A SHARP DECONFINEMENT TRANSITION FOR POTTS LATTICE GAUGE THEORY IN CODIMENSION TWO

PAUL DUNCAN AND BENJAMIN SCHWEINHART

Abstract. We prove that Wilson loop expectations in $(d-2)$-dimensional q-state Potts lattice gauge theory on $\mathbb{Z}^d$ undergo a sharp phase transition dual to that of the Potts model. This is a consequence of a more general theorem on the asymptotic probability that the boundary of a $(d-1)$-dimensional box is null-homologous in the $(d-1)$-dimensional plaquette random-cluster model. Our proof is unconditional for Ising lattice gauge theory, but relies on a regularity conjecture for the random cluster model in slabs when $q > 2$. As another special case, we generalize a theorem of Aizenman, Chayes, Chayes, Frölich, and Russo for 2-dimensional Bernoulli percolation on $\mathbb{Z}^3$ to $(d-1)$-dimensional Bernoulli plaquette percolation on $\mathbb{Z}^d$.

1. Introduction

Lattice gauge theories are a family of models studied in physics as discretizations of Yang–Mills Theory. They were introduced by Wilson [Wil74], with the special case of Ising lattice gauge theory being defined earlier by Wegner [Weg71]. Lattice gauge theories on $\mathbb{Z}^d$ assigns random spins from a complex matrix group G to the edges of that lattice. When $d = 4$ and G is taken to be one of the compact Lie group $U(1), SU(2)$, or $SU(3)$, these systems model the fundamental forces of the standard model of particle physics, and a detailed understanding of them would resolve some of the most important open questions in mathematical physics [Cha16]. However, the behavior of even the simplest non-trivial lattice gauge theories remains poorly understood from the perspective of rigorous mathematics.

We specialize to the cases where $d = 3$ and $G = \mathbb{Z}(2)$ or $\mathbb{Z}(3)$ is the group of second or third complex roots of unity, and to a separate family of models that they fit into called $(d-1)$-dimensional q-state Potts lattice gauge theory defined by by Kogut et al. [KPSS80]. While these models may not be directly physically relevant, they are well-studied in the physics literature (see e.g. [MP79, CJR79, KNY22]). In addition, there has been recently renewed interest in the mathematical properties of Ising lattice gauge theory and other lattice gauge theories with finite abelian gauge groups [Cao20, Cha20, FLV21, FV23]. We provide the first rigorous description of nontrivial behavior in a lattice gauge theory at middle temperatures, and the first proof of a sharp phase transition for such a system.

The most important random variables in lattice gauge theory are the Wilson loop variables. Roughly speaking, they measure the product of spins on the edges of a loop γ. When $G = SU(3)$, the asymptotics of Wilson loop expectations for rectangular loops γ are thought to be related to the phenomenon of quark confinement (hence the terminology "deconfinement transition"). In particular, they are conjectured to follow an "area law" and decay asymptotically in the area of γ as its dimensions are taken to ∞, for any value of β.

Different asymptotics are conjectured for q-state Potts lattice gauge theory on $\mathbb{Z}^d$ and — more generally — for k-dimensional Potts lattice gauge theories which assign spins to k-dimensional cells of a cell complex. In particular, it is thought that there is a critical threshold $\beta_c = \beta_c(q, k, d)$ so that Wilson loop expectations for the boundary of a $(k+1)$-dimensional box follow an "area law" when $\beta < \beta_c$ and decay exponentially in the volume of the box but exhibit a "perimeter law" when $\beta > \beta_c$ and decay exponentially in the surface area of the box. The special case of $k = 0$ is the sharpness of the phase transition for the classical q-state Potts model proven by Aizenman, Barsky, and Fernández [ABF87] in the special case $q = 2$ and by Duminil-Copin, Raoufi, and Tassion [DCRT19] in general; the "area" of a 1-dimensional box is the distance between its endpoints and its "perimeter" is a constant. Laanait, Messager, and Ruiz also demonstrated that such a transition occurs for sufficiently large q when $k = 1$ and $d = 4$ [LMR89]. Our main result is a proof of this conjecture when when $k = d - 2$ and $q = 2$, and a conditional proof for $k = d - 2$ and $q > 2$ assuming a conjecture of Pisztora [Pis96] on the behavior of the random-cluster model in slabs.

Our proof strategy begins by reducing the conjecture to a question concerning the stochastic topology of a random cell complex. Stochastic topology is a relatively new field, which studies the topological invariants of random cell complexes. Previous work in that area has concentrated on generalizing classical results from random graph theory to higher dimensional cell complexes, and to gaining a statistical understanding of noise and signal in the context of topological data analysis (see [Kah14, BK18, BK22] for an overview). Only a few recent papers have addressed connections with statistical physics and percolation theory [BS20, BS22, DKS20, DS22, Roa18, SW19]. We hope that the current work spurs further interest in the intersection of these fields.

The idea of representing 1-dimensional Potts lattice gauge theory with a 2-dimensional cell complex dates back to soon after its introduction [GGZ80, MO82]. These earlier attempts by physicists were limited by imprecise notions which counted degrees of freedom in terms of "independent surfaces of plaquettes" rather than homology — not accounting for the dependence of one-dimensional homology on the coefficient group. Aizenman and Frölich discovered that there were "topological anomalies" in Potts lattice gauge theory and its Wilson loop variables [AF84]. Specifically, they found that the weight assigned to 1-cochains consistent with a given plaquette configuration P is not always proportional to $q^{\operatorname{rank} H^1(P;\,\mathbb{Z})}$ (whereas, the classical Potts model weights 0-cochains consistent with a graph P' proportionally to $q^{\operatorname{rank} H^0(P';\,\mathbb{Z})}$ for any value of q). In addition, they constructed examples of plaquette systems for which a discrete analogue of Stokes' Theorem for Wilson loop variables fails. After these observations, this project seemed to have become dormant.

The plaquette random-cluster model was introduced as a higher dimensional generalization of the classical random-cluster model by Hiraoka and Shirai [HS16]. They demonstrated that it can be coupled with q-state Potts lattice gauge theory when q is a prime integer. This extends the well-known coupling of the random-cluster model with the Potts model, which, together with other graphical representations of spin models [DC16], have been powerful tools in statistical mechanics. In earlier work [DS22], we proved that — under this coupling — the Wilson loop expectation for a cycle γ equals the probability that γ is null-homologous in the plaquette random-cluster model when coefficients are taken in $\mathbb{Z}_q$ (roughly speaking, that γ is "bounded by a surface of plaquettes"). In other words, we showed that "topological anomalies" noticed in [AF84] can be accounted for by weighting the plaquette random cluster model by $q^{\operatorname{rank} H^i(P';\,\mathbb{Z}_q)}$ rather than $q^{\operatorname{rank} H^i(P;\,\mathbb{Z})}$. The situation is more subtle when q is non-prime, as $H^i(P;\,\mathbb{Z}_q)$ may have torsion. Here, we extend the plaquette random-cluster model to these cases by replacing $q^{\operatorname{rank} H^i(P;\,\mathbb{Z}_q)}$ with $\left|H^i(P';\,\mathbb{Z}_q)\right|$, thereby fully accounting for the "topological anomalies."

Our approach to prove the sharpness of the phase transition for the plaquette random-cluster model in codimension one is inspired by a seminal paper of Aizenman, Chayes, Chayes, Frölich, and Russo [ACC+83]. They showed a corresponding result for independent Bernoulli plaquette percolation on $\mathbb{Z}^3$. Their work was motivated by the analogy with lattice gauge theory, as mentioned in the introduction and in a later section on "The Relation to Interacting (Gauge) Systems." Adapting their argument to the general plaquette random-cluster model is non-trivial, both because of the dependence between disjoint plaquette events and because of the extension to higher dimensions. By adapting this proof and using it to show sharpness of the deconfinement transition for Potts lattice gauge theory, we demonstrate that higher-dimensional cellular representations of spin models possess some of the same power of graphical representations of spin models.

2. Background and Main Results

The k-dimensional Potts (hyper)lattice gauge theory assigns random spins in the abelian group $\mathbb{Z}_q$ to the k-cells of a cell complex X in a way so that reversing the orientation of a cell multiplies the spin by -1. Following the language of algebraic topology, we call such a spin assignment an k-cochain and denote the collection of them by $C^k\left(X;\ \mathbb{Z}_q\right)$. This collection has a natural structure as a $\mathbb{Z}_q$ module. For more detailed topological definitions, see Section A.1 in the appendix.

Definition 1 ([Weg71, KPSS80])**.** *The k-dimensional q-state Potts lattice gauge theory (or PLGT)* [Weg71, KPSS80] *on a finite cubical complex X is the measure on $C^k\left(X;\ \mathbb{Z}_q\right)$ given by*

$$\nu_{X,\beta,q,k}\left(f\right) := \frac{1}{\mathcal{Z}}e^{-\beta H(f)}$$

where β is a parameter called the inverse temperature, $\mathcal{Z} = \mathcal{Z}\left(X,\beta,q,k\right)$ is a normalizing constant, and H is the Hamiltonian

$$H\left(f\right) = -\sum_{\sigma} K\left(\delta f\left(\sigma\right),1\right)\ . \tag{1}$$

Here, δ is the coboundary operator $\delta f\left(\sigma\right) = f\left(\partial\sigma\right)$ and K is the Kronecker delta function. When $q=2$ or $q=3$, these measures coincide with the $\mathbb{Z}(2)$ and $\mathbb{Z}(3)$ Euclidean lattice gauge theories mentioned above, up to a rescaling of the parameter β. The special case $k=0$ is the classical q-state Potts model. When $k>1$, these models are sometimes called "hyperlattice" gauge theories, but we refer to them as lattice gauge theories to keep our language simple. The same methods we use to prove our main theorem for the case $k=1, d=3$ are equally applicable to more general case of $k=d-2, d\geq 3$. In the same vein, we will use terminology best suited for the special case $k=1$. In particular, we will refer to the number of $(k-1)$-plaquettes in the boundary of an k-dimensional box as its "perimeter" and the number of k-dimensional plaquettes in its interior as its "area," even though "surface area" and "volume" might be more appropriate when $k>1$.

We also consider Potts lattice gauge theory on subset of the integer lattice $\mathbb{Z}^d$ with boundary conditions. For convenience, we restrict ourselves to boxes $r\subset\mathbb{Z}^d$. We call the measure defined above Potts lattice gauge theory with free boundary conditions and denote it by $\nu^{\mathbf{f}}_{r,\beta,q,k}$. The other boundary conditions of interest in this paper specify that the cochain agrees with specified cocycle

$\eta \in Z^k(\partial r;\ \mathbb{Z}_q)$ on ∂r (a cocycle is a cochain η so that $\delta\eta = 0$). For example, we could choose η to assign a constant element of $\mathbb{Z}_q$ to each k-face of ∂r. It will turn out that expectations of gauge invariant quantities do not depend on the specific choice of η, and agree with a different type of boundary conditions for Potts lattice gauge theory that we call wired boundary conditions.

Fix $\eta \in Z^k(\partial r;\ \mathbb{Z}_q)$, let $\psi : C^k(r;\ \mathbb{Z}_q) \to C^k(\partial r;\ \mathbb{Z}_q)$ be the map which restricts a cochain on r to one on ∂r, and set $D_\eta(r;\ \mathbb{Z}_q) = \psi^{-1}(\eta)$.

Definition 2. *The k-dimensional q-state Potts lattice gauge theory (or PLGT) on a box $r \subset \mathbb{Z}^d$ with boundary conditions η is the restriction of $\nu_{r,\beta,q,k}$ to $D_\eta(r;\ \mathbb{Z}_q)$. That is, it is the Gibbs measure $\nu^\eta_{r,\beta,q,k}$ on $D_\eta(r;\ \mathbb{Z}_q)$ induced by the Hamiltonian (1). Similarly, the PLGT on r with wired boundary conditions is the restriction of $\nu_{r,\beta,q,k}$ to $\ker\delta \circ \psi$. It is denoted by $\nu^{\mathbf{w}}_{r,\beta,q,k}$.*

Definition 3. *Let γ be an k-cycle in $Z_k(X;\ \mathbb{Z}_q)$. In the PLGT, the Wilson loop variable W_γ is*

$$W_\gamma(f) = (f(\gamma))^{\mathbb{C}},$$

where the $\mathbb{C}$ superscript denotes that we are viewing the variable as a complex number by identifying the $\mathbb{Z}_q$ with the multiplicative group of complex q-th roots of unity.

The asymptotics of Wilson loop variables for 1-dimensional lattice gauge theories have been of great interest in both the mathematical and physical literatures. We mention only a few results which are relevant to the cases $G = \mathbb{Z}(2)$ and $G = \mathbb{Z}(3)$; see [Cha16] for a more thorough account. Classically, series expansions arguments were employed to demonstrate the existence of area law and perimeter law regimes at sufficiently extreme temperatures [OS78, Sei82]. Recently, these methods were used to produce a refined understanding of the low temperature asymptotics of lattice gauge theories with with finite gauge groups [Cha20, Cao20, FLV21]. There has been less specific interest in the PLGT, except for the special case of $q = 2$ [Cha20, FV23]. The series expansion techniques should be easily adaptable to show the existence of area law and perimeter law regimes for the k-dimensional PLGT at sufficiently extreme temperatures. This result can also be proven using a coupling with the plaquette random-cluster model and a comparison with Bernoulli plaquette percolation [DS22] (that reference considers the case of prime q, but the argument extends to the generalized plaquette random-cluster model introduced here). Finally, [LMR89] proved that the 1-dimensional PLGT on $\mathbb{Z}_4$ exhibits a sharp deconfinement transition for sufficiently large q. However, their results are not effective, and this result was not previously known for any specific example of lattice gauge theory.

Our main theorem characterizes the asymptotics of Wilson loop variables for the boundaries of $(d-1)$-dimensional boxes in $(d-2)$-dimensional Potts lattice gauge on $\mathbb{Z}^d$ at all but the critical value of β. Before stating it, we introduce some notation. For now, we let $\nu_{\mathbb{Z}^d,\beta,q,d-1}$ denote an infinite volume PLGT on $\mathbb{Z}^d$. In Section 4.2 we describe how to construct such measures as a weak limit of finite volume measures using both free, wired, and η boundary conditions. Our result holds for any such limiting measure.

Let r be a an i-dimensional box in $\mathbb{Z}^d$. That is, r is a set of the form $[0, N_1] \times \ldots [0, N_i] \times \{0\}^{d-i}$ or one obtained from it by symmetries of the lattice. When G is the additive (abelian) group of a ring with unity, we can identify ∂r with the chain $\sum_{\sigma\in\partial r} \sigma \in C_{i-1}\left(\mathbb{Z}^d;\ G\right)$, where the sum is taken over the (positively oriented) $(i-1)$-plaquettes of γ. This is an abuse of notation as, strictly speaking there is a difference between the set ∂r and the chain ∂r.

To obtain our sharpest result for the perimeter law regime, we require a minor regularity hypothesis on the boxes considered. For a box r, let $m(r)$ be its minimum dimension and let $M(r)$ be its maximum dimension. We say that a family of $(d-1)$-dimensional boxes r_l is suitable if its $(d-1)$ dimensions diverge to ∞ and if $m(r_l) = \omega\left(log\left(M\left(n\right)\right)\right)$. When r_l is suitable, we say that $\gamma_l = \partial r_l$ is a *suitable* family of rectangular boundaries.

Theorem 4. *Fix integers $q, d \geq 2$ and set $\nu = \nu_{\mathbb{Z}^d,\beta,q,d-1}$. There exist constants $0 < c_1(\beta, q), c_2(\beta, q) < \infty$ so that, if $\{\gamma_l\}$ is a suitable family of rectangular $(d-1)$-boundaries then*

$$\begin{aligned} -\frac{\log\left(\mathbb{E}_\nu(W_{\gamma_l})\right)}{\text{Area}(\gamma)} &\to c_1(\beta, q) & \qquad & \beta < \beta^*\left(\beta_{\text{slab}}(q)\right) \\ -\frac{\log\left(\mathbb{E}_\nu(W_{\gamma_l})\right)}{\text{Per}(\gamma)} &\to c_2(\beta, q) & & \beta > \beta^*\left(\beta_c(q)\right), \end{aligned}$$

where $\beta_c(q)$ is the critical inverse temperature for the Potts model on $\mathbb{Z}^d$, $\beta_{\text{slab}}(q) = -\log\left(1 - p_{\text{slab}}\left(q\right)\right)$ is the inverse temperature corresponding to the slab percolation threshold for the random-cluster model, and

$$\beta^*\left(\beta\right) = \log\left(\frac{e^\beta + q - 1}{e^\beta - 1}\right).$$

The constant $c_1(\beta, q)$ may depend on the infinite volume measure ν, but we show that $c_2(\beta, q)$ does not. The only place we use the assumption that γ_l is suitable is the proof of the existence of the sharp constant $c_2(\beta, q)$ in the perimeter law regime. Note that the special case of $d = 2$ is sharpness of the phase transition for the planar Potts model, a result due to Beffara and Duminil-Copin [BDC12].

It is a conjecture of Pisztora that $\beta_c(q) = \beta_{\text{slab}}(q)$ [Pis96] for all q. By the following result, we have an unconditional proof of a sharp phase transition for Ising lattice gauge theory.

Theorem 5 (Bodineau [Bod05])**.**

$$\beta_{\text{slab}}(2) = \beta_c(2) \ .$$

We prove Theorem 4 by adapting an argument of Aizenman, Chayes, Chayes, Frölich, and Russo for 2-dimensional Bernoulli plaquette percolation on $\mathbb{Z}^3$ to the $(d-1)$-dimensional plaquette random-cluster model in $\mathbb{Z}^d$. The i-dimensional Bernoulli plaquette percolation on a cell complex X with parameter p is the random subcomplex P of X that includes all cells of dimensions less than i and adds each i-cell independently with probability p. We recall the result of [ACC+83] for that independent model. In the statement of the following theorem, let V_γ be the event that $\gamma \in B_{i-1}(P;\, \mathbb{Z}_2)$, or in words the event that γ is a boundary of a set of plaquettes in P. As we will see below, there is some subtlety in choosing the correct definition of this event.

Theorem 6 (Aizenman, Chayes, Chayes, Frölich, Russo [ACC+83])**.** *For 2-dimensional Bernoulli plaquette percolation on $\mathbb{Z}^3$ there are constants $0 < c_3(p), c_4(p) < \infty$ so that*

$$\begin{aligned} -\frac{\log\left(\mathbb{P}_p(V_\gamma)\right)}{\text{Area}(\gamma)} &\to c_3(p) & & p < 1 - p_c(\mathbb{Z}^3) \\ -\frac{\log\left(\mathbb{P}_p(V_\gamma)\right)}{\text{Per}(\gamma)} &\to c_4(p) & & p > 1 - p_c(\mathbb{Z}^3)\,, \end{aligned}$$

for rectangular loops γ, as both dimensions of γ are taken to ∞.

As a historical note, this theorem originally relied on a conjecture about the continuity of the critical probability of percolation in slabs, that was later proven by Grimmett and Marstrand [GM90]. Our results are conditional on a more general conjecture when $q \neq 1, 2$.

The plaquette random-cluster model (or PRCM) with coefficients in a field $\mathbb{F}$ was defined in [HS16] to be the random i-dimensional subcomplex of a cell complex X so that

$$\mathbb{P}(P) \propto p^{|P|}(1-p)^{\left|X^{(i)}\right| - |P|}\, q^{\mathbf{b}_{i-1}(X;\, \mathbb{F})} \tag{2}$$

where $\left|X^{(i)}\right|$ denotes the number of i-cells of X, $|P|$ denotes the number of i-cells of P, and $\mathbf{b}_{i-1}(X;\, \mathbb{F})$ is the rank of the $(i-1)$-homology group $H_{i-1}(P;\, \mathbb{F})$. [HS16] and [DS22] show a number of results about these models,

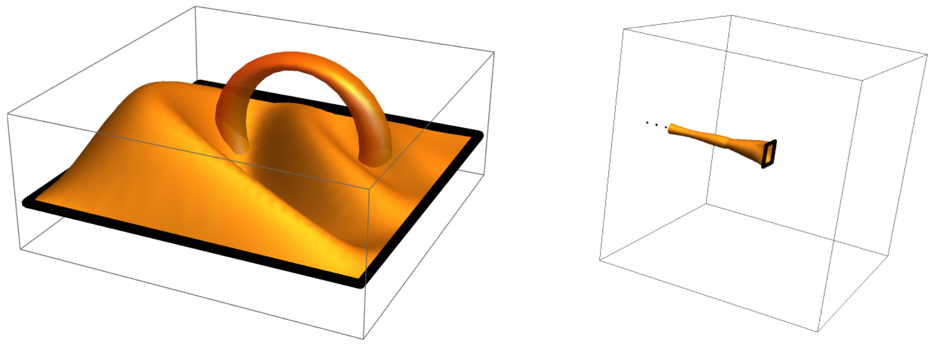


FIGURE 1. Illustrations of the events V_γ^{fin} and V_γ^{inf}.

and in particular that they are coupled with the $(i-1)$-dimensional q-state PLGT when q is a prime integer and $\mathbb{F} = \mathbb{Z}_q$. Here, we extend the definition of the PRCM to produce a model coupled with the q-state PLGT even when q is non-prime.

Definition 7. *Let X be a finite d-dimensional cell complex, $i < d$ and $p \in [0,1]$. The i-dimensional plaquette random cluster model on X with coefficients in a finite abelian group group G is the random i-complex P that includes the $(i-1)$-skeleton of X and is distributed as follows.*

$$\tilde{\mu}_{X,p,G,i}(P) := \frac{1}{Z} p^{|P|} (1-p)^{\left|X^{(i)}\right| - |P|} \left|H^{i-1}(P;\ G)\right| , \tag{3}$$

where $Z = Z(X,p,G,i)$ is a normalizing constant and $H^{i-1}(P;\ G)$ is the reduced cohomology of P with coefficients in G.

