathbb{P}_{H,\eta}$.

Theorem 8. *For a fixed infinite boundary condition η and local event A depending on the set λ, $\lim_{T\to 0} \mathbb{P}_{S,\eta,T}(A) = \mathbb{P}_{H,\eta}(A)$*

Proof. Without loss of generality, assume that λ is an $N \times N$ rectangle. Corollary 1 of [BM94] states:

$$|\mathbb{P}^M_{S,\eta_1,T}(A) - \mathbb{P}^M_{S,\eta_2,T}(A)| \leq \mathbb{P}_{p_i}(\text{there is an open path from some vertex in } N \text{ to } \partial M)$$

where $\mathbb{P}_{p_i}$ is Bernoulli site percolation with probability p_i, and p_i is the variational distance between neighbor boundary defined in the theorem on Disagreement Percolation. $p_i < 1/2$ at any non-zero temperature, so the probability of an open path is always less than at the critical probability, $1/2$. For fixed N, we can choose M_0 such that [Kes82]

$$\mathbb{P}_{1/2}(\text{there is an open path from some vertex in } N \text{ to } \partial M) < \epsilon/2$$

Notate by A^M_{EC} the event that no vertex in M is in the state F or H. Given M_0, we can choose T such that $\mathbb{P}^M_{S,\eta,T_0}(A^M_{EC}) > 1 - \epsilon/2$, by the previous theorem. $\mathbb{P}^M_{H,\eta}(A) = \mathbb{P}^M_{S,\eta,T}(A|A^M_{EC})$, for any T, by the definition of the hard hexagon model, so

$$|\mathbb{P}^M_{S,\eta,T}(A) - \mathbb{P}^M_{H,\eta}(A)| = \left|\mathbb{P}^M_{S,\eta,T}(A|A^M_{EC}) - \mathbb{P}^M_{S,\eta,T}(A|\neg A^M_{EC})\right| \cdot \mathbb{P}^M_{S,\eta,T}(\neg A^M_{EC}) \leq \epsilon/2$$

Now, consider a larger M' containing M.

$$\min_{\eta'} \mathbb{P}^M_{S,\eta',T}(A) \leq \mathbb{P}^{M'}_{S,\eta,T}(A) \leq \max_{\eta'} \mathbb{P}^M_{S,\eta',T}(A), \text{ and } \min_{\eta'} \mathbb{P}^M_{H,\eta'}(A) \leq \mathbb{P}^{M'}_{H,\eta}(A) \leq \max_{\eta'} \mathbb{P}^M_{H,\eta'}(A)$$

because $\eta_{M'}$ affects A only by inducing a probability distribution on η_M, by the Markov property. Thus

$$
\begin{aligned}
|\mathbb{P}^{M'}_{S,\eta,T}(A) - \mathbb{P}^{M'}_{H,\eta}(A)| &\leq \max_{\eta_1',\eta_2'} |\mathbb{P}^{M}_{S,\eta_1',T}(A) - \mathbb{P}^{M}_{H,\eta_2'}| \\
&= \max_{\eta_1',\eta_2'} |\mathbb{P}^{M}_{S,\eta_1',T}(A) - \mathbb{P}^{M}_{S,\eta_2',T}(A) + \mathbb{P}^{M}_{S,\eta_2',T}(A) - \mathbb{P}^{M}_{H,\eta_2'}| \\
&\leq \max_{\eta_1',\eta_2'} |\mathbb{P}^{M}_{S,\eta_1',T}(A) - \mathbb{P}^{M}_{S,\eta_2',T}(A)| + |\mathbb{P}^{M}_{S,\eta_2',T}(A) - \mathbb{P}^{M}_{H,\eta_2'}| \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}
$$

We can always choose T such that $|\mathbb{P}_{S,\eta,T}(A) - \mathbb{P}_{H,\eta}(A)| \leq \epsilon$, because the inequality holds for every sufficiently large M', and so $\lim_{T\to 0} \mathbb{P}_{S,\eta,T}(A) = \mathbb{P}_{H,\eta}(A)$ □