When $X = r$ is a box in $\mathbb{Z}^d$, we call this random cell complex the plaquette random cluster model with free boundary conditions and denote it by $\tilde{\mu}^{\mathbf{f}}_{r,p,G,i}$. By convention, we do not include the i-plaquettes in the boundary of r, and instead write $\overline{r}$ for the full induced subcomplex. We also define the PRCM with wired boundary conditions by replacing the term $|H^{i-1}(P;\ G)|$ in (3) with $|H^{i-1}(P \cup \partial r;\ G)|$. This has same effect as adding all the i-plaquettes in the boundary of r. This measure is denoted by $\tilde{\mu}^{\mathbf{w}}_{r,p,G,i}(P)$.

We prove that $\tilde{\mu}_{X,p,\mathbb{Z}_q,i}$ is coupled with the $(i-1)$-dimensional q-state PLGT (Proposition 12) in a way so that a Wilson loop expectation equals the probability that the loop is null-homologous. This generalizes one of the main theorems of [DS22], which covers the case when q is a prime integer. Until we specialize to the $(d-1)$-dimensional random-cluster model, we will follow the convention of [DS22] and reserve i for the dimension of the random-cluster model.

We would like to relate Wilson loop expectation W_γ to the probability that γ is "bounded by a surface of plaquettes." To do so, we need to account for the dependence on q, and also on the boundary conditions. Set $V_\gamma^{\text{fin}}(G)$ be the event that $\gamma \in B_{i-1}(P;\ G)$ and $V_\gamma^{\text{inf}}(G)$ to be the event that union of $V_\gamma^{\text{fin}}(G)$ and the event that γ is "homologous to infinity" in the sense that it is homologous to a cycle in the boundary of the cube $[-n,n]^d$ for arbitrarily large n. See Figure 1. When $G = \mathbb{Z}_q$ we simply write $V_\gamma^{\text{fin}}(q)$ and $V_\gamma^{\text{inf}}(q)$ for these events. To further simplify notation, we use $V_\gamma^{\text{fin}}(1)$ and $V_\gamma^{\text{inf}}(1)$ to denote these events for the choice $G = \mathbb{Z}$.

Theorem 8. *Let* $0 < i < d-1$, *let* γ *be an* $(i-1)$*-cycle in* $\mathbb{Z}^d$, $q \in \mathbb{N}+1$, *and* $(\#_1, \#_2) \in \{(\mathbf{f}, \text{fin}), (\mathbf{w}, \text{inf})\}$. *Then*

$$\mathbb{E}_\nu(W_\gamma) = \tilde{\mu}(V_\gamma) ,$$

where $\nu = \nu^{\#_1}_{\mathbb{Z}^d,\beta,q,i-1}$ *is the PLGT with boundary conditions* $\#_1$, $\tilde{\mu} = \tilde{\mu}^{\#_1}_{\mathbb{Z}^d,1-e^{-\beta},\mathbb{Z}_q,i}$ *is the corresponding random-cluster model, and* $V_\gamma = V^{\#_2}(q)$.

We note that the analogue of this theorem for finite cell complexes (Proposition 14 below) would fail if $V_\gamma(q)$ is replaced with $V_\gamma(1)$. This was one of the "topological anomalies" documented by Aizenman and Frölich; see the examples in Section 4.1 of [AF84]. Their construction in $\mathbb{Z}^3$ can be described as follows: Let $r = [0,N]^2 \times \{0\}$ and let $\gamma = \partial r$. Choose a tube T of width one that intersects r k times, each time in a single plaquette σ_j. Let P include all plaquettes in the exterior of T. Then the chains $[\partial\sigma_j]$ are homologous in $H_1(P;\mathbb{Z})$ and $[\gamma] = k[\sigma_1] \neq 0$ in that group. In particular, $V_\gamma(1)$ occurs but $V_\gamma(q)$ does not. In addition, conditional on P,

$$W_\gamma = W_{\sigma_1}^k .$$

Thus, if $q \mid k$, W_γ is uniformly distributed on a subgroup of $\mathbb{Z}(q)$ and

$$\mathbb{E}(W_\gamma | P) = 1 \neq I_{V_\gamma(1)}$$

but

$$\mathbb{E}(W_\gamma | P) = I_{V_\gamma(q)} .$$

For the special case of the PRCM in codimension one ($i = d-1$), we have that

$$H^{d-i-1}(P;\ G) \cong G^{\text{rank}(H_{d-i-1}(P;\ \mathbb{Q}))}$$

so the law in (3) takes the same form as in the classical random-cluster model (see Proposition 19 below). Let $\mu_{X,p,q,i}$ be the PRCM with coefficients in $\mathbb{Q}$ as defined in (2). Then we can place the models $\tilde{\mu}_{X,p,G,d-1}$ inside the larger family of measures $\mu_{X,p,q,i}$ where q can take on any positive real value. In this case $\tilde{\mu}_{X,p,G,d-1}$ coincides with $\mu_{X,p,|G|,d-1}$.

Let r be a d-dimensional box in $\mathbb{Z}^d$. By a small adaptation of Theorem 18 in [DS22], $\mu_{r,p,q,d-1}$ is dual to the classical (1-dimensional) random-cluster model on a dual box with appropriate boundary conditions. This allows us to adapt the proof of [ACC$^+$83], which crucially uses that the dual of Bernoulli plaquette percolation on $\mathbb{Z}^3$ is Bernoulli bond percolation. A similar duality relation holds for general i, but we do not require it here. We will include a proof in a separate manuscript.

Suppose $\gamma = \partial r$ and let V_γ be the event $V_\gamma^{\text{fin}}(m)$ or $V_\gamma^{\text{inf}}(m)$. Then the probability of the event V_γ bounded below by the probability that all plaquettes contained in r are included, which decays exponentially in the area of γ. On the other hand, V_γ is precluded if one of γ's constituent $(d-2)$-plaquettes is not adjacent to a $(d-1)$-plaquette of P. It is not difficult to show that the probability that there are no such "isolated" $(d-2)$-cells decays exponentially in the perimeter of γ. The following theorem establishes that there is a sharp transition between these two regimes (assuming the continuity of slab percolation thresholds). Here, $\mu_{\mathbb{Z}^d,p,q,d-1}$ will denote any infinite volume random cluster measure obtained as a weak limit of finite volume measures (see Section 5.2).

Theorem 9. *Let $\mu_{\mathbb{Z}^d,p} = \mu_{\mathbb{Z}^d,p,q,d-1}$ where $p \in [0,1]$ and $q \in [1,\infty)$. Also, and $V_\gamma = V_\gamma^{\text{fin}}(m)$ or $V_\gamma^{\text{inf}}(m)$ for some $m \in \mathbb{N}$. Then there exist constants and $0 < c_5(p,q), c_6(p,q) < \infty$ so that for any suitable family of rectangular $(d-1)$-boundaries γ_l*

$$\begin{aligned} -\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(V_{\gamma_l}\right)\right)}{\text{Area}(\gamma)} &\to c_5(p,q) && p < p^*\left(p_{\text{slab}}(q)\right) \\ -\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(V_{\gamma_l}\right)\right)}{\text{Per}(\gamma)} &\to c_6(p,q) && p > p^*\left(p_c(q)\right)\,, \end{aligned}$$

where $p_c(q)$ and $p_{\text{slab}}(q)$ are the critical thresholds for the classical (one-dimensional) random-cluster model on $\mathbb{Z}^d$ and in slabs in $\mathbb{Z}^d$, respectively, and

$$p^* = p^*(p) = \frac{(1-p)\,q}{(1-p)\,q + p}\,.$$

The constant $c_5(p,q)$ may depend on the infinite volume measure $\mu_{\mathbb{Z}^d,p,q,d-1}$ or the choice of the event V_γ but $c_6(p,q)$ does not depend on the infinite volume measure or on the choice of V_γ^{inf} or V_γ^{fin} (though it may depend on the coefficient group). By the preceding discussion, Theorem 4 follows by taking $q = m \in \mathbb{N}+$

1. Moreover, by the Grimmett–Marstrand Theorem [GM90, BGN91], $p_{\mathrm{slab}}(1) = p_c\left(\mathbb{Z}^d\right)$, so we have the the following generalization of Theorem 6.

Theorem 10. *For $(d-1)$-dimensional Bernoulli plaquette percolation on $\mathbb{Z}^d$ there are constants $0 < c_7(p), c_8(p) < \infty$ so that*

$$\begin{aligned} -\frac{\log\left(\mathbb{P}_p(V_\gamma)\right)}{\mathrm{Area}(\gamma)} &\to c_7(p) && p < 1 - p_c(\mathbb{Z}^d) \\ -\frac{\log\left(\mathbb{P}_p(V_\gamma)\right)}{\mathrm{Per}(\gamma)} &\to c_8(p) && p > 1 - p_c(\mathbb{Z}^d)\,, \end{aligned}$$

for rectangular $(d-1)$-boundaries γ as all dimensions of γ are taken to ∞, where $V_\gamma = V_\gamma^{\mathrm{fin}}(m)$ or $V_\gamma = V_\gamma^{\mathrm{inf}}(m)$ for some $m \in \mathbb{N}$.

The hypothesis that the limit be taken over a suitable family of rectangular boundaries is not necessary here, as discussed at the beginning of Section 9.

2.1. **Outline.** The paper is split into two parts. The first covers the properties of the generalized plaquette random cluster model and its relationship with Potts lattice gauge theory. We apply the tools of algebraic topology, which allow us to reduce the proofs of Theorems 4 and 9 to questions concerning the dual (dependent) bond percolation. We include a review of the definitions of homology and cohomology in the appendix (Section A.1) as well

Topics covered in the first half include the basic properties of the PRCM with coefficients in a finite group, construction of the infinite volume measures with free boundary conditions, the proof of Theorem 8, and a characterization of the event V_γ in terms of linking numbers with dual loops in codimension one. The infinite volume measures with wired boundary conditions are defined in Appendix B. The second half of the paper involves arguments more typical of the literature on percolation theory. A description of its content is included in the proof outline below.

We give a brief overview of our proof of the sharpness of the deconfinement transition for the PLGT (Theorem 4). We show that it is equivalent to a special case of our main result for the PRCM (Theorem 9) by demonstrating that Wilson loop expectations can be computed in terms of the probability of the topological event V_γ in the coupled plaquette percolation (Theorem 8). This is done in two steps: we begin by establishing the corresponding result for finite volume measures (Proposition 14 in Section 4.1). Then, we construct an infinite volume coupling between the PRCM and PLGT with free boundary conditions and explain how to extend the result on Wilson loop variables

(Proposition 18 in Section 4.2). The corresponding result for wired boundary conditions is Proposition 77 in Appendix B.1. These arguments work for the general i-dimensional PRCM, but to continue we specialize to the case $i = d - 1$. We show that, in codimension one, the event $V_\gamma(q)$ occurs exactly when there is no dual loop which has has non-zero linking number modulo q with γ (Proposition 29 in Section 6).

The remainder of the paper focuses on either constructing or precluding the existence of such dual loops. The overall proof outline is similar to that of [ACC$^+$83], though the arguments are complicated by the dependence of disjoint plaquette events and the generalization to higher dimensions. We demonstrate the two parts of Theorem 9 separately. In order to show an area law upper bound in the subcritical regime (Section 7), we construct a renormalized system of dual sites within a slab which stochastically dominates an independent site percolation with arbitrarily high parameter. We use this to prove the existence of a dual loop that has linking number one with γ, with sufficiently high probability to imply an area law upper bound for V_γ. This is done at the end of Section 7.2. As an aside, we demonstrate that the constant appearing in the area law exponent is well-defined in Lemma 44.

A first proof of the perimeter law in the supercritical regime is given in Section 8. Here, we use the almost sure finiteness of all components in the dual subcritical random cluster model (RCM) to construct a hypersurface of plaquettes with boundary γ with sufficiently high probability (Proposition 49). While this is relatively straightforward, showing that is a sharp constant requires more work and technical geometric arguments. This is done in Section 9, where we we work with both the plaquette system and the dual bond system to show that with high probability there is a "tube" of plaquettes surrounding γ together with additional plaquette sheets that prevent dual loops from linking with γ outside or inside of the tube. These constructions give matching upper and lower bounds for the perimeter law constant (Theorem 50, proven at the end of Section 9.3), thus concluding our proof of Theorem 9.

Part 1. The Plaquette Random Cluster Model with Coefficients in an Abelian Group

As defined in the introduction, the PRCM random cluster model with coefficients in an finite abelian group G is the random i-dimensional subcomplex of X so that

$$\tilde{\mu}_{X,p,G,i}(P) \propto p^{|P|} (1-p)^{\left|X^{(i)}\right| - |P|} \left|H^{i-1}(P;\, G)\right| .$$

In Section 3, we introduce the complexes and dual complexes in $\mathbb{Z}^d$ that we primarily work with in this article.

Section 4 returns to the general setting and studies the relationship between the PRCM with coefficients in $\mathbb{Z}_q$ and the q-state PLGT. These results hold for any i. We cover the case of finite complexes in Section 4.1. The definitions of the infinite volume measures constructed with free boundary conditions are given in Section 4.2. The details for similar results on finite and infinite complexes with wired boundary conditions can be found in B.2.

Next, in Section 5 we show that — when $i = d-1$ — the PRCM with coefficients in G coincides with the PRCM with coefficients in $\mathbb{Q}$ with parameter $q = |G|$. By earlier results of [HS16] and [DS22], it satisfies many nice properties, which we summarize below. It turns out these hold for general i when $G = \mathbb{Z}_q$, but we defer their proof until a later paper. We then discuss we discuss more general boundary conditions and infinite volume limits for the codimension one PRCM in Section 5.2. While these are not difficult to define in general, we use duality with the classical random cluster model to reduce technical overhead.

Finally, in Section 6 we relate the events V_γ to corresponding one for the dual RCM. While this might fit more logically in the second part, we leave it here as the arguments are more topological in nature.

3. Subcomplexes of $\mathbb{Z}^d$

We now specialize to the specific class of cubical complexes that we work with. The *k-skeleton* of a cubical complex X, written $X^{(k)}$, is the subcomplex consisting of the union of all cells of dimension at most k. Then an i-dimensional *percolation subcomplex* P of X is a complex satisfying $X^{(i-1)} \subset P \subset X^{(i)}$. In this article the complex X on which we consider the PRCM will usually either be $\mathbb{Z}^d$ or a subcomplex thereof. Recall that the i-dimensional cells of $\mathbb{Z}^d$ are the i-dimensional unit cubes with integer corner points. This complex has an associated dual complex $\left(\mathbb{Z}^d\right)^\bullet$, which is obtained from it by shifting by 1/2 in each coordinate direction. There is then a pairing that matches each i-plaquette of $\mathbb{Z}^d$ to the unique $(d-i)$-plaquette intersecting it at its center point. In particular, an i-dimensional percolation subcomplex P has an associated dual complex Q consisting of the union of the $(d-i-1)$-skeleton of $\left(\mathbb{Z}^d\right)^\bullet$ and the dual of each omitted plaquette of P.

Let $r = [a_1, b_1] \times \ldots \times [a_d, b_d]$ be a box in $\mathbb{Z}^d$. For convenience, we will abuse notation throughout and let r refer both to the aforementioned closed box and (when i is specified) the i-dimensional subcomplex of that box obtained from its i-skeleton by removing the i-dimensional plaquettes in ∂r. In other words, an i-dimensional subcomplex of r is not allowed it to contain i-plaquettes in ∂r. We will denote the full i-skeleton of r by $\overline{r}$.

The dual of a percolation subcomplex $P \subset r$ (or $\overline{r}$) will be a percolation subcomplex of a slightly shrunk (expanded) box. For $\epsilon \geq -1$, let r^ϵ be the box $[a_1 - \epsilon, b_1 + \epsilon] \times \ldots \times [a_d - \epsilon, b_d + \epsilon]$. The dual complex Q' of $P' \subset \overline{r}$ is defined exactly as above and is a percolation subcomplex of $r^\bullet := r^{1/2}$, while the dual complex Q of $P \subset r$ is a percolation subcomplex of $\overline{r^\bullet} := \overline{r^{-1/2}}$.

The topological properties of Q are closely related to those of the complement $\mathbb{R}^d \setminus P$.

Proposition 11. *Fix $0 < i < d$ and a box r in $\mathbb{Z}^d$. If P is a percolation subcomplex of $\overline{r}$ (r), Q is its dual complex, and r' is the box $r^\bullet$ (respectively $\overline{r^\bullet}$) then there is an isomorphism*

$$\mathcal{I} : H_i\left(P_r;\ \mathbb{Z}\right) \to H^{d-i-1}\left(Q \cup \partial r';\ \mathbb{Z}\right)$$

where $H_j\left(X;\ \mathbb{Z}\right)$ and $H^j\left(X;\ \mathbb{Z}\right)$ denote the j-dimensional reduced homology and reduced cohomology of X with integral coefficients, respectively.

A proof is included in Appendix A.3.

4. Relationship with the PLGT

The main goal of this section is to prove Theorem 8 for infinite volume measures constructed with free bounded conditions. We begin by coupling the PRCM and the PLGT on a finite cell complex X. Recall that if $X \subset \mathbb{Z}^d$, these measures are said to have free boundary conditions.

4.1. **Finite Volume Measures.**

The following statement is a generalization of the classical Edwards–Sokal coupling when $i = 1$ [ES88] and of previous results for the special case when q is a prime integer [HS16].

Proposition 12. *Let X be a finite cubical complex, $q \in \mathbb{N} + 1$, $\beta \in [0, \infty)$, and $p = 1 - e^{-\beta}$. Define a coupling on $C^{i-1}\left(X\right) \times \{0,1\}^{X^{(i)}}$ by*

$$\kappa\left(f, P\right) \propto \prod_{\sigma \in X^{(i)}} \left[\left(1-p\right) I_{\{\sigma \notin P\}} + p I_{\{\sigma \in P, \delta f(\sigma) = 0\}}\right] ,$$

Then κ has the following marginals.

- *The first marginal is $\nu_{X,\beta,q,i-1}$.*
- *The second marginal is $\tilde{\mu}_{X,p,\mathbb{Z}_q,i}$.*

Proof. The proof that the first marginal is the q-state PLGT is identical that for the case where q is a prime integer [HS16].

The computation of the second marginal proceeds similarly, with minor differences towards the end of the computation.

$$\begin{aligned}
\kappa_2(P) &\coloneqq \sum_{f\in C^{i-1}(X)} \kappa(f,P) \\
&\propto \sum_{f\in C^{i-1}(X)} \prod_{\sigma\in X^{(i)}} \left[(1-p)\, I_{\{\sigma\notin P\}} + p I_{\{\sigma\in P,\delta f(\sigma)=0\}}\right] \\
&= (1-p)^{\left|X^{(i)}\right|-|P|} p^{|P|} \sum_{f\in C^{i-1}(X)} \prod_{\substack{\sigma\in X^{(i)}\\ \sigma\in P}} I_{\{\delta f(\sigma)=0\}} \\
&= (1-p)^{\left|X^{(i)}\right|-|P|} p^{|P|} \left|Z^{i-1}(P;\ \mathbb{Z}_q)\right| \\
&= (1-p)^{\left|X^{(i)}\right|-|P|} p^{|P|} \left|H^{i-1}(P;\ \mathbb{Z}_q)\right| \left|B^{i-1}(P;\ \mathbb{Z}_q)\right| \\
&\propto (1-p)^{\left|X^{(i)}\right|-|P|} p^{|P|} \left|H^{i-1}(P;\ \mathbb{Z}_q)\right| ,
\end{aligned}$$

because $B^{i-1}(P;\ \mathbb{Z}_q)$ does not depend on P. □

The following characterization of the conditional measures of the coupling follows from the same proof of Proposition 21 of [DS22].

Corollary 13. *Let $p = 1 - e^{-\beta}$. Then κ has the following conditional measures:*

- *Given f, the conditional measure $\kappa(\cdot \mid f)$ is Bernoulli plaquette percolation with probability p on the set of plaquettes σ that satisfy $\delta f(\sigma) = 0$.*
- *Given P, the conditional measure $\kappa(\cdot \mid P)$ is the uniform measure on $(i-1)$-cocycles in $Z^{i-1}(P;\ \mathbb{Z}_q)$.*

We now show the analogue of Theorem 8 for finite cubical complexes. This is a generalization of Theorem 5 of [DS22].

Proposition 14. *Let X be a finite cubical complex, $0 < i < d-1$, $q \in \mathbb{N}+1$, and $\gamma \in Z_{i-1}(X;\ \mathbb{Z}_q)$. Then, if $H_{i-2}(X;\ \mathbb{Z}_q) = 0$,*

$$\mathbb{E}_\nu(W_\gamma) = \tilde{\mu}(V_\gamma) ,$$

where $\nu = \nu_{X,\beta,q,i-1,d}$ is the PLGT, $\tilde{\mu} = \tilde{\mu}_{X,1-e^{-\beta},\mathbb{Z}_q,i}$ is the corresponding PRCM, and V_γ is the event that $[\gamma] = 0$ in $H_{i-1}(P;\ \mathbb{Z}_q)$.

Proof. We compute $\mathbb{E}_\nu(W_\gamma)$ using the law of total conditional expectation:

$$\mathbb{E}_\nu(W_\gamma) = \mathbb{E}_\kappa(W_\gamma) = \mathbb{E}_\kappa(W_\gamma | V_\gamma)\,\kappa(V_\gamma) + \mathbb{E}_\kappa(W_\gamma \mid \neg V_\gamma)\,\kappa(\neg V_\gamma) .$$

The desired result will follow if we demonstrate that $\mathbb{E}_\kappa(W_\gamma | V_\gamma) = 1$ and $\mathbb{E}_\kappa(W_\gamma | \neg V_\gamma) = 0$.

First, if W_γ occurs then

$$\gamma = \partial\left(\sum_\sigma a_\sigma \sigma\right)$$

where the sum is taken over i-plaquettes σ so that $f(\partial\sigma) = 0$. By linearity, $f(\gamma) = 0$ and $W_\gamma = f(\gamma)^{\mathbb{C}} = 1$. Thus $\mathbb{E}_\kappa(W_\gamma | V_\gamma) = 1$.

Now, assume that $[\gamma] \neq 0$ in $H_{i-1}(P;\ \mathbb{Z}_q)$. By Corollary 67,

$$H^{i-1}(P;\ \mathbb{Z}_q) \cong \mathrm{Hom}\left(H_{i-1}(P;\ \mathbb{Z}_q), \mathbb{Z}_q\right) \cong H_{i-1}(P;\ \mathbb{Z}_q)\ ,$$

where we are using the assumption that $H_{i-2}(X;\ \mathbb{Z}_q) = 0$. Thus, there exists an $f \in Z^{i-1}(P;\ \mathbb{Z}_q)$ so that $f(\gamma) \neq 0$. It follows that the conditional random variable $(W_\gamma | P)$ does not vanish and in fact — by symmetry — is distributed on a non-trivial subgroup of the q-th complex roots of unity $\mathbb{Z}(q)$. The only such subgroups are those of the form $\mathbb{Z}(m)$ for $m \mid q$. The m-th roots of unity sum to zero so $\mathbb{E}_\kappa(W_\gamma \mid P) = 0$ for any P so that $\neg V_\gamma$. Therefore, by the law of total conditional expectation $\mathbb{E}_\kappa(W_\gamma | \neg V_\gamma) = 0$. ☐

4.2. **Infinite Volume Measures.** We construct infinite volume measures for the PRCM and PLGT with free boundary conditions, and study their relationship. First, we prove two technical lemmas.

Let P be a percolation subcomplex of $\mathbb{Z}^d$. For convenience, set $P_n = P \cap \Lambda_n$, where $\Lambda_n \coloneqq [-n, n]^d$. For $N > n$, define two restriction maps

$$\phi_{N,n} : Z^{i-1}(P \cap \Lambda_N;\ \mathbb{Z}_q) \to Z^{i-1}(P \cap \Lambda_n;\ \mathbb{Z}_q)$$

and

$$\phi_{\infty,n} : Z^{i-1}(P;\ \mathbb{Z}_q) \to Z^{i-1}(P \cap \Lambda_n;\ \mathbb{Z}_q)\ .$$

Set $Y_{N,n} = \mathrm{im}\,\phi_{N,n}$ and $Y_{\infty,n} = \mathrm{im}\,\phi_{\infty,n}$.

Lemma 15. *Let $\# \in \{N, \infty\}$ and $f \in Z^{i-1}(P_n;\ \mathbb{Z}_q)$. Then $f \notin \mathrm{im}\,\phi_{\#,n}$ if and only if there exists a cycle $\sigma \in Z_{i-1}(P_n;\ \mathbb{Z}_q)$ so that $f(\sigma) \neq 0$ and $\sigma \in B_{i-1}(P_\#;\ \mathbb{Z}_q)$.*

Proof. We begin by showing that the statement is equivalent to an analogous one for cohomology classes. Suppose that $[f_n] = [f'_n] \in H^{i-1}(P_n;\ \mathbb{Z}_q)$. Then there exists a $g_n \in C^{i-2}(P_n;\ \mathbb{Z}_q)$ so that $\delta g_n = f_n - f'_n$. We may extend g_n to obtain a cochain $g_\# \in C^{i-2}(P_\#;\ \mathbb{Z}_q)$ which vanishes on $(i-2)$-plaquettes outside of Λ_n. Then $f_n - f'_n = \phi_{\#,n}(\delta g_\#)$. It follows $f_n \in \mathrm{im}\,\phi_{\#,n} \iff f'_n \in \mathrm{im}\,\phi_{\#,n}$.

Let $\phi^*_{\#,n} : H^{i-1}(P_\#;\ \mathbb{Z}_q) \to H^{i-1}(P_n;\ \mathbb{Z}_q)$ be the induced map on cohomology. It suffices to demonstrate that $\mathrm{im}\,\phi^*_{\#,n}$ consists of of all cocycles of P_n which vanish on boundaries of $P_\#$ that are supported on P_n. This follows directly from the definition of the long exact sequence of the pair $(P_\#, P_n)$ and the associated boundary map (see page 199 of [Hat02]; more detail is given for the homological analogue on page 115).

□

Lemma 16. *Fix $n \in \mathbb{N}$.*

- *$Y_{N,n} = Y_{\infty,n}$ for all sufficiently large N.*
- *For N sufficiently large as in the previous statement, the pushforward by $\phi_{N,n}$ of the uniform measure on $Z^{i-1}(P_N;\, \mathbb{Z}_q)$ is the uniform measure on $Y_{\infty,n}$.*
- *For all $n < N$, $\phi_{N,n}(Y_{\infty,N}) = Y_{\infty,n}$ and the pushforward of the uniform measure on $Y_{\infty,N}$ by $\phi_{N,n}$ is the uniform measure on $Y_{\infty,n}$.*

Proof. The first statement follows quickly from the previous lemma. For each $f \in Z^{i-1}(P_n;\, \mathbb{Z}_q) \setminus \operatorname{im}\phi_{\infty,n}$ we can find a chain $\sigma \in C_i(P;\, \mathbb{Z}_q)$ so that $\partial\sigma \in Z_{i-1}(P_n;\, \mathbb{Z}_q)$ and $f(\partial\sigma) \neq 0$. By the definition of a chain, σ is finite so it is supported on all sufficiently large cubes in $\mathbb{Z}^d$. In fact, $Z^{i-1}(P_n;\, \mathbb{Z}_q)$ is itself finite so we may choose a cube Λ_N large enough to support boundaries for all such f. Then $Y_{N,n} = Y_{\infty,n}$.

To show the second statement, we use the decomposition

$$Z^{i-1}(P_n;\, \mathbb{Z}_q) \cong \ker\phi_{N,n} \oplus \operatorname{im}\phi_{N,n}\,.$$

In particular, a uniform element of a direct sum can be obtained by taking a uniform element of each summand, and so the pushforward is simply a uniform element of the second summand.

For the third statement, first notice that the equality $\phi_{N,n}(Y_{\infty,N}) = Y_{\infty,n}$ is immediate from the fact that a composition of restrictions is a restriction. We then have a similar decomposition

$$Y_{\infty,N} = \ker\left(\phi_{N,n}\,|_{Y_{\infty,N}}\right) \oplus \operatorname{im}\left(\phi_{N,n}\,|_{Y_{\infty,N}}\right)$$

and for similar reasons the pushforward of the uniform measure is again a uniform measure. □

The same proof goes through in a slightly more general setting. Let $r \subset \Lambda_N$ be a d-dimensional box. Let $\phi_{N,r} : Z^{i-1}(P \cap \Lambda_N;\, \mathbb{Z}_q) \to Z^{i-1}(P \cap r;\, \mathbb{Z}_q)$ and $\phi_{\infty,r} : Z^{i-1}(P;\, \mathbb{Z}_q) \to Z^{i-1}(P \cap r;\, \mathbb{Z}_q)$ be the restriction maps. Let $Y_{\infty,r} = \operatorname{im}(\phi_{\infty,r})$ and let $Y_{N,r} = \operatorname{im}(\phi_{N,r})$. We only require the analogue of the third bullet point above.

Corollary 17. *For all $n < N$, $\phi_{N,r}(Y_{\infty,N}) = Y_{\infty,r}$ and the pushforward of the uniform measure on $Y_{\infty,N}$ by $\phi_{N,r}$ is the uniform measure on $Y_{\infty,r}$.*

We now state and prove the main result of this section.

Proposition 18. *Let $0 < i < d-1$, $q \in \mathbb{N}+1$, $\beta \in (0,\infty)$, $p = 1-e^{-\beta}$, and*
$$(\#_1, \#_2) \in \{(\mathbf{f}, \mathrm{fin}), (\mathbf{w}, \mathrm{inf})\} \ .$$
The weak limits
$$\mu^{\#_1}_{\mathbb{Z}^d,p} = \lim_{N\to\infty} \mu^{\#_1}_{\Lambda_N,p,q,d-1}$$
and
$$\nu^{\#_1}_{\mathbb{Z}^d} = \lim_{N\to\infty} \nu^{\#_1}_{\Lambda_N,\beta,q,d-1}$$
exist and are translation invariant. Moreover, if γ is a $(i-1)$-cycle in $\mathbb{Z}^d$ then
$$\mathbb{E}_{\nu^{\#_1}_{\mathbb{Z}^d}}(W_\gamma) = \mu^{\#_1}_{\mathbb{Z}^d,p}\left(V^{\#_2}_\gamma\right) \ .$$

Note that the last statement is Theorem 8 for free boundary conditions.

Proof. The weak limit of $\mu^{\mathbf{f}}_{\Lambda_N,p,q,d-1}$ exists and is translation invariant by the same logic as that for the classical RCM, using the FKG inequality and a standard monotonicity argument (see Theorem 4.19 of [Gri99]). In fact, we may couple the PRCMs with $P(1) \subset P(2) \subset \ldots$ with $P(N) \sim \mu^{\mathbf{f}}_{\Lambda_N,p,q,d-1}$ and $P = \cup_N P(N) \sim \mu^{\mathbf{f}}_{\mathbb{Z}^d,p}$.

We will construct a coupling whose first marginal is $\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}$ and whose second marginal is the weak limit of $\nu^{\mathbf{f}}_{\Lambda_N,\beta,q,d-1}$ as $N\to\infty$. Let $\Omega = \{0,1\}^{\left(\mathbb{Z}^d\right)^{(i)}}$ and $\Sigma = C^{i-1}\left(\mathbb{Z}^d;\ \mathbb{Z}_q\right)$. The σ-algebra on Ω is generated by cylinder events of the form
$$\mathcal{K}(P_n) \coloneqq \left\{P \subset \mathbb{Z}^d : P \cap \Lambda_n = P_n\right\} \ .$$
Similarly, the cylinder events for Σ are
$$\mathcal{L}(f_n) \coloneqq \left\{f \in C^{i-1}\left(\mathbb{Z}^d;\ \mathbb{Z}_q\right) : f\mid_{\Lambda_n} = f_n\right\}$$
where $f_n \in C^{i-1}(\Lambda_n;\ \mathbb{Z}_q)$. Then the product σ-algebra on $\Omega\times\Sigma$ is generated by the semi-ring consisting of the the products $\mathcal{K}(P_n)\times\mathcal{L}(f_n)$.

We define a coupling $\kappa^{\mathbf{f}}_{\beta,q}$ on $\Omega\times\Sigma$ by first specifying it on cylinder events. Set
$$\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}(P_n)\times\mathcal{L}(f_n)\right) = \sum_{H\subseteq Z^{i-1}(P_n;\ \mathbb{Z}_q)} \frac{I_{\{f_n\in H\}}}{|H|}\mu_{\mathbb{Z}^d,p}\left(Y_{\infty,n} = H\cap\mathcal{K}(P_n)\right) \ ,$$

where we note that $\{Y_{\infty,n} = H\}$ is measurable because $Y_{\infty,n} = \bigcap_{N>n} Y_{N,n}$. In words, given P_n we can sample f_n by revealing $Y_{\infty,n}$ and selecting a uniform random cocycle from $Y_{\infty,n}$.

We now need to check that our partial definition of κ extends to a measure on $\Omega \times \Sigma$. By Carathéodory's extension theorem, the remaining requirement is countable additivity on cylinder sets.

Note that since $\Omega \times \Sigma$ is a product of countably many finite spaces, no cylinder set is an infinite countable disjoint union of cylinder sets (see Chapter 8.6 of [Lin17]). Thus, it suffices to check finite additivity. We claim that it is enough to show that if $n < N$ then

$$\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_n\right) \times \mathcal{L}\left(f_n\right)\right) = \sum_{\left(P_N, f_N\right) \in \mathcal{R}\left(P_n, f_n, N\right)} \kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_N\right), \mathcal{L}\left(f_N\right)\right) \tag{4}$$

where

$$\mathcal{R}\left(P_n, f_n, N\right) = \left\{\left(P_n, f_n\right) : P_N \cap \Lambda_n = P_n, \phi_{N,n}\left(f_N\right) = f_n\right\} .$$

Suppose that

$$\mathcal{K}\left(P_n\right) \times \mathcal{L}\left(f_n\right) = \bigsqcup_{k=1}^{l} C_k$$

where each C_k is a cylinder event for a a cube Λ_{N_k}. Then, if we choose $N = \max_k N_k$, we can use (4) to reduce the statement

$$\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_n\right) \times \mathcal{L}\left(f_n\right)\right) \stackrel{?}{=} \sum_{k=1}^{l} \kappa^{\mathbf{f}}_{\beta,q}\left(C_k\right) \tag{5}$$

to one of the form

$$\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_n\right) \times \mathcal{L}\left(f_n\right)\right) \stackrel{?}{=} \sum_{l=1}^{L} \kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P^l_N\right) \cap \mathcal{L}\left(f^l_N\right)\right) \tag{6}$$

where the P^l_N are subcomplexes of Λ_N and the f^l_N are cochains in $C^{i-1}\left(\Lambda_n;\ \mathbb{Z}_q\right)$ so that

$$\mathcal{K}\left(P_n\right) \times \mathcal{L}\left(f_n\right) = \bigsqcup_{l=1}^{L} \mathcal{K}\left(P^l_N\right) \cap \mathcal{L}\left(f^l_N\right) .$$

Now, the only way to achieve this decomposition is if $P^l_N \cap \Lambda_n = P_n$ and $f^l_N \mid_{\Lambda_n} = f_n$ for each l, and the pairs $\left(P^l_N, f^l_N\right)$ cover all possible plaquette/spin combinations in the annulus $\Lambda_N \setminus \Lambda_n$. That is, we must have

$$\left\{\left(P^l_N, f^l_N\right)\right\}_{l=1}^{L} = \mathcal{R}\left(P_n, f_n, N\right)$$

and then (5) and (6) follow from (4).

Now, we demonstrate (4).

$$\begin{aligned}
&\sum_{(P_N,f_N)\in\mathcal{R}(P_n,f_n,N)} \kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_N\right),\mathcal{L}\left(f_N\right)\right)\\
=&\sum_{\substack{(P_N,f_N)\in\mathcal{R}(P_n,f_n,N)\\ H\subseteq Z^{i-1}(P_N;\,\mathbb{Z}_q)}} \frac{I_{\{f_N\in H\}}}{|H|}\mu_{\mathbb{Z}^d,p}\left(\{Y_{\infty,N}=H\}\cap\mathcal{K}\left(P_N\right)\right)\\
=&\sum_{\substack{P_N:P_N\cap\Lambda_n=P_n\\ H\subseteq Z^{i-1}(P_N;\,\mathbb{Z}_q)}} \frac{|\{f_N\in H : f_N\mid_{\Lambda_n}=f_n\}|}{|H|}\mu_{\mathbb{Z}^d,p}\left(\{Y_{\infty,N}=H\}\cap\mathcal{K}\left(P_N\right)\right)\\
=&\sum_{\substack{P_N:P_N\cap\Lambda_n=P_n\\ H'\subseteq Z^{i-1}(P_n;\,\mathbb{Z}_q)}} \frac{I_{\{f_n\in H'\}}}{|H'|}\mu_{\mathbb{Z}^d,p}\left(\phi_{N,n}\left(\{Y_{\infty,N}\}=H'\right)\cap\mathcal{K}\left(P_N\right)\right)\\
=&\sum_{H'\subseteq C^{i-1}(\Lambda_n;\,\mathbb{Z}_q)} \frac{I_{\{f_n\in H'\}}}{|H'|}\sum_{P_N:P_N\cap\Lambda_n=P_n}\mu_{\mathbb{Z}^d,p}\left(\{Y_{\infty,n}=H'\}\cap\mathcal{K}\left(P_N\right)\right)\\
=&\sum_{H'\subseteq C^{i-1}(\Lambda_n;\,\mathbb{Z}_q)} \frac{I_{\{f_n\in H'\}}}{|H'|}\mu_{\mathbb{Z}^d,p}\left(\{Y_{\infty,n}=H'\}\cap\mathcal{K}\left(P_n\right)\right)\\
=&\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_n\right)\times\mathcal{L}\left(f_n\right)\right)\,,
\end{aligned}$$

where we used the third bullet point in Lemma 16 to move from the third to the fourth line of the computation. Thus, we have defined a measure $\kappa^{\mathbf{f}}_{\beta,q}$ on $\Omega\times\Sigma$. Notice that a similar computation can be used to prove translation invariance. That is, if Λ_n' is a translate of Λ_n, P_n' and f_n' are the corresponding shifts of P_n and f_n, and N is sufficiently large then

$$\begin{aligned}
&\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_n'\right)\times\mathcal{L}\left(f_n'\right)\right)\\
=&\sum_{\substack{P_N:P_N\cap\Lambda_n'=P_n\\ H'\subseteq Z^{i-1}(P_{n'};\,\mathbb{Z}_q)}} \frac{I_{\{f_n'\in H'\}}}{|H'|}\mu_{\mathbb{Z}^d,p}\left(\phi_{N,\Lambda_n'}\left(\{Y_{\infty,N}\}=H'\right)\cap\mathcal{K}\left(P_N\right)\right)\\
=&\sum_{\substack{P_N:P_N\cap\Lambda_n=P_n\\ H'\subseteq Z^{i-1}(P_n;\,\mathbb{Z}_q)}} \frac{I_{\{f_n\in H'\}}}{|H'|}\mu_{\mathbb{Z}^d,p}\left(\phi_{N,n}\left(\{Y_{\infty,N}\}=H'\right)\cap\mathcal{K}\left(P_N\right)\right)\\
=&\kappa^{\mathbf{f}}_{\beta,q}\left(\mathcal{K}\left(P_n\right)\times\mathcal{L}\left(f_n\right)\right)
\end{aligned}$$

where we used the translation invariance of $\mu_{\mathbb{Z}^d,p}$ and Corollary 17.

Next, we verify that the marginals are as claimed. That the first one is $\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}$ is immediate from the definition. For the second marginal, consider the conditional distribution obtained by restricting to Λ_n.

By definition, it assigns to $f_n \in C^{i-1}(\Lambda_n;\ \mathbb{Z}_q)$ the probability

$$\mathbb{E}_{\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}}\left[\sum_{H\subseteq Z^{i-1}(P_n;\ \mathbb{Z}_q)} \frac{I_{\{f_n\in H, Y_{\infty,n}=H\}}}{|H|}\right]. \tag{7}$$

On the other hand, by Corollary 13, the restriction of $\nu^{\mathbf{f}}_{\Lambda_n,\beta,q,d-1}$ to Λ_n gives the probability as

$$\mathbb{E}_{\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}}\left[\sum_{H\subseteq Z^{i-1}(P(n)\cap\Lambda_n;\ \mathbb{Z}_q)} \frac{I_{\left\{f_n\in H, Y_{N,n}=H\right\}}}{|H|}\right]. \tag{8}$$

Fix P. We may choose N large enough so that $P(N)\cap\Lambda_n = P_n$ and $Y_{N,n} = Y_{\infty,n}$, where the first claim follows because $P(N) \nearrow P$ and the second is the first item of Lemma 16. Thus, the inner term of (8) converges to the inner term of (7) pointwise as $N \to \infty$ as a function of P. Therefore, by bounded convergence theorem, the second marginal of $\kappa^{\mathbf{f}}_{\beta,q}$ is the weak limit of the measures $\nu^{\mathbf{f}}_{\Lambda_N}$.

Let $V(N)$ be the event that γ is a boundary in $P(N)$. Then as $P(N) \nearrow P$, $V(N) \nearrow V^{\text{fin}}_\gamma$ and

$$\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(V^{\text{fin}}_\gamma\right) = \lim_{N\to\infty} \mu^{\mathbf{f}}_{\Lambda_N,p}(V(N)) = \lim_{N\to\infty} \mathbb{E}_{\nu^{\mathbf{f}}_{\Lambda_N}}(W_\gamma) = \mathbb{E}_{\nu^{\mathbf{f}}_{\mathbb{Z}^d}}(W_\gamma)\,,$$

where the third equality follows from Proposition 14 and the last equality is implied by weak convergence.

□

5. The PRCM in Codimension One

5.1. **Basic Properties.** Recall that the PRCM with coefficients in the rational numbers $\mathbb{Q}$ is the random i-dimensional percolation subcomplex of X satisfying

$$\mu_{X,p,q,i}(P) \propto p^{|P|}(1-p)^{\left|X^{(i)}\right|-|P|}\, q^{\mathbf{b}_{i-1}(P;\mathbb{Q})}.$$

Proposition 19. *Let r be a box in $\mathbb{Z}^d$. Then $(d-1)$-dimensional PRCM $\tilde{\mu}_{r,p,G,d-1}$ on r with coefficients in a finite abelian group G is equal in distribution to the PRCM $\mu_{r,p,|G|,d-1}$ with coefficients in $\mathbb{Q}$. In particular,*

$$\tilde{\mu}_{r,p,\mathbb{Z}_q,d-1} \overset{d}{=} \mu_{r,p,q,d-1}\,.$$

Proof. This follows from the fact that, if P is a percolation subcomplex of r then

$$H^{d-2}(P;\ \mathbb{Z}_q) \cong \mathbb{Z}_q^{\mathbf{b}_{d-2}(P;\ \mathbb{Q})}\,.$$

This is Proposition 68 in the appendix. The proof uses several standard tools from algebraic topology, including the universal coefficients theorems for homology and cohomology and Alexander duality. See Section A.2. □

We review some properties of the PRCM with coefficients in $\mathbb{Q}$. First, it satisfies the FKG inequality.

Theorem 20 ([HS16])**.** *Let $p \in (0,1)$, and $q \geq 1$, $i \in \mathbb{N}$, and X a finite cubical complex. Then $\mu_{X,p} = \mu_{X,p,q,i}$ satisfies the FKG inequality. That is, if A and B are increasing events then*

$$\mu_{X,p}\left(A \cap B\right) \geq \mu_{X,p}\left(A\right)\mu_{X,p}\left(B\right).$$

Next, we have the following duality relation.

Theorem 21 ([DS22])**.**

$$\mu_{r,p,q,i}\left(P^{\bullet}\right) = \mu^{\mathbf{w}}_{r^{\bullet},p^{*},q,d-i}\left(P\right) .$$

Proof. The proof is nearly identical to that of Theorem 18 in [DS22] but uses Proposition 11 instead of Theorem 14 of that paper. □

5.2. **Boundary Conditions and Infinite Volume Measures.** The duality between the $(d-1)$- and 1-dimensional random cluster models allows us to take a shortcut to defining more general boundary conditions for random cluster measures on finite subsets of $\mathbb{Z}^d$. We do not go into too much detail on this topic, as it has been proven that the resulting infinite volume measures are unique except at at most countably many values of p [Gri95].