Corollary. *For all A, $\mathbb{P}_{H,\eta}(A)$ is unique regardless of η, so the even-distribution hard hexagon model has a unique infinite volume limit.*

Proof. $\lim_{T\to 0} \mathbb{P}_{S,\eta,T}(A) = \mathbb{P}_{H,\eta}(A)$, and we already established that $\mathbb{P}_{S,\eta,T}(A)$ is independent of η, so $\mathbb{P}_{H,\eta}(A)$ is independent of η. Note that for $z < 1$, the result follows immediately from disagreement percolation. □

4 Special Cases

To our knowledge, the only finite-energy model in the Hadwiger class other than the Ising model in a field that has been exactly solved is the triplet model, sometimes called the Ising model with three-spin interactions [Bax74]. This has Hamiltonian $-\sum_v h_{v1}h_{v2}h_{v3}$, where h_{v1}, h_{v2}, h_{v3} are the spins in $\{-1, 1\}$ of the hexagons adjacent to v, which results in a model at the midpoint of the $C - F$ or $E - H$ transition lines, depending on sign convention. This makes it a natural analogue of the Ising model in the configuration space, which is the midpoint of the only other Peierls transition line. The triplet model exhibits a sharp phase transition, which, given the sharp transitions in the Ising model as well, is reason to believe that in fact the entire space of Hadwiger functions experiences such a transition, except along the non-Peierls transition lines.

At the pure Euler point, the Hamiltonian has a number of very nice properties, particularly on the hexagonal lattice. Euler Characteristic is the difference of components and holes, so with fixed boundary conditions there is an isomorphism between the pure Euler model and a model defined on loop configurations ω, with energy $O(\omega) - I(\omega)$, with $O(\omega)$ and $I(\omega)$ denoting the number of loops contained within an even and odd number of loops respectively (generally loops will not be deeply nested, so this corresponds to "outer" and "inner" loops). Adding the perimeter term results in a local analogue of the loop $O(n)$ model. At low temperatures, where nested loops are very rare, these models display similar behavior, and the loop $O(n)$ results can be used to make an alternate proof of three distinct Gibbs states[H D17]. However, due its nonlocality the loop $O(n)$ model is much less amenable to traditional techniques, and so does not provide the completeness results of Pirogov–Sinai-Zahradnik.

5 Further Directions

As any finite-range model displays only a single undominated Gibbs state at sufficiently high temperatures [FV17], models in the C and H regions must experience a change in the number of Gibbs state at some intermediate temperature. However, precisely where this change occurs, and if there are temperatures with a different number of Gibbs states, is yet unknown. For any set of parameters not on the non-Peierls transition lines, a transition occurs in the sense that some configurations dominate at low temperature, while at high temperatures no configuration dominates. However, the nature of this transition is also not known.

This paper relies almost entirely on very generic features of the Hadwiger models. However, the Hadwiger property is a very strong restriction on the space of 2-dimensional models, and should make this class amenable to much stronger results using these subtler properties. With face assignments of $\{1, -1\}$, the area

term is $\sum_i x_i$, the perimeter term is $\sum_{(i,j)\in e} x_i x_j$ and the Euler term is $\sum_{(i,j,k)\in v} x_i + x_j + x_k - 3x_i x_j x_k$, so all Hadwiger models are generalized Ising models, as mentioned in the introduction. However, any model with nonzero Euler term is not ferromagnetic, so results about the general class of ferromagnetic Ising models fail to hold.

The structure of strictly locally geometric models depends significantly on the underlying lattice. As the maximally symmetric 2D infinite graph, the hexagonal graph provides the simplest dynamics, but the model can also be considered on other lattices. On the usual square lattice, Hadwiger models do not span the set of all vertex state assignments, because there are 4 nonenmpty assignments and only 3 Hadwiger basis vectors. Thus in that domain the space of ground states is not nearly as evident, nor is there an obvious natural choice of basis.