First, we recall boundary conditions for the classical random cluster model on a graph. A boundary condition on a subgraph induced by a vertex set S is a percolation subcomplex ξ on $\left(\mathbb{Z}^d \setminus S\right) \cup \partial S$. Let P^{ξ} be the union of P and the edges of ξ. The idea is to define a random-cluster measure on S with the additional edges of P^{ξ} added for the purpose of counting connected components. Of course, P^{ξ} will have infinitely many connected components in general, but finitely many of them are connected to S.

More precisely, there is a corresponding random-cluster measure on S with boundary condition ξ written as $\mu^{\xi}_{S,p,q,1}\left(P\right)$, where the term $\mathbf{b}_0\left(P\right)$ counting the number of connected components of P in S is replaced by the number of connected components of P^{ξ} that intersect S. The free and wired boundary conditions discussed previously can be thought of as the extremal cases of ξ containing no edges or all possible edges, respectively.

We define boundary conditions for the $(d-1)$-dimensional PRCM on a box $r \subset \mathbb{Z}^d$ by duality. Let ξ be a boundary condition and denote by $\xi^\bullet$ the dual configuration of edges of the dual lattice corresponding to the plaquettes not included in ξ.

Definition 22. *The measure $\tilde{\mu}^{\xi}_{r,p,\mathbb{Z}_q,1}$ is defined by*

$$\mu^{\xi}_{r,p,q,d-1}(P) \coloneqq \mu^{\xi^\bullet}_{r^\bullet,p^*,q,1}(Q)\ .$$

This is equivalent to setting

$$\mu^{\xi}_{r,p,q,d-1}(P) \propto p^{|P|}(1-p)^{\left|X^{(i)}\right|-|P|} q^{\operatorname{rank}\phi^*}$$

where $\phi^* : H^{d-1}\left(P^\xi;\ \mathbb{Q}\right) \to H^{d-1}(P;\ \mathbb{Q})$ is the map on cohomology induced by the inclusion $P \hookrightarrow P^\xi$. A proof of this fact and the definition of an analogous notion for the PRCM with coefficients in an abelian group will be contained in another paper. Alternatively, the PRCM with boundary conditions can be obtained as the restriction to r of the PRCM on a sufficiently large cube Λ, conditioned on the states of the plaquettes in $\Lambda \setminus r$.

The free boundary conditions (denoted by $\mathbf{f}$) contain no $(d-1)$-plaquettes of $\mathbb{Z}^d \setminus r$ and the wired boundary conditions (denoted by $\mathbf{w}$) contain all $(d-1)$-plaquettes in ∂r (this has the same effect as taking all complementary plaquettes, but is more convenient). By construction, duality maps the PRCM with free boundary conditions to the classical random cluster model with wired boundary conditions and vice versa. Also, as a consequence of Theorem 21, the PRCM on a box with free boundary conditions coincides with the PRCM on the finite complex r defined above.

Proposition 23. *Let $\{\xi_n\}$ be a sequence of boundary conditions for the $(d-1)$-dimensional PRCM on the cube Λ_n. The weak limit*

$$\lim_{n\to\infty} \mu^{\xi_n}_{\Lambda_n,p,q,d-1}$$

exists if and only if the dual weak limit

$$\lim_{n\to\infty} \mu^{\xi_n^\bullet}_{\overline{\Lambda_n^{1/2}},p^*,q,1}$$

does and the resulting infinite volume measures are dual.

Proof. This is immediate from Definition 22. $\square$

The following two propositions are corollaries of the analogous, well-known results for the dual classical random cluster model.

Proposition 24. *Let $p \in (0,1)\,, q \geq 1, d \in \mathbb{N}$. Also, fix a box r in $\mathbb{Z}^d$ with boundary conditions ξ. Then $\mu_r^\xi = \mu_{r,p,q,d-1}^\xi$ satisfies the FKG inequality. That is, if A and B are increasing events then*

$$\mu_{r,p}^{\xi}\left(A \cap B\right) \geq \mu_{r,p}^{\xi}\left(A\right) \mu_{r,p}^{\xi}\left(B\right).$$

Proposition 25. *Let r be a box in $\mathbb{Z}^d$ and let ξ be any boundary conditions. Then*

$$\mu_{r,p,q,d-1}^{\mathbf{f}} \leq_{\mathrm{st}} \mu_{r,p,q,d-1}^{\xi} \leq_{\mathrm{st}} \mu_{r,p,q,d-1}^{\mathbf{w}}.$$

In addition, if $\mu_{\mathbb{Z}_d,p,q,d-1}$ is any infinite volume plaquette random cluster measure obtained as a weak limit of finite volume measures

$$\mu_{r,p,q,d-1}^{\mathbf{f}} \leq_{\mathrm{st}} \mu_{\mathbb{Z}_d,p,q,d-1}^{\mathbf{f}} \leq_{\mathrm{st}} \mu_{\mathbb{Z}_d,p,q,d-1} \leq_{\mathrm{st}} \mu_{\mathbb{Z}_d,p,q,d-1}^{\mathbf{w}} \leq_{\mathrm{st}} \mu_{r,p,q,d-1}^{\mathbf{w}}.$$

Compare this statement with Corollary 30.

6. Duality and V_γ

We can use Alexander duality to characterize the events V_γ in terms of the dual RCM. We begin by describing this relationship for homology with integer coefficients. In this case, $[\gamma] = 0$ in $H_i\left(P;\, \mathbb{Z}\right)$ if and only if $\mathcal{I}\left(\gamma\right) = 0$ in $H^{d-i}\left(Q;\, \mathbb{Z}\right)$. When $i = d-1$ we obtain a more precise statements using linking numbers.

Fix k_1, k_2 so that $d = k_1 + k_2 + 1$. Let $\gamma_1 \in Z_{k_1}\left(S^d;\, \mathbb{Z}\right)$ and let γ_2 be an oriented embedding of S^{k_2} into $S^d \setminus \gamma_1$. Define the *linking number* $l\left(\gamma_1, \gamma_2\right)$ to be k if γ_1 is homologous to k times the generator of $H_{k_1}\left(S^d \setminus \gamma_2;\, \mathbb{Z}\right) \cong \mathbb{Z}$. This is equivalent to setting

$$l\left(\gamma_1, \gamma_2\right) = \mathcal{I}\left(\left[\gamma_2\right]\right)\left(\left[\gamma_1\right]\right)$$

where

$$\mathcal{I} : H_{k_2}\left(\gamma_2\right) \to H^{k_1}\left(S^d \setminus \gamma_2\right)$$

is the Alexander duality isomorphism. This is true because both notions define isomorphisms from $H_{k_1}\left(S^d \setminus \gamma_2;\, \mathbb{Z}\right)$ to $\mathbb{Z}$, so they send generators of the former group to ± 1 (the ambiguity in sign is resolved by choosing the generator of $H_{k_1}\left(S^d \setminus \gamma_2;\, \mathbb{Z}\right)$ appropriately). For more on this and other definitions of the linking number, see Chapter 5 of [Rol03].

We require a standard property of linking numbers, namely that if γ_1 is an oriented embedding of S^{k_1} into S^d then the linking number is either symmetric or anti-symmetric:

$$l\left(\gamma_1, \gamma_2\right) = (-1)^{k_1 k_2 + 1}\, l\left(\gamma_2, \gamma_1\right).$$

This has the following corollary.

Corollary 26. *Let γ_1 and γ_2 be disjoint, oriented embeddings of S^{k_1} and S^{k_2} into S^d, respectively. If either γ_1 is contractible in the complement of γ_2 or γ_2 is contractible in the complement of γ_1 then $l\left(\gamma_1, \gamma_2\right) = 0$.*

Next, we prove that — in codimension one — the homology class of a $(d-2)$-cycle is determined by linking numbers. Recall that a basis for a free $\mathbb{Z}$-module is a linearly independent generating set.

Proposition 27. *Let P be a $(d-1)$-dimensional percolation subcomplex of a box $\overline{r} \subset \mathbb{Z}^d$. There are simple cycles $\alpha_1, \ldots, \alpha_n$ of $Q' :== Q \cup \partial r^{\bullet}$ so that the homomorphism $L : H_{d-1}\left(P;\, \mathbb{Z}\right) \to \mathbb{Z}^n$ defined by*

$$L\left([\gamma]\right) = \left(l\left(\gamma, \alpha_1\right), \ldots, l\left(\gamma, \alpha_n\right)\right)$$

is an isomorphism.

Proof. First, we find a basis for $H^1\left(Q';\, \mathbb{Z}\right)$. Let T' be a spanning tree for the one-skeleton of $\partial r^{\bullet}$ and let $Q'' = Q \cup T'$. The inclusion $i : Q'' \hookrightarrow Q'$ induces an isomorphism $i_* : H_1\left(Q'';\, \mathbb{Z}\right) \to H_1\left(Q';\, \mathbb{Z}\right)$ (adding T' to Q has the same effect on homology as merging all vertices in $\partial r^{\bullet}$). As Q'' is a simple graph, $Z_1\left(Q'';\mathbb{Z}\right) = H_1\left(Q'';\, \mathbb{Z}\right)$ has a basis of simple cycles. We may construct such a basis by finding a minimum spanning tree T for Q'' and choosing a simple cycle α_j for each edge of $Q'' \setminus T$. $H_0\left(Q';\, \mathbb{Z}\right)$ is a free $\mathbb{Z}$-module so the duals $\left[\alpha_1^*\right], \ldots, \left[\alpha_n^*\right]$ form a basis for $H^1\left(Q';\, \mathbb{Z}\right)$ by Corollary 67.

For each $j \in \{1, \ldots, n\}$ we define three maps on (co)homology. Let $\phi_j : H_{d-2}\left(P;\, \mathbb{Z}\right) \to H_{d-2}\left(S^d \setminus \alpha_j\right)$ and $\psi_j : H^1\left(Q';\, \mathbb{Z}\right) \to H^1\left(\alpha_j\right)$ be the maps induced by the inclusions $P \hookrightarrow S^d \setminus \alpha_j$ and $\alpha_j \hookrightarrow Q'$, respectively. Also, denote by $\mathcal{I}_j : H_{d-2}\left(S^d \setminus \alpha_j\right) \to H^1\left(\alpha_j;\, \mathbb{Z}\right)$ the Alexander duality isomorphism. Alexander duality is functorial, so the following diagram commutes in the sense that $\mathcal{I}_j \circ \phi_j = \psi_j \circ \mathcal{I}$.

$$\begin{array}{ccc}
H_{d-2}\left(P;\, \mathbb{Z}\right) & \xrightarrow{\phi_j} & H_{d-2}\left(S^d \setminus \alpha_j;\, \mathbb{Z}\right) \\
{\scriptstyle \mathcal{I}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{I}_j} \\
H^1\left(Q';\, \mathbb{Z}\right) & \xrightarrow{\psi_j} & H^1\left(\alpha_j;\, \mathbb{Z}\right)
\end{array}$$

Our next step is to combine the horizontal maps in the previous diagram from different values of i. Before doing so, note that if we choose a generator of $H_{d-2}\left(S^d \setminus \alpha_j;\, \mathbb{Z}\right)$ (say $\mathcal{I}^{-1}\left(\left[\alpha_j^*\right]\right)$), we obtain an isomorphism L_j :

$H_{d-2}\left(S^d \setminus \alpha_j;\ \mathbb{Z}\right) \to \mathbb{Z}$ by sending $[\gamma]$ to the linking number $l\left(\gamma, \alpha_j\right)$. Consider the commutative diagram.

$$\begin{tikzcd}
H_{d-2}\left(P;\ \mathbb{Z}\right) \arrow[r, "\oplus_j \phi_j"] \arrow[d, "\mathcal{I}"'] & \oplus_j H_{d-2}\left(S^d \setminus \alpha_j;\ \mathbb{Z}\right) \arrow[r, "\oplus_j L_j"] \arrow[d, "\oplus_j \mathcal{I}_j"] & \mathbb{Z}^n \\
H^1\left(Q';\ \mathbb{Z}\right) \arrow[r, "\oplus_j \psi_j"] & \oplus_j H^1\left(\alpha_j;\ \mathbb{Z}\right) &
\end{tikzcd}$$

The map $\oplus_j \psi_j$ is an isomorphism because the cohomology classes $\left[\alpha_j^*\right]$ are a basis for $H^1\left(Q';\ \mathbb{Z}\right)$. The downward maps are also isomorphisms, so $\oplus_j \phi_j$ is as well, by commutativity of the diagram. Finally, we may conclude that that $L = \left(\oplus_j L_j\right) \circ \left(\phi_j\right)$ is an isomorphism, because each L_j is an isomorphism.

□

We have the following immediate corollary.

Corollary 28. *Assume the same hypotheses as in the previous proposition, and γ_1 be an oriented embedding of S^{d-2} in P. Then $[\gamma_1] \neq 0$ in $H_{d-2}\left(P;\ \mathbb{Z}\right)$ if and only if there exists a simple, oriented loop γ_2 in Q' so that $l\left(\gamma_1, \gamma_2\right) \neq 0$.*

Next, we find an analogous criterion for homology with coefficients in $\mathbb{Z}_q$.

Proposition 29. *Assume the same hypotheses as above. Then $[\gamma_1] \neq 0$ in $H_{d-2}\left(P;\ \mathbb{Z}_q\right)$ if and only if there exists a simple, oriented loop γ_2 in Q' so that $l\left(\gamma_1, \gamma_2\right) \not\equiv 0 \pmod{q}$.*

Proof. We can relate homology with $\mathbb{Z}$ and $\mathbb{Z}_q$ coefficients using the sequence

$$0 \to H_{d-2}\left(P;\ \mathbb{Z}\right) \xrightarrow{\times q} H_{d-2}\left(P;\ \mathbb{Z}\right) \xrightarrow{(\mathrm{mod}\ q)} H_{d-2}\left(P;\ \mathbb{Z}_q\right) \to 0 \tag{9}$$

which is exact in the sense that the image of each map is the kernel of the next. Exactness follows from the Universal Coefficient Theorem for Homology (Theorem 3A.3 in [Hat02]) using the properties of $\otimes \mathbb{Z}_q$, and the fact that $\mathrm{Tor}\left(H_{d-3}\left(P;\ \mathbb{Z}\right)\right) = 0$ (as $H_{d-3}\left(P;\ \mathbb{Z}\right) = 0$). More detail on this topics is included in Section A.2 in the appendix. (Equivalently, the Bockstein homomorphism $H_{d-2}\left(P;\ \mathbb{Z}_q\right) \to H_{d-2}\left(P;\ \mathbb{Z}\right)$ vanishes; see the beginning of Chapter 10 of [McC01]).

In words, exactness of the sequence is equivalent to the statement that $[\gamma_1] = 0$ in $H_{d-2}\left(P;\ \mathbb{Z}_q\right)$ if and only if there exists a $\gamma_3 \in Z_{d-2}\left(P;\ \mathbb{Z}\right)$ so that $[\gamma_1] = q\left[\gamma_3\right]$

in $H_{d-2}(P;\ \mathbb{Z})$. Thus, if $[\gamma_1] = 0$ in $H_{d-2}(P;\ \mathbb{Z}_q)$ then

$$l(\gamma_1, \gamma_2) = l(q\gamma_3, \gamma_2) = ql(\gamma_3, \gamma_2) \equiv 0 \pmod q$$

for all $\gamma_2 \in Z_1(Q';\ \mathbb{Z})$. On the other hand, if $l(\gamma_1, \gamma_2) \equiv 0 \pmod q$ for all simple closed loops γ_2 in Q' then there are integers $b_1, \ldots, b_n$ so that $l(\gamma_1, \alpha_i) = qb_i$ for $i = 1, \ldots, n$, for the simple cycles $\alpha_1, \ldots, \alpha_n$ constructed in Proposition 27. In particular, we have that $[\gamma_1] = qL^{-1}(b_1, \ldots, b_n)$ in $H_{d-2}(P;\ \mathbb{Z})$ so so $[\gamma_1] = 0$ in $H_{d-2}(P;\ \mathbb{Z}_q)$. □

We apply these results using the following two statements. The first is true for any i, but we state it for $i = d-1$.

Corollary 30. *Let $m \in \mathbb{N}$ and let γ be the boundary of a $(d-1)$-dimensional box r' of $\mathbb{Z}^d$. Then*

$$V_\gamma^{\text{fin}}(1) \subset V_\gamma^{\text{fin}}(m) \subset V_\gamma^{\text{inf}}(m)\ ,$$

where we recall the notation $V_\gamma^{\text{fin}}(1) = V_\gamma^{\text{fin}}(\mathbb{Z})$.

Proof. The first containment is an immediate consequence of (9), and the second follows from the definition of $V_\gamma^{\text{inf}}(m)$. □

Corollary 31. *Let P be a $(d-1)$-dimensional percolation subcomplex of $\mathbb{Z}^d$ and let $\gamma_1 = \partial r'$ be the boundary of a $(d-1)$-dimensional box r' of $\mathbb{Z}^d$.*

- *If there exists a simple loop γ_2 of Q so that $l(\gamma_1, \gamma_2) = \pm 1$ then $V_{\gamma_2}^{\text{inf}}(m)$ does not occur for any $m \in \mathbb{N}$.*
- *If there is a box r containing γ_1 so that $l(\gamma_1, \gamma_2) = 0$ for all simple closed loops γ_2 of $Q' := (Q \cap r^\bullet) \cup \partial r^\bullet$ then $V_\gamma^{\text{fin}}(m)$ occurs for every $m \in \mathbb{N}$.*

Proof. For the first statement, suppose that $V_\gamma^{\text{inf}}(m)$ occurs and let γ_2 be a simple loop of Q. Let $\Lambda_n := [-n, n]^d$ and denote by P_n the percolation subcomplex $(P \cap \Lambda_n) \cup \partial\Lambda_n$. We have that $[\gamma] = 0$ in $H_{d-2}(P_n;\ G)$ for all sufficiently large n. By choosing n large enough so that $\gamma_2 \subset \Lambda_n$, we can conclude that $l(\gamma_1, \gamma_2) \neq \pm 1$ by either Proposition 29 or Corollary 26.

The second statement follows from Corollary 28 and Corollary 30. □

By standard results, Corollaries 30 and 31 (and in fact Theorem 9) hold when homology coefficients are taken in the additive group G of a ring with unity. That is, we may state them for the events $V_\gamma^{\text{inf}}(G)$ and $V_\gamma^{\text{fin}}(G)$.

Part 2. Proof of the Deconfinement Transition

We now proceed to the proof of Theorem 9 for the plaquette random cluster model with coefficients in the rational numbers. As a consequence of Theorem 8 and Proposition 19, this suffices to demonstrate Theorem 4 on the deconfinement transition in Potts lattice gauge theory. The proof is divided into three parts.

In Section 7, we show that an area law upper bound holds for V_γ when $p < p^*\left(p_{\mathrm{slab}}(q)\right)$, meeting the trivial lower bound found by including all plaquettes in a minimal null-homology. Our work in the previous section has significantly simplified this task. Specifically, it suffices to construct a dual loop $\gamma^\bullet$ so that $l\left(\gamma, \gamma'\right) = 1$ with high enough probability, by Corollary 31. Moreover, as an application of Proposition 25, we only need consider the infinite volume PRCM constructed with wired boundary conditions. That is, we have reduced the proof to a question concerning the supercritical RCM with $\mathbf{f}$ boundary conditions. We will resolve it via a renormalization argument, as described below.

We provide two proofs of the perimeter law for the supercritical PRCM, in Sections 8 and 9. First, we construct a a null-homology for γ as the boundary of a union of components in the dual RCM. This provides a perimeter law lower bound, complementing the obvious upper bound. Our other proof is substantially more complex, but has the advantage of demonstrating that the existence of a sharp constant in the exponent of the perimeter law. Our strategy is to build a hypersurface of plaquettes in the PRCM which precludes the existence of a dual loop linking with γ. Another application of Corollary 31 then yields the desired result.

7. The Area Law Regime

Before getting into details, we sketch what we will accomplish in this section. As shown in Proposition 29, the events V_γ are obstructed by the existence of a dual loop $\gamma^\bullet$ so that $l\left(\gamma, \gamma^\bullet\right) = 1$. Recall that γ is supported on the boundary of a $(d-1)$-dimensional box r_γ, and assume without loss of generality that r_γ has nonzero thickness in all but the first coordinate. We construct $\gamma^\bullet$ within a slab: a thickened copy of $\mathbb{Z}^2$ that is infinite in the first coordinate and one of the other $d-1$ coordinates. Then, γ meets a suitably chosen slab in two "pins" orthogonal to its infinite dimensions. We find $\gamma^\bullet$ by constructing constructing a loop within the slab that winds around one of the "pins" but not the other. Towards that end, we create a supercritical renormalized site system with vertex set $\mathbb{Z}^2$, and show that it contains loops separating distant vertices with high probability. Considering a maximal set of disjoint slabs which intersect γ in two "pins" yields an area law upper bound.

It suffices to prove an area law upper bound for the infinite volume PRCM with wired boundary conditions. Here, we will work almost exclusively with the dual free RCM. For convenience, we will sometimes write $\mu^{\bullet}_{X,p^*} = \mu^{\mathbf{f}}_{X,p^*(p),q,1}$ for this measure when the parameters are understood.

Let $\mathcal{S}_M = \mathbb{R}^2 \times [-M+1/2, M+1/2]^{d-2}$ be the dual slab of thickness M. As in Section 5.2, we can define the random-cluster model on the slab as the weak limit

$$\mu^{\bullet}_{\mathcal{S}_M,p^*,q,1} := \lim_{n\to\infty} \mu^{\mathbf{f}^{\bullet}}_{(\Lambda_n\cap\mathcal{S}_M),p^*,q,1}\,.$$

In the remainder, we will assume that $p^* > p_c(\mathcal{S}_M, q)$, where $p_c(\mathcal{S}_M, q)$ is the critical probability for the classical random-cluster model with parameter q on $\mathcal{S}_M$. Our strategy will be to construct a renormalized system of cells that faithfully represents the connectivity of the original supercritical slab RCM, and stochastically dominates an independent site percolation whose activation probability is arbitrarily close to one. While the outline is inspired by that of the proof of the area law in [ACC+83], we will use a different renormalization scheme related to those constructed by [Bod05].

Before proceeding, we recall a standard tool in showing stochastic domination of measures. A family of real valued random variables $(X_i)_{i\in I}$ stochastically dominates another family $(Y_i)_{i\in I}$ if there is a coupling so that $X_i \le Y_i$ for each $i \in I$ almost surely. In this case we write $(X_i)_{i\in I} \le_{\mathrm{st}} (Y_i)_{i\in I}$.

Theorem 32 (Holley's Inequality)**.** *Let I be a finite set and let $(X_i)_{i\in I}, (Y_i)_{i\in I} \in \{0,1\}^I$ be distributed according to strictly positive probability measures μ_1 and μ_2. Suppose that for each pair $(W_i)_{i\in I}, (Z_i)_{i\in I} \in \{0,1\}^I$ with $W_i \le Z_i$ for each $j \in I$,*

$$\begin{aligned}\mu_1\left(X_j = 1 : X_i = W_i \text{ for all } i \in I\setminus\{j\}\right)\\ \le \mu_2\left(Y_j = 1 : Y_i = Z_i \text{ for all } i \in I\setminus\{j\}\right).\end{aligned}$$

Then $\mu_1 \le_{\mathrm{st}} \mu_2$.

We will often use the following statement. In particular, it allows us to combine upper bounds for the the probabilities of decreasing events occurring in disjoint slabs with free boundary conditions.

Lemma 33. *Let $X_1, X_2 \subset X$ be finite edge-disjoint subcomplexes of $\mathbb{Z}^d$. If A_1, A_2 are events that depend only on the edges of X_1 and X_2 respectively and*

A_1 is increasing, then

$$\mu^{\mathbf{f}}_{X,p^*}(A_1|A_2) \geq \mu^{\mathbf{f}}_{X_1,p^*}(A_1) \ . \tag{10}$$

In particular, if A_2 is also increasing then

$$\mu^{\mathbf{f}}_{X,p^*}(A_1^c \cap A_2^c) \leq \mu^{\mathbf{f}}_{X_1,p^*}(A_1^c)\, \mu^{\mathbf{f}}_{X_2,p^*}(A_2^c) \ .$$

Proof. Let B_2 be the event that no edge from the edge boundary of X_2 is included in P. Then by Theorem 32,

$$\mu^{\bullet}_{X,p^*}(A_1|A_2) \geq \mu^{\bullet}_{X,p^*}(A_1|B_2) \geq \mu^{\bullet}_{X_1,p^*}(A_1) \ .$$

To obtain the second statement, we compute

$$\begin{aligned}\mu^{\mathbf{f}}_{X,p^*}(A_1^c \cap A_2^c) =& \mu^{\mathbf{f}}_{X,p^*}(A_1^c|A_2^c)\, \mu^{\mathbf{f}}_{X,p^*}(A_2^c) \\ \leq& \mu^{\mathbf{f}}_{X_1,p^*}(A_1^c)\, \mu^{\mathbf{f}}_{X,p^*}(A_2^c) \\ \leq& \mu^{\mathbf{f}}_{X_1,p^*}(A_1^c)\, \mu^{\mathbf{f}}_{X_2,p^*}(A_2^c)\end{aligned}$$

where the second line follows from the first by applying (10) to A_1 and A_2^c and taking complements. □

7.1. **The Renormalized Construction.** Bodineau [Bod05] defined two renormalization constructions for the random-cluster model in slabs, in the regime $p^* > p_{\mathrm{slab}}(q)$. We take inspiration from the second, which consists of a system of renormalized sites corresponding to $\mathbb{Z}^2$. It is shown that sites are successively traversed by paths connecting each of their faces with high probability, leading to percolation. The construction is designed to work without sprinkling, at the cost of not faithfully representing loops in the underlying bond system. Our strategy involves the construction of such a loop, so we will build a different renormalized system that takes places in the same system of sites, but only uses crossing estimates implied by the results of [Bod05].

For parameters $M, N \in \mathbb{Z}$, define a system $\mathbb{V}$ of cells as the collection of translates of $[-N+1/2, N+1/2]^2 \times [-M+1/2, M+1/2]^{d-2}$ by vectors in $\mathrm{span}(2N\boldsymbol{e}_1, 2N\boldsymbol{e}_2)$. Identify $\mathbb{V}$ with $\mathbb{Z}^2$.

For a box r, let $R_j(r)$ be the event that there is a crossing of r in the $\mathbf{e}_j$ direction. When the crossing is in the direction of the unique longest dimension of r, we will sometimes write $R(r)$ instead. The following statement is a partial summary of section 5 of [Bod05], where both properties we specify follow from inequality (5.4) of that paper.

Theorem 34 (Bodineau)**.** *Let $p^* > p_{\mathrm{slab}}(q)$. For a renormalized site $\mathbb{S} \in \mathbb{V}$, let $\mathcal{F}(\mathbb{S})$ be the event that there is a connected set of bonds witnessing both $R_1(\mathbb{S})$*

and $R_2(\mathbb{S})$. *Then for any* $\rho < 1$ *there exist parameters* $M, N \in \mathbb{Z}$ *so that*

$$\mu^{\bullet}_{\mathbb{S},p^*}(\mathcal{F}(\mathbb{S})) \geq \rho\,.$$

Moreover, if $\mathbb{S}$ *and* $\mathbb{S}'$ *are adjacent,*

$$\mu^{\bullet}_{\mathbb{S}\cup\mathbb{S}',p^*}(R(\mathbb{S}\cup\mathbb{S}')) \geq \rho^2\,.$$

We will construct our renormalized system by finding connected horizontal and vertical crossings with high probability from Theorem 34, and then connecting adjacent sites via a path in a different slab that is then connected to these crossings via sprinkling.

Fix $p_{\mathrm{slab}}(q) < p^* < s^*$. For a box $r \subset \mathcal{S}_M$, set $\hat{r} := r + 2M\mathbf{e}_3$ and $\tilde{r} := r \cup \hat{r}$. One should think of r as a renormalized site or a union of renormalized sites, and $\hat{r}$ as the translation of r into the second slab. We will build connections in two steps, first finding crossings of renormalized sites in $\mathcal{S}_M$ at parameter p^* then increasing to s^* and connecting crossings in neighboring renormalized sites by finding overlapping crossings in $\mathcal{S}_M + 2M\mathbf{e}_3$ and sprinkling. Consider the random graphs Q and Q' given by $\mu^{\bullet}_{\tilde{\mathcal{S}}_M,p^*}$ and $\mu^{\bullet}_{\tilde{\mathcal{S}}_M,s^*}$ respectively. There is a coupling $\kappa^{\bullet}_{p^*,s^*}$ between Q and Q' so that $Q \subset Q'$ almost surely by Theorem 3.21 of [Gri06]. We then say that a set of renormalized sites $\{\mathbb{S}_k\}_{k\in J}$ is *faithful* in Q' if there is a set of vertices $\{v_k\}$ of the underlying lattice so that $v_k \in \mathbb{S}_k$ and for any adjacent pair $\mathbb{S}_k, \mathbb{S}_{k'}$, with $k, k' \in J$,

$$v_k \xleftrightarrow[Q'\cap\left(\tilde{\mathbb{S}}_k\cup\tilde{\mathbb{S}}_{k'}\right)]{} v_{k'}\,.$$

We now give the construction of the random set $O = O(p^*, s^*) \subset \mathbb{V}$. To begin with, we reveal $Q \cap \mathbb{V}$. We then say that a site $\mathbb{S}$ is *full* if $\mathcal{F}(\mathbb{S})$ holds (in Q). In this case we fix a witness $E(\mathbb{S})$ for that event. Otherwise, we say $\mathbb{S}$ is *empty.*

We now add the additional connections. For a box r and two subgraphs $E_1, E_2 \subset r$ define

$$C(E_1, E_2, r) := \left\{E_1 \xleftrightarrow[r\cap Q']{} E_2\right\}\,.$$

Enumerate the neighbors of a full site $\mathbb{S}_i$ as $\left(\mathbb{S}_{i_j}\right)$. Then we say that $\mathbb{S}_i$ is *open* if for each $\mathbb{S}_{i_j}$, $C\left(E(\mathbb{S}_i), E\left(\mathbb{S}_{i_j}\right), \tilde{\mathbb{S}}_i \cup \tilde{\mathbb{S}}_{i_j}\right)$ holds, otherwise $\mathbb{S}_i$ is *closed.* See Figure 2. Write $\mathcal{O}(\mathbb{S}_i)$ for the event that $\mathbb{S}_i$ is open. Empty sites are always declared to be closed. Then we define O to be the set of open sites.

In order to compare O with a Bernoulli site percolation, we will use a general theorem on stochastic domination found in [LSS97]. Let $B_k(v)$ denote the set of vertices of graph distance at most k from v.

FIGURE 2. (A) A full renormalized site contains a designated set of microscopic bonds which is connected and crosses the site in both horizontal and vertical directions. (B) Such a site is declared open if its designated bond set is connected with that of each of its full neighbors. We construct open sites by finding rectangle crossings in an adjacent slab, and adding "pins" of bonds at overlap points.

Theorem 35 (Liggett, Schonmann, Stacey)**.** *Let $\tilde{p} < 1$ and $k \in \mathbb{N}$. Also, let $(X_v)_{v \in \mathbb{Z}^d}$ be random variables taking values in $\{0, 1\}$. Then there is a $\rho = \rho(\tilde{p}, k)$ so that if*

$$\mathbb{P}(X_v = 1 | A) \geq \rho$$

for all $v \in \mathbb{Z}^d$ and all events $A \in \sigma\left((X_w)_{w \in \mathbb{Z}^d \setminus B_k(v)}\right)$,

$$(X_v)_{v \in \mathbb{Z}^d} \geq_{\mathrm{st}} \mathrm{Ber}(\tilde{p})^{\mathbb{Z}^d} .$$

We also use Theorem 3.45 of [Gri06] comparing the probabilities of events with the probability of their neighborhoods in the Hamming cube. For an event A, let the random variable $\mathcal{H}(A)$ be the Hamming distance from the event A (i.e. the minimum number of bonds whose states would need to be changed to reach a configuration in A).

Theorem 36 (Grimmett [Gri06])**.** *Let G be a graph and $0 < p^* < s^* < 1$. For any nonempty, decreasing event A,*

$$\mu^{\bullet}_{G,p^*}(A) \geq \left(\frac{s^* - p^*}{qs^*}\right)^k \mu^{\bullet}_{G,s^*}(\mathcal{H}(A) \leq k) .$$

For clarity, we include a rearrangement of this inequality as a corollary.

Corollary 37. *Let G be a graph and let $0 < p^* < s^* < 1$ and let B be an increasing event so that the complementary event $\neg B$ is nonempty. Then*

$$\mu^{\bullet}_{G,s^*}\left(\mathcal{H}\left(\neg B\right) > k\right) \geq 1 - \left(\frac{qs^*}{s^* - p^*}\right)^k \left(1 - \mu^{\bullet}_{G,p^*}\left(B\right)\right) .$$

Next we prove a lemma inspired by Proposition 4.6 of [ACC+83] that connects overlapping crossings in Q by sprinkling additional edges from Q'.

Lemma 38. *Fix $0 < p^* < s^* < 1$ and fix $M \in \mathbb{N}$. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be adjacent cells of $\mathbb{V}$ and set $r_{1,2} = \mathbb{S}_1 \cup \mathbb{S}_2$.*

Then there exists a function $f_{p^*,s^*} : [0,1) \to [0,1]$ satisfying

$$\lim_{x \to 1} f_{p^*,s^*}\left(x\right) = 1$$

so that for any $N \in \mathbb{N}$, any configuration $Q_{1,2}$ of edges in $Q \cap r_{1,2}$, and any $E_1, E_2 \subset Q_{1,2}$ witnessing $\mathcal{F}\left(\mathbb{S}_1\right)$ and $\mathcal{F}\left(\mathbb{S}_2\right)$ respectively,

$$\kappa^{\bullet}_{\tilde{r}_{1,2},p^*,s^*}\left(C\left(E_1, E_2, \tilde{r}_{1,2}\right) | Q \cap r_{1,2} = Q_{1,2}\right) \geq f_{p^*,s^*}\left(\mu^{\bullet}_{\hat{r}_{1,2},p^*}\left(R\left(\hat{r}_{1,2}\right)\right)\right) .$$

Proof. Let $\pi_{1,2}$ denote the projection onto the first two coordinates of $\mathbb{R}^d$. The images under $\pi_{1,2}$ of E_1 and of any witness for $R\left(\hat{r}_{1,2}\right)$ must intersect. Thus, it suffices to add at most $(2M)^{d-2}$ additional bonds in order to connect them. The same is true for a witness for $R\left(\hat{r}_{1,2}\right)$ and E_2. The desired limit statement will follow from the fact that when $R\left(\hat{r}_{1,2}\right)$ occurs with high probability, there must be many such points of overlap.

For any vertex $v \in \mathbb{Z}^2 \times \{0\}^{d-2}$, we define the pin at v to be the subgraph in $\mathbb{Z}^d$ contained in $\pi^{-1}_{1,2}\left(v\right)$. Write $\mathcal{P}\left(v\right)$ for the event that all bonds in the pin at v are included in Q'. Notice that the set of sites in a pin can be disconnected from the rest of $\mathcal{S}_M$ with the removal of at most $4dM^{d-2}$ bonds.

Until the very end of the proof, we fix $L \in \mathbb{N}$ and condition on the event $\left\{\mathcal{H}\left(\neg R_d\left(\hat{r}_{1,2}\right)\right) > L\right\}$. In this case, the crossing $R_d\left(\hat{r}_{1,2}\right)$ happens even if we remove any set of L bonds. Now, let $\mathcal{V}$ be a maximal set of pairs of sites of $\mathbb{V}$ so that

$$C\left(E_1, E_2, \tilde{r}_{1,2}\right) \subset \mathcal{P}\left(v_1\right) \cap \mathcal{P}\left(v_2\right)$$

where each site can appear in at most one pair. Any witness for $R_d\left(\hat{r}_{1,2}\right)$ must contain a pair of such sites. Thus, if we remove all bonds appearing in the pins corresponding to each vertex in each pair of $\mathcal{V}$, $R_d(\hat{r}_{1,2})$ cannot occur. It follows that

$$\mathcal{H}(\neg R_d(\hat{r}_{1,2})) \leq 8dM^{d-2}|\mathcal{V}|$$

and

$$|\mathcal{V}| > \frac{L}{8dM^{d-2}}.$$

Now we consider the probability of an edge $e \in \tilde{r}_{1,2}$ being included in Q', conditional on $Q_{1,2}$. From the definition of the random-cluster model, this probability is minimized when e is in $r_{1,2}$, not in Q, and the addition of e reduces the number of connected components of Q. This gives a lower bound

$$\min_{e,Q_{1,2}} \kappa^{\bullet}_{\tilde{r}_{1,2},p^*,s^*}(e \in Q' | Q \cap r_{1,2} = Q_{1,2}) \geq \frac{s^* - p^*}{s^* - p^* + q(1 - s^* + p^*)}.$$

For convenience, denote the quantity on the right by $g(q, s^*, p^*)$.

Then we can compute

$$\kappa^{\bullet}_{\tilde{r}_{1,2},p^*,s^*}(\mathcal{P}(v)) \geq g(s^*, p^*)^{(2M)^{d-2}}.$$

Therefore, by the FKG inequality and a union bound, the probability that there are a pair of pins contained in Q' is at a minimum

$$\delta(p^*, s^*, M, L) := \left[1 - \left(1 - g(s^*, p^*)^{(2M)^{d-2}}\right)^{L/\left(8dM^{d-2}\right)}\right]^2.$$

Then using Theorem 36 and FKG we can compute

$$\begin{aligned}
&\kappa^{\bullet}_{\tilde{r}_{1,2},p^*,s^*}(C(E_1, E_2, \tilde{r}_{1,2}) | Q \cap r_{1,2} = Q_{1,2}) \\
&\quad \geq \delta(p^*, s^*, M, L)\, \mu^{\bullet}_{\tilde{r}_{1,2},s^*}(\mathcal{H}(\neg R_d(\hat{r}_{1,2}) > L) | Q \cap r_{1,2} = Q_{1,2}) \\
&\quad \geq \delta(p^*, s^*, M, L)\, \mu^{\bullet}_{\hat{r}_{1,2},s^*}(\mathcal{H}(\neg R_d(\hat{r}_{1,2})) > L) \\
&\quad \geq \delta(p^*, s^*, M, L) \left[1 - \left(\frac{qs^*}{s^* - p^*}\right)^L \left(1 - \mu^{\bullet}_{\hat{r}_{1,2},s^*}(R_d(\hat{r}_{1,2}))\right)\right] \\
&\quad := f'_{p^*,s^*,L}\left(\mu^{\bullet}_{\hat{r}_{1,2},p^*}(R_d(\hat{r}_{1,2}))\right),
\end{aligned}$$

where we use Lemma 33 to change the domain of the random-cluster model between lines 2 and 3. Now it suffices to find a function $L(x)$ so that

$$f_{p^*,s^*}(x) := \max\left(f'_{p^*,s^*,L(x)}(x), 0\right) \to 1$$

as $x \to 1$. Since $\delta \to 1$ as $L \to \infty$, the only constraint is that $L(x)$ must grow sufficiently slowly so that $c^{L(x)}(1-x) \to 0$ where

$$c = \frac{qs^*}{s^* - p^*} > 1\,.$$

For example, if we choose $L(x) = \lfloor \log\log(1/(1-x)) \rfloor$ then f has the required properties. □

Theorem 39. *Let $p_{\text{slab}}(q) < p^* < s^*$. Let $\tilde{p} < 1$ and let F be the open sites of Bernoulli site percolation on $\mathbb{V}$ with parameter $\tilde{p}$. Then there are $M, N \in \mathbb{Z}$ so that $O = O(p^*, s^*)$ is almost surely faithful and satisfies*

$$O \geq_{\text{st}} F\,.$$

Proof. By construction, if $\mathbb{S}_i$ and $\mathbb{S}_j$ are adjacent and open, the union of the connected sets $E(\mathbb{S}_i)$ and $E(\mathbb{S}_j)$ must be connected in $Q' \cap \left(\tilde{\mathbb{S}}_i \cup \tilde{\mathbb{S}}_j\right)$, so O is faithful.

We now analyze the probability that a site is declared open. Let $\rho = \rho(\tilde{p}, 1)$ be as in Theorem 35 and $\rho < \rho' < 1$ to be determined later. By Theorem 34 we may choose N, M large enough so that

$$\mu^{\bullet}_{\mathbb{S},p^*}(\mathcal{F}(\mathbb{S})) \geq \rho'$$

and

$$\mu^{\bullet}_{\mathbb{S}\cup\mathbb{S}',p^*}(R(\mathbb{S}\cup\mathbb{S}')) \geq (\rho')^2$$

for adjacent sites $\mathbb{S}$ and $\mathbb{S}'$.

For convenience, set $S_{i,j} = \mathbb{S}_i \cup \mathbb{S}_j$ and $\mathcal{C}_{i,j} = C\left(E(\mathbb{S}_i), E(\mathbb{S}_j), \tilde{S}_{i,j}\right)$. Let U_i denote the union of $\mathbb{S}_i$ and its neighboring sites and let U_i^{full} denote the union of $\mathbb{S}_i$ and its full neighbors. As a preliminary computation, we have for full, adjacent sites $\mathbb{S}_i, \mathbb{S}_j$ that

$$\begin{aligned}\kappa^{\bullet}_{\tilde{U}_i,p^*,s^*}(\mathcal{C}_{i,j} | Q \cap U_i = Q_i) \geq &\kappa^{\bullet}_{\tilde{S}_{i,j},p^*,s^*}(\mathcal{C}_{i,j} | Q \cap S_{i,j} = Q_i \cap S_{i,j}) \\ \geq &f_{p^*,s^*}\left((\rho')^2\right)\end{aligned}$$

by Lemmas 33 and 38 and Theorem 34. Strictly speaking Lemma 33 is stated for μ instead of κ, but its proof only requires that the measure satisfy positive association, which κ also does.

We can now calculate the probability that a renormalized site is open given that it is full. There is a minor subtlety in that adding edges may change

$E(\mathbb{S}_i)$, so the event $\mathcal{O}(\mathbb{S}_i)$ is not monotone. We will therefore need to take the infimum over possible boundary conditions. A union bound gives

$$\begin{aligned}\inf_{\xi^\bullet} \kappa^{\xi^\bullet}_{\tilde{U}_i,p^*,s^*}\left(\mathcal{O}(\mathbb{S}_i)|\mathcal{F}(\mathbb{S}_i)\right) \geq & \min_{Q:Q_i\cap\mathbb{S}_i\in\mathcal{F}(\mathbb{S}_i)} \kappa^{\bullet}_{\tilde{U}_i,p^*,s^*}\left(\bigcap_{\mathbb{S}_j\in U_i^{\mathrm{full}}} \mathcal{C}_{i,j} \middle| Q\cap U_i = Q_i\right) \\ \geq & 1-\left(1-4f_{p^*,s^*}\left((\rho')^2\right)\right).\end{aligned}$$

Combining this result with Theorem 34, we have for any event

$$A \in \sigma\left(\{\mathcal{O}(\mathbb{S})\}_{\mathbb{S}\in\mathbb{V}\setminus B_1(\mathbb{S}_i)}\right)$$

that

$$\begin{aligned}\kappa^{\mathbf{f}}_{\tilde{\mathcal{S}}_M,p^*,s^*}\left(\mathcal{O}(\mathbb{S}_i)|A\right) \geq & \inf_{\xi^\bullet} \kappa^{\xi^\bullet}_{\tilde{\mathcal{S}}_M,p^*,s^*}\left(\mathcal{O}(\mathbb{S}_i)|A\right) \\ \geq & \inf_{\xi^\bullet} \kappa^{\xi^\bullet}_{\tilde{U}_i,p^*,s^*}\left(\mathcal{O}(\mathbb{S}_i)\right) \\ \geq & \mu^{\bullet}_{\mathbb{S}_i,p}\left(\mathcal{F}(\mathbb{S}_i)\right) \inf_{\xi} \kappa^{\xi}_{\tilde{U}_i,p^*,s^*}\left(\mathcal{O}(\mathbb{S}_i)|\mathcal{F}(\mathbb{S}_i)\right) \\ \geq & \rho'\left(1-\left(1-4f_{p^*,s^*}\left((\rho')^2\right)\right)\right) \\ \geq & \rho\end{aligned}$$

for ρ' sufficiently close to 1. Then applying Theorem 35 gives the desired stochastic domination.