In 3 dimensions, the space of Hadwiger functions is 4-dimensional: Euler characteristic, mean width, surface area, and volume [Lot+20]. Here, "mean width" refers to the average projected length of the set, over the space of 1-dimensional subspaces. The natural analogue to the lattice of hexagons is the lattice of truncated octahedra; for both, $d+1$ identical cells meet at each vertex. However, unlike the hexagonal lattice, for which the Euler characteristic is antisymmetric, the Euler characteristic is now symmetric, paired with the symmetric surface area term. Volume remains antisymmetric, along with the new mean-width terms. Up to temperature normalization, this generates two 1-dimensional subspaces of purely symmetric or anti-symmetric energy functions, granting the model many more potential symmetries to rely on.

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References

[Bax74] R. J. Baxter. "Ising Model on a Triangular Lattice with Three-spin Interactions. I. Free Energy and Correlation Length". In: *Australian Journal of Physics* 27 (1974), pp. 369–382.

[Bax89] Rodney Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic Press, 1989. ISBN: 9780486462714.

[BH82] H W J Blote and H J Hilborst. "Roughening transitions and the zero-temperature triangular Ising antiferromagnet". In: *Journal of Physics A: Mathematical and General* 15.11 (1982), p. L631.

[BM94] J. Van Den Berg and C. Maes. "Disagreement Percolation in the Study of Markov Fields". In: *The Annals of Probability* 22.2 (1994), pp. 749–763. (Visited on 12/04/2024).

[FV17] Sacha Friedli and Yvan Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, 2017. ISBN: 978-1-107-18482-4. DOI: 10.1017/9781316882603.

[Gor21] Vadim Gorin. *Lectures on Random Lozenge Tilings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2021.

[Gri73] G. R. Grimmett. "A Theorem about Random Fields". In: *Bulletin of the London Mathematical Society* 5.1 (Mar. 1973), pp. 81–84.

[H D17] R. Peled W. Samotij Y. Spinka H. Duminil-Copin. "Exponential decay of loop lengths in the loop $O(n)$ model with large n". In: *Communications in Mathematical Physics* 349 (2017), pp. 777–817.

[Had56] H. Hadwiger. "Integralsatze im Konvexring". In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 20 (1956), pp. 136–154.

[HP22] Daniel Hadas and Ron Peled. *Columnar order in random packings of 2×2 squares on the square lattice*. 2022. arXiv: 2206.01276 [math-ph]. URL: https://arxiv.org/abs/2206.01276.

[Kes82] H. Kesten. *Percolation Theory for Mathematicians*. Progress in probability and statistics. Birkhäuser, 1982. ISBN: 9783764331078.

[Lot+20] Martin Lotz et al. *Geometric Aspects of Functional Analysis: Israel Seminar (GAFA) 2017-2019 Volume II*. Ed. by Bo'az Klartag and Emanuel Milman. Springer International Publishing, 2020, pp. 139–167.

[Pei36] R. Peierls. "On Ising's Model of Ferromagnetism". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 32 (1936), pp. 477–81.

[PS75] S. A. Pirogov and Ya. G. Sinai. "Phase Diagrams of the Classical Lattice Systems". In: *Teor. Mat. Fiz.* 25 (1975), pp. 358–369. DOI: 10.1007/BF01040127.

[RL 85] S.B. Shlosman R.L. Dobrushin. "The problem of translation invariance of Gibbs states at low temperatures". In: *Soviet Scientific Reviews. Section C: Mathematical Physics Reviews* 5 (1985), pp. 53–195.

[Sla87] J. Slawny. "Low temperature properties of classical lattice systems". In: *Phase transitions and critical phenomena* 11 (1987), pp. 127–205.

[Zah84] M. Zahradnik. "An alternate version of Pirogov-Sinai theory". In: *Communications in Mathematical Physics* 93 (1984), pp. 559–581.