□

7.2. **Proof of the Area Law.** The remainder of our argument will be similar to that given in section 4iii of [ACC+83]. Our strategy will be to construct a loop in a thickened slab $\tilde{\mathcal{S}}_M$ with suitably high probability that prevents V_γ from occurring. Without loss of generality, assume that γ is the boundary of a box that lies in the hyperplane $\{x_1 = 0\}$. Then the intersection of the slab and γ is two pins contained in sites $\tilde{\mathbb{S}}$ and $\tilde{\mathbb{S}}'$. Let $\Gamma_{\mathbb{V}}(\mathbb{S},\mathbb{S}')$ be the event that there is a loop in O that separates $\tilde{\mathbb{S}}$ and $\tilde{\mathbb{S}}'$.

We prove that that this event precludes V_γ. Since $V_\gamma^{\mathrm{fin}} \subset V_\gamma^{\mathrm{inf}}$, it is enough to consider the latter.

Lemma 40. *For any $q' \in \mathbb{N}$*

$$V_\gamma^{\mathrm{inf}}(q') \subset \neg\Gamma_{\mathbb{V}}(\mathbb{S},\mathbb{S}').$$

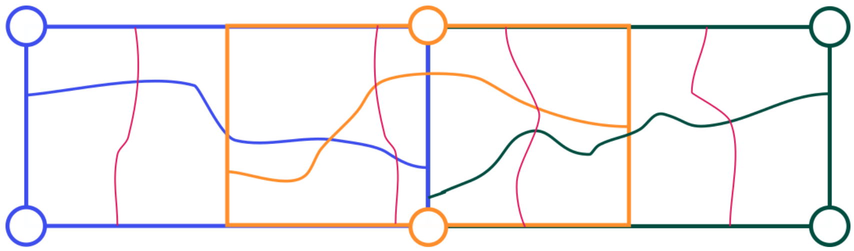


FIGURE 3. The event $\mathcal{R}_1(4a,a)$ is implied by three translates of the event $\mathcal{R}_1(2a,a)$ and four translates of the event $\mathcal{R}_2(a,a)$.

Proof. Let $\{\mathbb{S}_{i_1},\ldots,\mathbb{S}_{i_k}\}$ be a loop of open renormalized sites witnessing the event $\Gamma_{\mathbb{V}}(\mathbb{S},\mathbb{S}')$ (where $\mathbb{S}_{i_k}$ is adjacent to $\mathbb{S}_{i_1}$. We may assume without loss of generality that γ intersects the plane $\mathbb{R}^2\times\{1/2\}^{d-2}$ in the points $\{0\}\times\{0\}\times\{1/2\}^{d-2}$ and $\{0\}\times\{N\}\times\{1/2\}^{d-2}$, and that the origin is in the bounded component of the complement of $\{\mathbb{S}_{i_1},\ldots,\mathbb{S}_{i_k}\}$.

By construction, there is a simple loop $\gamma^\bullet$ of Q formed by concatenating paths contained in consecutive renormalized sites $\mathbb{S}_{i_j}$. We may continuously deform $\gamma^\bullet$ in $\Lambda\setminus\gamma$ to produce a simple loop contained inside $\mathbb{R}^2\times\{1/2\}^{d-2}$. This transformation may be performed first by moving the points of intersection with the boundaries of the renormalized cells into the plane, then deforming the path inside each cell. As the transformed $\gamma^\bullet$ surrounds the origin in the plane, its winding number with that point is ±1. It does not surround $[0,N]\times\{1/2\}^{d-2}$, so its winding number with that point is zero.

Therefore, as homotopy classes of loops in the twice punctured plane are determined by these winding numbers, we may continuously deform $\gamma^\bullet$ again so that it coincides with the circle of radius $N/2$ centered at the origin. This is easily seen to be a generator of $H_1\left(S^d\setminus\gamma;\ \mathbb{Z}\right)$ ($(S^d\setminus\partial\gamma$ deformation retracts to it). Thus, $l(\gamma^\bullet,\gamma)=\pm1$ and $V_\gamma^{\inf}$ does not occur by Corollary 31, $\square$

To bound the probability of $\neg\Gamma_{\mathbb{V}}(\mathbb{S},\mathbb{S}')$, we begin by showing that we have crossings of large boxes with high probability. As a consequence of Theorem 39, it suffices to consider Bernoulli site percolation. For a,b both even, let $r(a,b)=[0,a]\times[0,b]$ and let $\mathcal{R}_i(a,b)$ denote the event that there is a path of open sites crossing $r(a,b)$ in the i-th coordinate direction that avoids both the corners and midpoints of the faces of $r(a,b)$.

Proposition 41. *Let $a\in\mathbb{N}$, let $0<p,\lambda<1$, and let β be the real root of $16x^3+24x^2+9x-1$ (so $\beta\approx0.089$.) If*

$$\mathbb{P}_p(\mathcal{R}_1(2a,a))\geq 1-\beta\lambda,$$

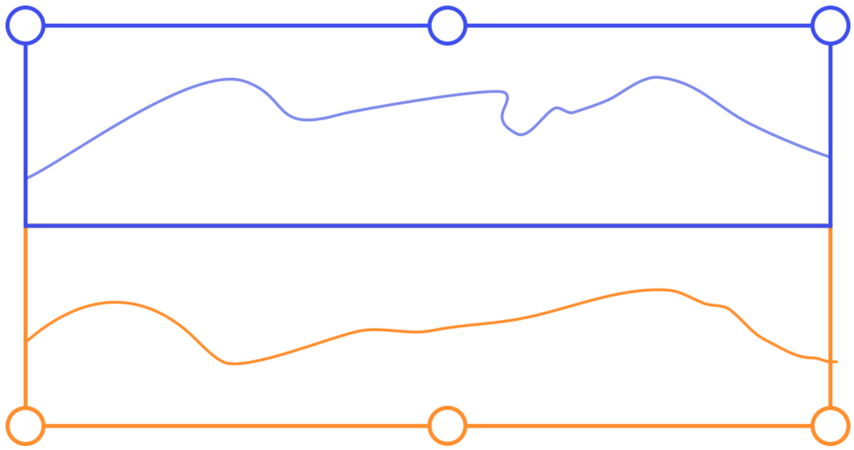


FIGURE 4. A horizontal crossing of either thin rectangle implies one for their union.

and

$$\mathbb{P}_p\left(\mathcal{R}_2\left(a,a\right)\right) \geq 1-\beta^2\lambda$$

then

$$\mathbb{P}_p\left(\mathcal{R}_1\left(4a,2a\right)\right) \geq 1-\beta\lambda^2$$

and

$$\mathbb{P}_p\left(\mathcal{R}_2\left(2a,2a\right)\right) \geq 1-\beta^2\lambda^2 .$$

Proof. As illustrated in Figure 3, the event $\mathcal{R}_1\left(4a,a\right)$ is implied by three translates of the event $\mathcal{R}_1\left(2a,a\right)$ and four translates of the event $\mathcal{R}_2\left(a,a\right)$. Then from a union bound we have

$$\mathbb{P}_p\left(\mathcal{R}_1\left(4a,a\right)\right) \geq 1-3\mathbb{P}_p\left(\neg\mathcal{R}_1\left(2a,a\right)\right)-4\mathbb{P}_p\left(\neg\mathcal{R}_2\left(a,a\right)\right) \geq 1-3\beta\lambda-4\beta^2\lambda .$$

Next, $r\left(4a,2a\right)$ equals the disjoint union of two copies of $r\left(4a,a\right)$. A horizontal crossing in either of these rectangles would imply one for the thicker rectangle, as depicted in Figure 4. Thus,

$$\mathbb{P}_p\left(\mathcal{R}_1\left(4a,2a\right)\right) \geq 1-\mathbb{P}_p\left(\neg\mathcal{R}_1\left(4a,a\right)\right)^2 \geq 1-\left(3\beta+4\beta^2\right)^2\lambda^2 = 1-\beta\lambda^2 .$$

Finally, $r\left(2a,2a\right)$ is the disjoint union of two translated copies $r\left(a,2a\right)$ so a similar argument yields

$$\mathbb{P}_p\left(\mathcal{R}_2\left(2a,2a\right)\right) \geq 1-\mathbb{P}_p\left(\neg\mathcal{R}_2\left(a,2a\right)\right)^2 \geq 1-\beta^2\lambda^2 .$$

□

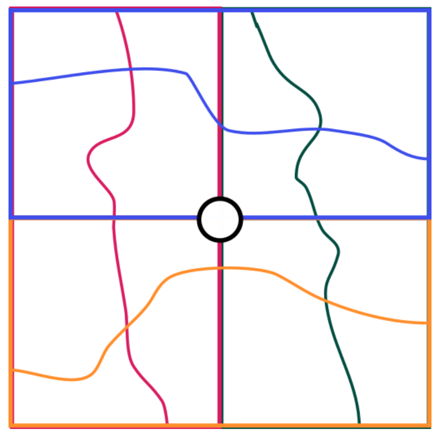


FIGURE 5. The event $A(l)$ is implied by the intersection of four events of the form $R\left(2^{l+1}, 2^{l}\right)$.

Corollary 42. *There is a $\tilde{p}$ so that for all $a \in \mathbb{N}$,*

$$\mathbb{P}_{\tilde{p}}\left(\mathcal{R}_1\left(2^{a+1}, 2^{a}\right)\right) \geq 1 - \beta\lambda^{2^{a-2}}.$$

We now show that $\Gamma_{\mathbb{V}}(\mathbb{S}, \mathbb{S}')$ occurs with sufficiently high probability. In the following lemma we revert to using p^* in the role for which we have previously been using s^* (so there is an implicit parameter $p^k_{\text{slab}}(q) < t^* < p^*$ from the previous argument, but this will not be used from here forward).

Lemma 43. *Let $p^* > p^k_{slab}(q)$. Then for sufficiently large M, N there is a constant $\zeta > 0$ so that for each pair $\mathbb{S}, \mathbb{S}' \in \mathbb{V}$,*

$$\mu^{\bullet}_{\tilde{\mathcal{S}}_M \cap \Lambda_L, p^*}\left(\Gamma_{\mathbb{V}}(\mathbb{S}, \mathbb{S}')\right) \geq 1 - \exp\left(-\zeta\left|\mathbb{S} - \mathbb{S}'\right|_{\infty}\right),$$

where $L = L(\mathbb{S}, \mathbb{S}') = \max\left\{2\left|\mathbb{S}\right|_{\infty}, 2\left|\mathbb{S}'\right|_{\infty}\right\}$.

Proof. Consider Bernoulli site percolation F in $\mathbb{Z}^2$ with parameter $\tilde{p}$ as in Corollary 42. Let $\Gamma(v, w)$ be the event that there is an open path separating v from w. By Theorem 39, it is enough to show that for sufficiently large $\tilde{p}$, there is a ζ' so that

$$\mathbb{P}_{\tilde{p}}\left(\Gamma(v, w)\right) \geq 1 - \exp\left(-\zeta'\left|v - w\right|_{\infty}\right)$$

for every $v, w \in \mathbb{Z}^2$.

Let $\mathcal{A}(l)$ be the event that there are four translations of $\mathcal{R}_1\left(2^{l+1}, 2^{l}\right)$ rotated by $\pi/2$ radians and arranged in a square as illustrated in Figure 5, centered at v. Notice that if $l \leq \log_2\left(\left|v - w\right|_{\infty}\right)$, $\mathcal{A}(l)$ implies $\Gamma(v, w)$. Then by Theorem 39 and Corollary 42, we have

$$
\begin{aligned}
\mathbb{P}_{\tilde{p}}\left(\Gamma\left(v,w\right)\right) &\geq \mathbb{P}_{\tilde{p}}\left(\mathcal{A}\left(l\right)\right)\\
&\geq \left(1-\beta\lambda^{2^{l-2}}\right)^{4}\\
&\geq \left(1-\beta\lambda^{|v-w|_{\infty}/8}\right)^{4}\\
&\geq 1-\exp\left(\zeta'\left|v-w\right|_{\infty}\right)
\end{aligned}
$$

for some $\zeta' > 0$.

$\square$

Proof of the Area Law. Suppose without loss of generality that

$$\gamma = \partial r_\gamma \coloneqq \partial\left(\{0\}\times[0,n_2]\times\ldots\times[0,n_d]\right)\,.$$

Let

$$\left\{\tilde{\mathcal{S}}_M^l\right\}_{l=1}^{n} = \left\{\tilde{\mathcal{S}}_M + (0,0,4l_3M,2l_4M,\ldots,2l_dM) : l_jM < n_j \text{ for } 3\leq j\leq d\right\}$$

be an enumeration of thickened slab translates overlapping with r_γ. Note that

$$n = \prod_{j=3}^{d}\lfloor\frac{n_i}{M}\rfloor$$

so

$$\mathrm{Area}\left(\gamma\right) = \mathrm{Vol}_{d-1}\left(r_\gamma\right) = 2^{d-1}M^{d-2}n_2n + o\left(n_2n\right)\,.$$

For each l, $\partial r_\gamma \cap \tilde{\mathcal{S}}_M^l$ is a pair of pins of the form $\{0,0\}\times\boldsymbol{x}$ and $\{0,n_2\}\times\boldsymbol{x}$ for $\boldsymbol{x}\in\mathbb{Z}^{d-2}$. In particular, the renormalized sites $\tilde{\mathbb{S}}_1^l$ and $\tilde{\mathbb{S}}_2^l$ containing each pin are at distance at least $\lfloor\frac{n_2}{N}\rfloor$ in $\mathbb{V}$. Let Γ^l be the event that $\tilde{\mathbb{S}}_1^l$ and $\tilde{\mathbb{S}}_2^l$ are separated by a renormalized loop in the slab $\tilde{\boldsymbol{\mathcal{S}}}_M^l$ as in Lemma 43. By Lemma 40,

$$V_\gamma^{\inf} \subset \bigcap_{1\leq l\leq n}\neg\Gamma^l\,.$$

Let $\Lambda\left(\gamma\right) = \Lambda_{\max\{2n_j:2\leq j\leq d\}}$. We choose this cube in order to have a finite set containing the construction of Lemma 43 in each slab. Applying the second part of Lemma 33 to the decreasing events Γ^l yields that

$$\mu^{\bullet}_{\mathbb{Z}^d,p^*,q,1}\left(\bigcap_{l=1}^{n}\neg\Gamma^l\right) \leq \mu^{\bullet}_{\Lambda(\gamma),p^*,q,1}\left(\bigcap_{l=1}^{n}\neg\Gamma^l\right) \leq \prod_{l=1}^{n}\mu^{\bullet}_{\tilde{\mathcal{S}}^l\cap\Lambda(\gamma),p,q,1}\left(\neg\Gamma^l\right)\,.$$

Finally, by Lemma 43 there is a $\zeta > 0$ so that

$$\begin{aligned}\mu^{\mathbf{w}}_{\mathbb{Z}^d,p,q,d-1}\left(V^{\mathrm{inf}}_{\gamma}\right) &\leq \mu^{\bullet}_{\Lambda(\gamma),p^*,q,1}\left(\bigcap_{l=1}^{n}\neg\Gamma^{l}\right)\\ &\leq \mu^{\bullet}_{\tilde{\mathcal{S}}^1\cap\Lambda(\gamma),p^*,q,1}\left(\neg\Gamma^1\right)^n\\ &\leq \exp\left(-\zeta n\lfloor\frac{n_2}{N}\rfloor\right)\\ &\leq \exp\left(-\zeta''\mathrm{Area}\left(\gamma\right)\right)\end{aligned}$$

for any $\zeta'' < \zeta/\left(N2^{d-1}M^{d-2}\right)$ and all sufficiently large $n_2,\ldots,n_d$. $\square$

7.3. **The Area Law Constant.** Finally, we show that the constant appearing in the area law exponent is well-defined.

Lemma 44. *Let $\mu_{\mathbb{Z}^d,p}$ be an infinite volume PRCM and $V_\gamma = V^{\mathrm{fin}}_{\gamma}(q')$ or $V^{\mathrm{inf}}_{\gamma}(q')$. Then there is a constant c_5 so that*

$$\lim_{l\to\infty}\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(V_{\gamma_l}\right)\right)}{\mathrm{Area}(\gamma_l)} = c_5$$

for any sequence $\{\gamma_l\}$ of hyperrectangular $(i-1)$-boundaries whose dimensions diverge with l. c_5 may depend on the choice $\mu_{\mathbb{Z}^d,p}$ and V_γ, but not on $\{\gamma_l\}$.

Proof. We proceed as in Proposition 2.4 of [ACC+83]. For a loop γ, let $r(\gamma)$ be the box whose boundary is the support of γ. Let

$$\mathcal{E} := \left\{\{\gamma_l\} : m\left(r\left(\gamma_l\right)\right) \xrightarrow{l\to\infty} \infty\right\}$$

where we recall that $m(r)$ is the minimum of the i dimensions of r, and let

$$c_5(p,q) := \liminf_{\{\gamma_l\}\in\mathcal{E}}\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(V_{\gamma_l}\right)\right)}{\mathrm{Area}(\gamma_l)}.$$

Now let $\{\gamma'_k\}\in\mathcal{E}$. We may tile $r(\gamma_l)$ with $m := \lfloor\frac{\mathrm{Area}(\gamma_l)}{\mathrm{Area}(\gamma'_k)}\rfloor$ translates of $r(\gamma'_k)$ (call them $r_1,\ldots,r_m$) with the exception of $o(\mathrm{Area}(\gamma_l))$ i-plaquettes (call the set of such plaquettes T.) Notice that

$$\partial r(\gamma) = \sum_{j=1}^{m}\partial r_m + \sum_{\sigma\in T}\partial\sigma$$

when the chains are oriented appropriately. It follows that V_{γ_l} is implied by at most $\lfloor\frac{\mathrm{Area}(\gamma_l)}{\mathrm{Area}(\gamma'_k)}\rfloor$ translates of $V_{\gamma'_k}$ together with $o(\mathrm{Area}(\gamma_l))$ additional plaquettes. Then by the FKG inequality we have

$$\begin{aligned}
&\liminf_{\{\gamma_l\}\in\mathcal{E}} \frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(V_{\gamma_l}\right)\right)}{\operatorname{Area}(\gamma_l)} \\
&\quad\geq \limsup_{\{\gamma'_k\}\in\mathcal{E}} \liminf_{\{\gamma_l\}\in\mathcal{E}} \frac{1}{\operatorname{Area}(\gamma_l)} \left\lfloor \frac{\operatorname{Area}(\gamma_l)}{\operatorname{Area}(\gamma'_k)} \right\rfloor \log\left(\mu_{\mathbb{Z}^d,p}\left(V_{\gamma'_k}\right)\right) + o\left(\operatorname{Area}(\gamma_l)\right) \\
&\quad= c_5 + o(1)\ .
\end{aligned}$$

□

8. The Perimeter Law Regime

We show that, in the supercritical regime, a perimeter law holds for $(d-2)$-cycles obtained as the boundaries of connected, hyperplanar regions of $\mathbb{Z}^d$. For a set X that is the union of i-dimensional plaquettes, write $\rho_X = \sum_{\sigma\in X}\sigma$ where the sum is taken over the (positively oriented) plaquettes σ that compose X. In this section, γ will be a $(d-2)$-dimensional cycle of the form $\rho_{\partial X}$ where X is a connected union of $(d-1)$-dimensional plaquettes $\{\sigma_1,\dots,\sigma_N\}$ contained in a hyperplane of $\mathbb{Z}^d$. We may assume without loss of generality that γ is contained in $\{x_d = 0\}$.

The proof of the perimeter law is not substantially different than that for independent plaquette percolation [ACC+83], but we include it here for completeness. We provide more detail in the proof of the key geometric argument — phrasing it in the language of homology — which is a good warm-up for what follows. Complementarily to the area law section, it is enough to give a perimeter law bound for the PRCM with free boundary conditions. For convenience, we will denote the dual (wired) random cluster measure by $\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*} = \mu^{\mathbf{w}}_{\mathbb{Z}^d,p^*,q,1}$.

We require the following exponential decay result for the supercritical random cluster model.

Theorem 45 (Duminil-Copin, Raoufi, Tassion [DCRT19])**.** *Fix $d\geq 2$ and $q\geq 1$. Let $\theta(p^*) = \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}(0\leftrightarrow\infty)$ and let $p_c = p_c\left(\mathbb{Z}^d,q\right)$. Then*

- *there exists a $c>0$ so that $\theta(p^*)\geq c\,(p^*-p_c)$ for any $p^*\geq p_c$ sufficiently close to p_c;*
- *for any $p^*<p_c$, there exists a b_{p^*} so that for every $n\geq 0$,*

$$\mu^{\bullet,\mathbf{w}}_{\Lambda_n,p^*}\left(0\leftrightarrow\partial\Lambda_n\right)\leq \exp\left(-b_{p^*}n\right)\ .$$

Here, this theorem will be applied via the following corollary.

Corollary 46. *Let $\mathcal{C}_0$ be the component of the origin in the classical random cluster model on $\mathbb{Z}^d$. Then, if $p^* < p_c\left(\mathbb{Z}^d, q\right)$,*

$$\mathbb{E}_{\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}}\left(|\mathcal{C}_0|\right) < \infty\,.$$

Proof. See Theorem 5.86 in [Gri06]. □

Let $p^* < p_c\left(\mathbb{Z}^d, q\right)$. We start by demonstrating that the positive $\boldsymbol{e}_d$-axis is disconnected from the hyperplane $W = \{\boldsymbol{e}_d = -1/2\}$ with positive probability in Q. Let $K = \left\{(1/2, \ldots, 1/2, 1/2 + z) : z \in \mathbb{Z}^{\geq 0}\right\}$ and $K_h = K \cap \{x_d \geq 1/2 + h\}$. Denote by F_h the event that W is connected to K_h in Q.

Proposition 47. *If $p^* < p_c\left(\mathbb{Z}^d, q\right)$ then $\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(F_0\right) < 1$.*

Proof. By translation invariance, we have that

$$\sum_{v \in K} \sum_{w \in W} \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}(v \leftrightarrow w) = \mathbb{E}_{\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}}\left(|\mathcal{C}_0|\right) < \infty\,,$$

using Corollary 46. Therefore

$$\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(F_h\right) \leq \sum_{v \in K_h} \sum_{w \in W} \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}(v \leftrightarrow w) \xrightarrow{h \to \infty} 0\,,$$

since the sum is the tail of a convergent series. Let h be large enough so that $\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(F_h\right) < 1/2$. Notice that if the entire edge boundary of $K \setminus K_h$ is omitted, then $K \setminus K_h$ is disconnected from W. Thus

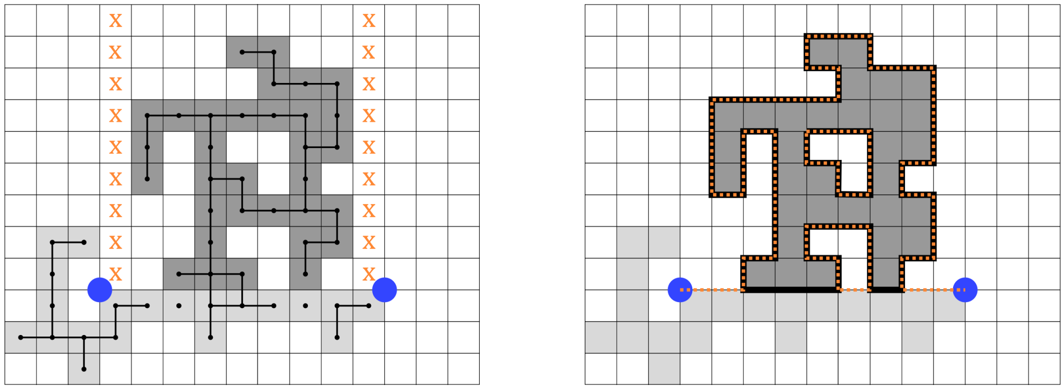
$$\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(K \setminus K_h \leftrightarrow W\right) \leq 1 - \left(\frac{p}{q}\right)^{4h+2}\,.$$

Then, by the FKG inequality,

$$\begin{aligned}
\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(\neg F_0\right) &\geq \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(\neg F_h \bigcap \neg\left(K \setminus K_h \leftrightarrow W\right)\right) \\
&\geq \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(\neg F_h\right) \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(\neg\left(K \setminus K_h \leftrightarrow W\right)\right) \\
&\geq \frac{1}{2}\left(\frac{p}{q}\right)^{4h+2} > 0\,.
\end{aligned}$$

□

Let $\sigma_{i_1}, \ldots, \sigma_{i_M}$ be the subset of the $(d-1)$-plaquettes σ_i which share a $(d-2)$-dimensional face with γ. For each $1 \leq i \leq N$, let a_i be the center of σ_i and let $F^i = F_0 + a_i$ (that is, $F^i = \{K + a_i \leftrightarrow W\}$).



(A) (B)

FIGURE 6. The construction in the proof of Lemma 48 for the case $d = 2$ (or a cross-section of it in higher dimensions). In both figures γ is shown by the large blue dots, and the cubes $\tau_1, \ldots, \tau_J$ are colored gray with the subset $\tau_{j_1}, \ldots, \tau_{j_K}$ given a darker shade. In (A) C is depicted by the small black dots connected by bonds. Note that the cubes marked with the orange X's are excluded from c by the event $\bigcap_{j \leq M} \neg F^{i_j}$. (B) shows the support of the chains α_1 (the dotted orange paths) and α_2 (the thick black paths). This figure was adapted from Figure 3 of [ACC+83].

Lemma 48.

$$\bigcap_{j \leq M} \neg F^{i_j} \subset V_\gamma^{\text{fin}}(\mathbb{Z})$$

Proof. Assume that the event

$$\bigcap_{j \leq M} \neg F^{i_j} \tag{11}$$

occurs. Let $\mathcal{C}_i$ denote the connected component of a_i in Q and let $\hat{\mathcal{C}}_i$ be the closure $\mathcal{C}_i$ in $\mathbb{R}^d \setminus P$, for $i \leq 1 \leq N$. Set $\mathcal{C} = \cup_{i=1}^N \mathcal{C}_i$. By Lemma 69, $\cup_{i=1}^N \hat{\mathcal{C}}_i$ is the union of the d-cubes dual to the sites of $\mathcal{C}$. There are finitely many such cubes because $p < p_c\left(\mathbb{Z}^d\right)$; denote them by $\tau_1, \ldots, \tau_J$.

We may write

$$\partial \sum_{i=1}^J \tau_j = \alpha_0 + \alpha_1$$

where α_1 and α_0 are supported on $(X \setminus \gamma) \times [0, \infty)$ and the complement of that set, respectively. Any plaquette in the support of $\partial \sum \tau_j$ is contained in P, so $\alpha_1 \in C_{d-1}(P;\ \mathbb{Z})$. We will show that $\partial \alpha_1 = (-1)^{d-1} \gamma$.

Let

$$\{\tau_{j_k}\}_{k=1}^{K} = \{\tau :\in \{\tau_1, \ldots, \tau_J\} : \tau \subset X \times [0, \infty)\}$$

We will compute the difference between α_1 and $\alpha_2 \coloneqq \partial \sum_{j=1}^{K} \tau_{j_k}$. For $j \in \{1, 2\}$, write

$$\alpha_j = \sum_{\sigma} a_{\sigma}^{j} \sigma$$

where the sum is taken over the $(d-1)$-plaquettes of $\mathbb{Z}^d$. Each $(d-1)$-plaquette σ is contained in exactly two d-cubes, so $a_{\sigma}^{j} \in \{-1, 0, 1\}$ for $j = 1, 2$. In addition, if $a_{\sigma}^{1} \neq 0$ and $a_{\sigma}^{2} \neq 0$ then $a_{\sigma}^{1} = a_{\sigma}^{2}$.

Let σ be a $(d-1)$ plaquette of $\mathbb{Z}^d$. If $a_{\sigma}^{1} \neq 0$ and $a_{\sigma}^{2} = 0$ then σ is dual to an edge connecting a site of $X \times [0, \infty) \setminus \mathcal{C}$ with a site of $\mathcal{C}$ outside of that cylinder. As we excluded plaquettes contained in $\gamma \times [0, \infty)$ from the support of α_1, it follows that σ is one of the plaquettes $\{\sigma_1, \ldots, \sigma_N\}$ from the definition of γ. On the other hand, if $a_{\sigma}^{2} \neq 0$ and $a_{\sigma}^{1} = 0$ then σ corresponds to an edge between two sites of $\mathcal{C}$, one inside the cylinder and one outside of it. By the assumption in (11), $\sigma \in \{\sigma\}_{i=1}^{N}$. Morever, each σ_i must fall into one of these two classes. Therefore

$$a_{\sigma_i}^{1} \neq 0 \iff a_{\sigma_i}^{2} = 0$$

for $i = 1, \ldots, N$.

In addition, by (13), σ_i either appears with the sign $(-1)^{d-1}$ in α_1 or it appears with the sign $(-1)^{d}$ in α_2. Thus

$$\alpha_1 = \alpha_2 + (-1)^{d-1} \sum_{i=1}^{N} \sigma_i \,,$$

and, recalling the definitions of α_2 and the plaquettes σ_i,

$$\partial \alpha_1 = \partial \circ \partial \left(\sum_{k=1}^{K} \tilde{\tau}_k \right) + (-1)^{d-1} \partial \left(\sum_{i=1}^{N} \sigma_i \right) = (-1)^{d-1} \gamma \,.$$

Therefore γ is null-homologous in P. □

Proposition 49. *Let* $p^* < p_c\left(\mathbb{Z}^d, q\right)$*, let* $m \in \mathbb{N}$*, and let* $V_\gamma = V_\gamma^{\mathrm{inf}}(m)$ *or* $V_\gamma = V_\gamma^{\mathrm{fin}}(m)$*. Then there is a* $\beta > 0$ *so that*

$$\mu^{\mathbf{f}}_{\mathbb{Z}^d, p, q, d-1}(V_\gamma) \geq \exp\left(-\beta \mathrm{Per}(\gamma)\right)$$

for any boundary γ *of a* $(d-1)$*-dimensional connected, hyperplanar region of* $\mathbb{Z}^d$*.*

Proof. By Corollary 30, it suffices to show the statement for $V_\gamma = V_\gamma^{\mathrm{fin}}(\mathbb{Z})$. Using the FKG inequality, Proposition 47 and Lemma 48, we have

$$\begin{aligned}\mu^{\mathbf{f}}_{\mathbb{Z}^d,p,q,d-1}(V_\gamma) &\geq \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(\bigcap_{i\leq N}\neg F^i\right)\\ &\geq \mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*}\left(\neg\left(F_0\right)\right)^{\mathrm{Per}(\gamma)}\\ &\geq \exp\left(-\beta\mathrm{Per}(\gamma)\right),\end{aligned}$$

for some $\beta > 0$. □

9. A Sharp Constant for the Perimeter Law

In this section, we gain a finer understanding of the supercritical asymptotics of Wilson loop variables corresponding the boundaries of $(d-1)$-dimensional boxes of $\mathbb{Z}^d$. We may assume without loss of generality that these boxes are of the form $[0, M_1] \times \ldots \times [0, M_{d-1}] \times 0$. This section is the only place where we require control over how the dimensions of the boxes grow to ∞. Recall from the introduction that a family of k-dimensional boxes r_l is suitable if its k dimensions diverge to ∞ and if $m(r_l) = \omega(\log(M(n)))$, where $M(r)$ and $m(r)$ denote the largest and smallest dimensions. When r_l is a suitable, we say that $\gamma_l = \partial r_l$ is a *suitable* family of $(k-1)$-dimensional rectangular boundaries. This hypothesis allows us to interpolate between boundary conditions for the PRCM on a box, and is unnecessary in the case $q = 1$ of Bernoulli plaquette percolation.

Theorem 50. *Let $p > p^*\left(p_c\left(\mathbb{Z}^d, q\right)\right)$ and $m \in \mathbb{N}$. There is a constant $c = c(p, q, d, m) > 0$ so that if γ_l is a suitable family of $(d-2)$-dimensional rectangular boundaries, $\mu_{\mathbb{Z}^d,p} = \mu^{\xi}_{\mathbb{Z}^d,p,q,d-1}$ is any infinite volume random cluster measure, and $V_\gamma = V_\gamma^{\mathrm{fin}}(m)$ or $V_\gamma = V_\gamma^{\mathrm{inf}}(m)$ then*

$$\lim_{l\to\infty}\frac{\mu_{\mathbb{Z}^d,p}\left(V_\gamma\right)}{\mathrm{Per}\left(\gamma_l\right)} = -c\,.$$

The events are positive, so by Proposition 25, it suffices to show that that there exists a $c > 0$ so that

$$\liminf_{l\to\infty}\frac{\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{fin}}_{\gamma_l}(m)\right)}{\mathrm{Per}\left(\gamma_l\right)} \geq -c$$

and

$$\liminf_{l\to\infty}\frac{\mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{inf}}_{\gamma_l}(m)\right)}{\mathrm{Per}\left(\gamma_l\right)} \leq -c\,.$$

For the remainder of this section, fix an infinite volume PRCM $\mu_{\mathbb{Z}^d,p} = \mu^{\xi}_{\mathbb{Z}^d,p,q,d-1}$ and denote the dual RCM by $\mu^{\bullet,\mathbf{w}}_{\mathbb{Z}^d,p^*} = \mu^{\xi^\bullet}_{\mathbb{Z}^d,p^*(p),q,1}$. The outline of our proof is similar to that of Theorem 3.9 of [ACC+83], though we must modify it to work for higher dimensional plaquette percolation and to handle the dependence between disjoint plaquette events. We adapt the notation of that paper. A non-expert reader may benefit from reading that account first, as the arguments are simpler.

We split the proof into three sections. In Section 9.1, we show a technical lemma on interpolating between $\mu_{\mathbb{Z}^d,p}$ and the plaquette random-cluster measure with wired boundary conditions in a box. Next, we construct a plaquette event that precludes the existence of a dual loop that links with γ in Section 9.2. The event V_γ^{fin} implies the occurrence of a related event. We conclude in Section 9.3 by "sandwiching" the probability of V_γ between the probabilities of these two events, thereby demonstrating the existence of the constant c.

In several places in this section, we will use the following notion of a box crossing.

Definition 51. *Let r be a box in $\mathbb{Z}^d$. The i-box crossing event for r, denoted $R_i^\square(r)$, is the event that there is a hypersurface of plaquettes contained in the interior of r which separates the two faces of r orthogonal to the $\boldsymbol{e}_i$-axis.*

Here (and later), a hypersurface of plaquettes contained in a box r will be a collection of plaquettes S so that $\partial \rho_S$ is supported on ∂r.

9.1. **An Interpolation Lemma.** Let $r = [0, M_1] \times \ldots [0, M_d]$ be a box in $\mathbb{Z}^d$, let r^L be the enlarged box $r = [-L, M_1 + L] \times \ldots [-L, M_d + L]$, and let A be an increasing event depending only on the edges of r. We will compare the asymptotics of $\mu_{\mathbb{Z}^d,p}(A)$ with those of $\mu^{\mathbf{w}}_{r^L,p}(A)$ in the supercritical regime. Towards this end, we will show that, when the dimensions of the boxes are grown appropriately, there is a high probability that ∂r^L is separated from r by a hypersurface of plaquettes. First, we prove a technical lemma.

Lemma 52. *Let $p > p^*(p_c(q))$. There exists a $b_p > 0$ so that*

$$\mu_{\mathbb{Z}^d,p}\left(R_d^\square(r)\right) \geq 1 - e^{-b_p M_d} \prod_{j=1}^{d-1} M_j \,.$$

Proof. Let D^+, D^- be the top and bottom faces of r, respectively (with respect to the d-th coordinate direction). By definition,

$$\mu_{\mathbb{Z}^d,p}\left(\neg R_d^\square(r)\right) = \mu^{\bullet}_{\mathbb{Z}^d,p^*}\left(D^+ \xleftrightarrow{Q\cap r} D^-\right)$$

where $\mu^{\bullet}_{\mathbb{Z}^d,p^*}$ is the dual 1-dimensional random cluster model and $p^* = p^*(p)$.

Let v be a dual vertex contained in $D^+ - \frac{1}{2}\boldsymbol{e}_d$. Then

$$\begin{aligned}\mu^{\bullet}_{\mathbb{Z}^d,p^*}\left(v \underset{r^{\bullet}}{\longleftrightarrow} D^- + \frac{1}{2}\boldsymbol{e}_d\right) \leq & \mu^{\bullet}_{\mathbb{Z}^d,p^*}\left(v \underset{\Lambda_{M_d}(v)}{\longleftrightarrow} \partial\Lambda_{M_d}(v)\right) \\ \leq & \mu^{\bullet,\mathbf{w}}_{\Lambda_{M_d}(v),p^*}\left(v \underset{\Lambda_{M_d}(v)}{\longleftrightarrow} \partial\Lambda_{M_d}(v)\right) \\ \leq & e^{-b_p M_d}.\end{aligned}$$

as a consequence of Theorem 45. As

$$\neg R^{\square}_d(r) = \bigcup_{v \in D^+ - \frac{1}{2}\boldsymbol{e}_d} v \underset{r}{\longleftrightarrow} D^- + \frac{1}{2}\boldsymbol{e}_d$$

the desired statement follows by the union bound. □

Corollary 53. *Let $\Xi_{r,L}$ be the event that ∂r^L is separated from r by a hypersurface of plaquettes contained in the annular region $r^L \backslash r$. Then, if $p > p^*(p_c(q))$,*

$$\mu_{\mathbb{Z}^d,p}(\Xi_{r,L}) \geq \prod_{i=1}^{d}\left(1 - e^{-b_p L}\prod_{j\neq i} M_j\right)^2.$$

Proof. We define a box crossing event for each $(d-1)$-dimensional face D of r. For the special case $D = [0, M_1]\times \ldots [0, M_{d-1}]\times\{0\}$, set $r_{D,L} = [-L, M_1 + L]\times \ldots [-L, M_{d-1} + L]\times[-L, -M_d]$ and $\hat{R}^{\square}(D, L)$ to be the event $R^{\square}_d(r_{D,L})$. More generally, define $\hat{R}^{\square}(D, L)$ by symmetry. Then $\Xi_{r,L}$ is implied by the occurrence of the $2d$ events $R^{\square}_d(r_{D,L})$, one for each face of r. Since these events are positive, the statement follows from the preceding lemma by the FKG inequality. □

Proposition 54. *Let A be an increasing event that depends only on the plaquettes of r. If $p > p^*(p_c(q))$ then*

$$\mu_{\mathbb{Z}^d,p}(\Xi_{r,L})\,\mu^{\mathbf{w}}_{r^L,p}(A) \leq \mu_{\mathbb{Z}^d,p}(A) \leq \mu^{\mathbf{w}}_{r^L,p}(A).$$

Proof. A standard application of Theorem 32 yields that restriction of the conditional measure $\left(\mu_{\mathbb{Z}^d,p}\middle|\Xi_{r,L}\right)$ to r stochastically dominates the restriction of $\mu^{\mathbf{w}}_{r_L,p}$ to r. This implies the first inequality. The second inequality is also an immediate consequence of Theorem 32. □

9.2. **Geometric Lemmas.** In this section, we will prove several geometric results that will be useful for the proof of Theorem 50. First, we show that box crossings behave nicely under intersections. For convenience, we state the following lemma for d-box crossings. The corresponding statement for i-box crossings follows by symmetry.

Lemma 55. *Let r_1 and r_2 be boxes of the form $[0, M_1] \times \ldots \times [0, M_d]$ and $[N_1, M_1'] \times \ldots \times \times \left[N_{d-1}, M_{d-1}'\right] \times [0, M_d]$, with $0 \leq N_i < M_i' \leq M_i$ for $i = 1, \ldots, d-1$. That is, r_2 is contained in r_1 and has the same height as it. Then $R_d^{\square}(r_1) \implies R_d^{\square}(r_2)$.*

Proof. Let D_1^+ and D_1^- be the two faces of r_1 orthogonal to $\boldsymbol{e}_d$ and let D_2^+ and D_2^- be faces of r_2 orthogonal to $\boldsymbol{e}_d$ so that $D_2^+ \subset D_1^+$ and $D_2^- \subset D_1^-$. Then if $R_d^{\square}(r_2)$ does not occur, there is a dual path between D_2^+ and D_2^- in r_2. But then since $r_2 \subset r_1$, such a path also connects D_1^+ from D_1^- in r_1, so $R_d^{\square}(r_2)$ cannot occur. $\square$

We say that a set of plaquettes J is a minimal witness for a crossing event $R_i^{\square}(r)$ if there is no subset $J' \subset J$ which is also a witness for $R_i^{\square}(r)$. We now investigate properties of a minimal box crossing.

Lemma 56. *Let r be a box and let $\hat{B}_i(r)$ be the union of the $(d-1)$-faces of r which are not orthogonal to $\boldsymbol{e}_i$. Assume $R_i^{\square}(r)$ occurs and that J is a minimal collection of plaquettes witnessing that event. Then the map on homology $H_{d-2}(\partial J;\ \mathbb{Z}) \to H_{d-2}\left(\hat{B}_i(r);\ \mathbb{Z}\right) \cong \mathbb{Z}$ induced by inclusion is an isomorphism. In particular, if $\rho_J = \sum_{\sigma \in J} \sigma$ then $\partial \rho_J$ generates $H_i\left(\hat{B}_i(r);\ \mathbb{Z}\right) \cong \mathbb{Z}$ and is null-homologous in P.*

Proof. Let D_i^+ and D_i^- be the two $(d-1)$-faces of r orthogonal to $\boldsymbol{e}_i$. Let A be the union of d-cells of the connected component containing D_i^- of $r \setminus J$. Since J is minimal, it cannot contain any plaquettes of ∂r and it must all plaquettes of ∂A that are not supported on r. Thus, we can write ∂A as the union of three disjoint sets of plaquettes

$$\partial A = D_i^- \cup J \cup E\,,$$

where E is a union of plaquettes contained in $\hat{B}_i(r)$. Recall that ρ_Y is the sum of the positively oriented plaquettes composing Y. Thus

$$\partial \rho_{D_i^-} + \partial \rho_E + \partial \rho_J = \partial \partial \rho_A = 0\,.$$

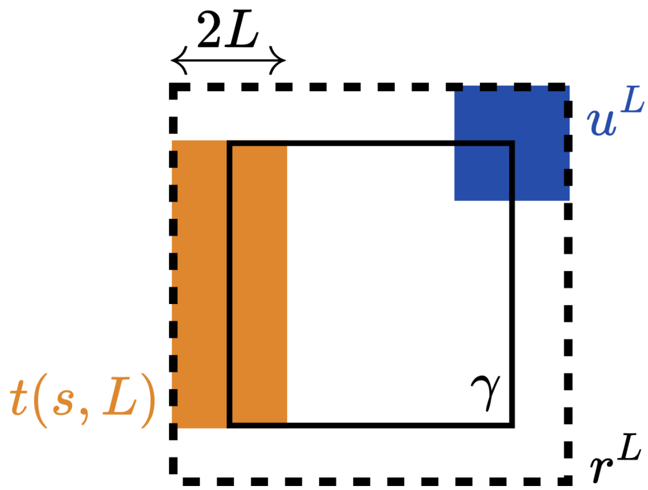


FIGURE 7. An illustration of some of the notation developed in this section, shown for $d = 3$ in the plane containing γ.

So since $0 = [\partial \rho_E] \in H_{d-2}\left(\hat{B}_i\left(r\right);\ \mathbb{Z}\right)$, it follows that

$$\left[\partial \rho_{D_i^-}\right] = -\left[\partial \rho_J\right] \in H_{d-2}\left(\hat{B}_i\left(r\right);\ \mathbb{Z}\right).$$

Now, $\left[\partial \rho_{D_i^-}\right]$ is a generator for $H_{d-2}\left(\hat{B}_i\left(r\right);\ \mathbb{Z}\right)$, so $[\partial \rho_J]$ is as well. Thus, the map $H_{d-2}\left(\partial J;\ \mathbb{Z}\right) \to H_{d-2}\left(\hat{B}_i\left(r\right);\ \mathbb{Z}\right) \cong \mathbb{Z}$ is an isomorphism. ☐

Given a box of the form $t = s' \times [-L, L]^2$ of $\mathbb{Z}^d$, let s be the $(d-2)$-dimensional box $S' \times \{0\}^2$. We say that $t = t\left(s, L\right)$ is the tube around s of width L. Often s will be a $(d-2)$-face of a rectangular boundary γ. Denote by C_t be the event that there a $(d-1)$-chain $\tau \in C_{d-1}\left(P \cap t;\ G\right)$ so that $\partial \tau = \rho_s + \alpha$ where α is supported on ∂t. See Figure 8. Compare this with the definition of the corresponding event in Section 3(iii) of [ACC+83] when $d = 3$.

Lemma 57. *Let t, s, and s' be as in the previous paragraph, and suppose that $s'' \subset s'$ is a $(d-2)$-dimensional box in $\mathbb{Z}^{d-2} \times \{0\}^2$. Then if $t'' = s'' \times [-L, L]^2$, $C_t \implies C_{t''}$.*

Proof. Assume that the event C_t occurs. Then there exists a chain $\tau = \sum_{\sigma \subset t} a_\sigma \sigma$ so that $\partial \tau = \rho_s + \alpha$ where α is supported on the boundary of t. If $\tau' = \sum_{\sigma \subset t''} a_\sigma \sigma$ then $\partial \tau' = \rho_s + \alpha'$ where α' is supported on the boundary of t''. ☐

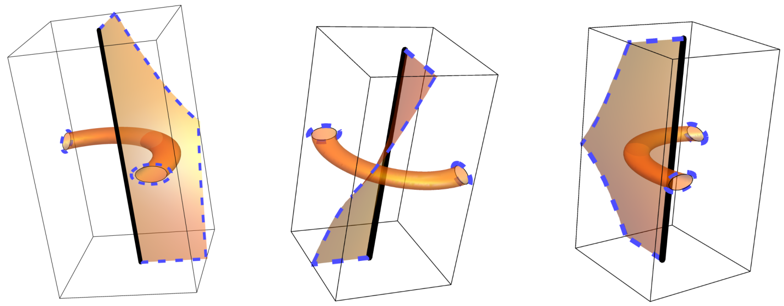


FIGURE 8. A witness for the event C_t, shown from three different viewpoints. The support of τ is the orange hypersurface, the support of α is shown by a dotted blue curve, and s is depicted with a thick black line. A handle is included to emphasize the possible complexity of set of plaquettes. See also Figure 13 of [ACC+83].

Consider the four boxes $y_1 = s' \times [-L, -L/2] \times [-L, L]$, $y_2 = s' \times [L/2, L] \times [-L, L]$, $y_3 = s' \times [-L, L] \times [-L, -L/2]$, and $y_4 = s' \times [-L, L] \times [L/2, L]$ that surround s. Set

$$D_t = R^{\square}_{d-1}(y_1) \cap R^{\square}_{d-1}(y_2) \cap R^{\square}_{d}(y_3) \cap R^{\square}_{d}(y_4) \ .$$

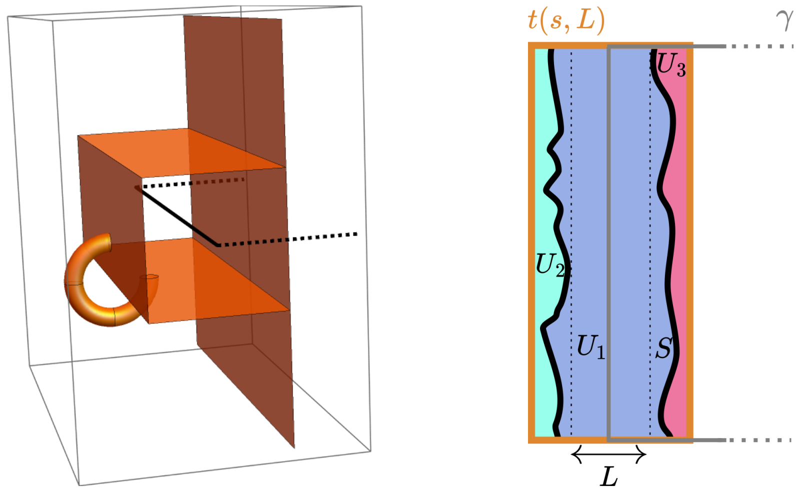
D_t implies that s is separated from the faces of t parallel to it by a surface of plaquettes. Let $\tilde{B}$ to be the union of the four faces of t that are parallel to s.

Lemma 58. *If D_t occurs, then there is a connected hypersurface S of plaquettes of $P \cap t$ that separates t into three components $U_1 = U_1(s), U_2 = U_2(s)$, and $U_3 = U_3(s)$ with the following properties.*

- *$t(s, L/2)$ is contained in U_1 and $\tilde{B}$ is contained in $U_2 \cup U_3$.*
- *U_2 contains the $(d-1)$-face of t that is contained in the boundary of r^L.*
- *U_3 is the component of $y_2 \setminus S_2$ containing the shared face of y_2 and t, where S_2 is a minimal witness for $R^{\square}_{d}(y_2)$.*
- *$r \cap t(s, L)$ is contained in $U_1 \cup U_3$.*

See Figure 9.

Proof. Let S_1, S_2, S_3, and S_4 be minimal witnesses for the crossing events $R^{\square}_{d-1}(y_1), R^{\square}_{d-1}(y_2), R^{\square}_{d}(y_3)$, and $R^{\square}_{d}(y_4)$, respectively. S_2 separates $t \setminus S_2$ into two components U_3 and U_4, where the former contains both γ and the



(A) (B)

FIGURE 9. (A) An example of the surface S constructed in Lemma 58, shown in orange. The surface includes a handle to emphasize the possible complexity of S. s is shown by a solid black line, and the neighboring parts of γ by a dashed black line. (B) A cross section of the partition of the tube $t(s,L)$ by the hypersurface S, in the plane containing γ. S is shown in black, and γ in gray.

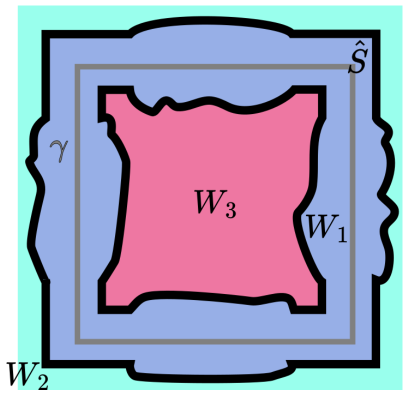
face of t contained in $\partial r^L\ (s \times [-L, L] \times \{-L\})$ and the latter contains the opposite face of t.

The union $\cup_{i=1}^4 S_i$ separates γ from $\tilde{B}$, so we may find a minimal hypersurface S_5 composed of plaquettes of $(S_1 \cup S_3 \cup S_4) \cap U_4$ so that $S_2 \cup S_5$ does the same. As S_5 is minimal, it must divide U_4 into two components U_1 and U_2, where U_1 contains s.

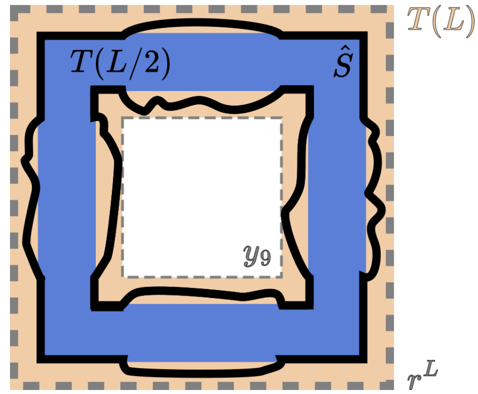
Set $S = S_2 \cup S_5$. The first two properties are satisfied by construction, and the third follows from the observation that $r \cap t(s,L) \subset t(s, L/2) \cup y_2$.

$\square$

If s is a $(d-2)$-dimensional box in $\mathbb{Z}^d$, there is a rigid motion ρ of $\mathbb{Z}^d$ so that $\rho(s)$ lies in $\mathbb{Z}^{d-2} \times \{0\}^2$. Let $t(s,L) = \rho^{-1}(t(\rho s, L))$. Similarly, set $C_{t(s,L)}$ and $D_{t(s,L)}$ to be the events $\rho^{-1}\left(C_{t(\rho s,L)}\right)$ and $\rho^{-1}\left(D_{t(\rho s,L)}\right)$, respectively. Also, denote by $\overline{C}_{t(s,L)}$ the event $C_{t(s,L)} \cap D_{t(s,L)}$. When s is not specified, it will be assumed to be contained in $\mathbb{Z}^{d-2} \times \{0\}^2$.



(A)



(B)

FIGURE 10. A cross section of the hypersurface $\hat{S}$ constructed in Lemma 60 for the case $d = 3$, shown in the plane containing γ. It is depicted in two different contexts: (A) with the regions W_1, W_2, and W_3 in pastel purple, green, and pink, respectively and (B) with the regions $T(L)$ (light orange), $T(L/2)$ (dark purple), r^L (the outer box, bounded by a thick dashed line), and y_8 (the inner box, bounded by a thin dashed line). Observe that S may be taken to coincide with the boundary of $T(L/2)$ in a neighborhood of a corner of γ; this is possible because of the occurence of the events $E(u, L)$.

We are now ready to state the main topological result of this section. Let r be a $(d-1)$-dimensional box in $\mathbb{Z}^d$ and let $\gamma = \partial r$. Also, set $T(L) = \cup_s s^L$, where s ranges over all $(d-2)$-faces of $r(\gamma)$, so T is a solid $(d-1)$-torus surrounding γ. Finally, for a box u, denote by $E_{u,L}$ be the event that all plaquettes in u^L are contained in P.

Proposition 59. *Set* $y_6 = [-L, M_1 + L] \times \ldots \times [-L, M_{d-1} + L] \times [-L, -L/2]$ *and* $y_7 = [-L, M_1 + L] \times \ldots \times [-L, M_{d-1} + L] \times [L/2, -L]$. *Then, for any* $m \in \mathbb{N}$

$$R_d^{\square}(y_6) \bigcap R_d^{\square}(y_7) \bigcap \cap_s \overline{C}_{t(s,L)} \bigcap \cap_u E_{u,L} \implies V_\gamma^{\text{fin}}(m), \tag{12}$$

where s *ranges over all* $(d-2)$*-dimensional faces of* γ *and* u *ranges over all* $(d-3)$*-faces of* γ.

While it would suffice to replace $R_d^{\square}(y_6) \cap R_d^{\square}(y_7)$ with a single occurrence of $R_d^{\square}(r^L)$, the proof is simpler for this formulation of the proposition. We begin with a lemma extending the construction in Lemma 58.

Lemma 60. *Assume the hypotheses of Proposition 59. Then there exists a hypersurface $\hat{S}$ of plaquettes of $P \cap r^L$ that separates r^L into three regions W_1, W_2, and W_3 satisfying:*

- $T(L/2) \subset W_1 \subset T(L)$.
- *W_3 is contained in the shrunken box* $y_8 := [L/2, M_1 - L/2] \times \ldots \times [L/2, M_{d-1} - L/2] \times [-L, L]$.
- *r is contained in $W_1 \cup W_3$.*

See Figure 10.

Proof. Set $W_1 = \cup_s U_1(s)$, where $U_1(s)$ was defined in the statement of Lemma 58 (and we extend the definition to general $(d-2)$-dimensional boxes by using translations/rotations). By construction, every plaquette in the boundary of W_1 is contained in P and $T(L/2) \subset W_1 \subset T(L)$.

Let S' and S'' be minimal witnesses for the crossings $R_d^{\square}(y_6)$ and $R_d^{\square}(y_7)$. They are disjoint, so by Lemma 56, they divide r^L into three components, one of which contains the center of r^L. Call this component W_0. Set $y_9 = [L, M_1 - L] \times \ldots \times [L, M_{d-1} - L] \times [-L, L]$, i.e. the closure of $r^L \setminus T$, and

$$W_3 = W_0 \cap \left(y_9 \cup \bigcup_s U_3(s) \right).$$

Notice that the second desired containment property is satisfied since it is holds for both y_9 and each $U_3(s)$. It is also not difficult to check that $r \subset W_1 \cup W_3$. Every plaquette in ∂W_3 is contained in P because each face of ∂y_9 is contained in the component $U_3(s)$ for some s. That is, ∂W_3 is a union of subsets of S', S'', and hypersurfaces of the form S_5 constructed in the proof of Lemma 58.

Set $\hat{S} = \partial W_1 \cup \partial W_3$ and $W_2 = r^L \setminus \left(\hat{S} \cup W_1 \cup W_3\right)$. Then the regions W_1, W_2, and W_3 satisfy the required conditions. ☐

Proof of Proposition 59. By Corollary 31 it suffices to show that no dual loop of $\overline{r^{L+1/2}}$ can be linked with γ. Any such loop $\gamma^\bullet$ must be contained in one of the three components W_1, W_2, and W_3.

In the first case, $\gamma^\bullet$ is in the interior of one of the tubes $t(s, L)$, as the events $E_{u,L}$ precludes it from entering more than one tube. The occurrence of the event $C_{t(s,L)}$ implies that γ is homologous to a cycle γ' contained in $T \setminus \text{interior}(t(s, L))$. The interior of $t(s, L)$ is contractible in $\mathbb{R}^d \setminus (T \setminus \text{interior}(t(s, L)))$ so $\gamma^\bullet$ cannot be linked with γ. See Corollary 26.

If $\gamma^\bullet$ is contained in W_2 then $\gamma^\bullet$ cannot be linked with γ because $r \subset \mathbb{R}^d \setminus W_2$ and γ is contractible in r.

Finally, γ is contractible in $\mathbb{R}^d \setminus y_8$ so it cannot be linked with any loop contained in W_3. □

9.3. **Proof of Theorem 50.** Now that we have finished the technical lemmas, the proof proceeds similarly to those of [ACC+83]. Their arguments often use the independence of events defined on disjoint edge sets in Bernoulli percolation. In lieu of that, we employ the following lemma for wired boundary boundary conditions.

Lemma 61. *Let X be a subcomplex of $\mathbb{Z}^d$, and suppose $r_1, r_2 \subset X$ are boxes which contain no shared d-cubes. If A_1, A_2 are increasing events that depend only on the edges of r_1 and r_2 respectively, then*

$$\mu^{\mathbf{w}}_{X,p}\left(A_1 \cap A_2\right) \leq \mu^{\mathbf{w}}_{r_1,p}\left(A_1\right) \mu^{\mathbf{w}}_{r_2,p}\left(A_2\right) .$$

Lemma

Proof. Let $P_1 \subset r_1$ and $P_2 \subset r_2$. As an application of the Mayer–Vietoris sequence,

$$H^{d-2}\left(P_1^{\mathbf{w}} \cup P_2^{\mathbf{w}};\ \mathbb{Z}_q\right) \cong H^{d-2}\left(P_1^{\mathbf{w}};\ \mathbb{Z}_q\right) \oplus H^{d-2}\left(P_2^{\mathbf{w}};\ \mathbb{Z}_q\right)$$

so

$$\left|H^{d-2}\left(P_1^{\mathbf{w}} \cup P_2^{\mathbf{w}};\ \mathbb{Z}_q\right)\right| = \left|H^{d-2}\left(P_1^{\mathbf{w}};\ \mathbb{Z}_q\right)\right| \left|H^{d-2}\left(P_2^{\mathbf{w}};\ \mathbb{Z}_q\right)\right|$$

It follows that $\mu^{\mathbf{w}}_{r_1 \cup r_2,p}$ is the independent product measure $\mu^{\mathbf{w}}_{r_1,p} \times \mu^{\mathbf{w}}_{r_2,p}$. (Alternatively, one could prove this by counting components of the dual graphs.)

Let B be the event that all plaquettes of $\partial r_1 \cup \partial r_2$ are contained in P. Then, by the FKG inequality,

$$\begin{aligned} \mu^{\mathbf{w}}_{X,p}\left(A_1 \cap A_2\right) \leq & \mu^{\mathbf{w}}_{X,p}\left(A_1 \cap A_2 \mid B\right) \\ = & \mu^{\mathbf{w}}_{X_1 \cup X_2,p}\left(A_1 \cap A_2\right) \\ = & \mu^{\mathbf{w}}_{X_1,p}\left(A_1\right) \mu^{\mathbf{w}}_{X_2,p}\left(A_2\right) . \end{aligned}$$

□

Next we show the analogue of Proposition 3.6 of [ACC+83], closely following the argument therein.

Proposition 62. *For $p \in [0,1]$,*

$$c := -\lim_{n\to\infty} \frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)\right)}{(2n)^{d-2}}$$

exists, and is positive when $p > p^*\left(p_c\left(q\right)\right)$.

Proof. Let n and m be positive integers with $n > m$. We can find $k := \lfloor \frac{n}{m} \rfloor^{d-2}$ disjoint cubes of width m which are contained in Λ_n and are centered at points of s. Call these cubes $\Lambda^1, \ldots, \Lambda^k$ and let D be the event that all plaquettes in the boundaries of those cubes are occupied. By Lemma 57, if C_{Λ_n} occurs then the events $C_{\Lambda^1}, \ldots, C_{\Lambda^k}$ happen as well. Then,

$$\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right) \leq \mu^{\mathbf{w}}_{\Lambda_m,p}\left(C_{\Lambda_m}\right)^k \,,$$

as a consequence of Lemma 61.

Taking logs and rearranging yields

$$\frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)\right)}{n^{d-2}} \leq \left(1 + a\frac{m^{d-2}}{n^{d-2}}\right) \frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_m,p}\left(C_{\Lambda_m}\right)\right)}{m^{d-2}}$$

where

$$a = k - \left(\frac{n}{m}\right)^{d-2}$$

satisfies $|a| < 1$ so

$$\limsup_{n\to\infty} \frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)\right)}{n^{d-2}} \leq \frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_m,p}\left(C_{\Lambda_M}\right)\right)}{m^{d-2}}$$

and we may conclude by taking the limit infimum as $m \to \infty$. □

Note that the definition of the event C_t depends on the choice of abelian group G for homology coefficients, and so c may be contingent on it as well.

Proposition 63. *For $p \in [0,1]$,*

$$\lim_{n\to\infty} \frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_n,p}\left(\overline{C}_{\Lambda_n}\right)\right)}{(2n)^{d-2}} = -c\,.$$

Proof. The proof is identical to that of Proposition 3.7 in [ACC⁺83]:

$$\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)\mu^{\mathbf{w}}_{\Lambda_n,p}\left(\overline{D}_r\right) \leq \mu^{\mathbf{w}}_{\Lambda_n,p}\left(\overline{C}_{\Lambda_n}\right) \leq \mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)$$

by the FKG inequality, where we are using the definition of $\overline{C}_{\Lambda_n}$. The event $\overline{D}_r$ is the intersection of $2d$ box crossing events whose probability goes to 1 as $N \to \infty$ by Lemma 52. As such, the desired result follows by taking logarithms and dividing by N.

□

Proposition 64. *Let r_l be a family of boxes of the form $[0,n_1(l)]\times\ldots\times[0,n_{d-2}(l)]\times[-m(l),m(l)]^2$ all of whose dimensions diverge to ∞. Then*

$$\lim_{l\to\infty}\frac{\log\left(\mu_{r_l,p}^{\mathbf{w}}\left(\overline{C}_{r_l}\right)\right)}{\prod_{i=1}^{d-2}n_i(l)}=-c\,.$$

In addition, if r_l is suitable then

$$\lim_{l\to\infty}\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(\overline{C}_{r_l}\right)\right)}{\prod_{i=1}^{d-2}n_i(l)}=-c\,.$$

Proof. The general idea is that we can fit the appropriate number of cubes along s in r_l for large l and we can also fit the appropriate number of copies of r_l along s in a large cube. First, for a fixed value of N, we can find $\prod_{i=1}^{d-2}\lfloor\frac{n_i(l)}{2N}\rfloor$ disjoint $2N$-cubes so that the occurrence of C_{r_l} implies that an event of the form C_{Λ_N} in each cube (when l is sufficiently large). The same argument as in the proof of Proposition 62 yields that

$$\limsup_{l\to\infty}\frac{\log\left(\mu_{r_l,p}^{\mathbf{w}}\left(C_{r_l}\right)\right)}{\prod_{i=1}^{d-2}n_i(l)}\leq-c\,.$$

On the other hand, if we fix $r=[0,n_1]\times\ldots\times[0,n_{d-2}]\times[-m,m]^2$ and choose $N>m$, then C_{Λ_N} entails that translates of the event C_r happen in $\prod_{i=1}^{d-2}\lfloor\frac{N}{n_i}\rfloor$ disjoint boxes (which do not depend on the specific witness for C_{Λ_N}). Thus

$$\liminf_{l\to\infty}\frac{\log\left(\mu_{r_l,p}^{\mathbf{w}}\left(C_{r_l}\right)\right)}{\prod_{i=1}^{d-2}n_i(l)}\geq-c$$

so

$$\lim_{l\to\infty}\frac{\log\left(\mu_{r_l,p}^{\mathbf{w}}\left(C_{r_l}\right)\right)}{\prod_{i=1}^{d-2}n_i(l)}=-c\,.$$

The proof that this limit coincides with the corresponding one for the event $\overline{C}_{r_l}$ is identical to that of the previous proposition.

We apply Proposition 54 to show the second claim. As $r(l)$ is a suitable family of boxes, we may choose a thickening parameter $L(l)$ so that $L(l)=\omega\left(\log\left(M\left(r_l\right)\right)\right)$ and $L(l)=o\left(m\left(r_l\right)\right)$. For convenience, set $\tilde{r}_l=r_l^{L(l)}$ and $f(r)$ to be the product of the first $(d-2)$ dimensions of a box r. By Proposition 54,

$$\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(\Xi_{r,L}\right)\right)+\log\left(\mu_{r^L,p}^{\mathbf{w}}\left(\overline{C}_{r_l}\right)\right)}{f\left(r_l\right)}\leq\frac{\mu_{\mathbb{Z}^d,p}\left(\overline{C}_{r_l}\right)}{f\left(r_l\right)}\leq\frac{\mu_{r^L,p}^{\mathbf{w}}\left(\overline{C}_{r_l}\right)}{f\left(r_l\right)}\,.$$

It follows from Corollary 53 that

$$\left|\log\left(\mu_{\mathbb{Z}^d,p}\left(\Xi_{r,L}\right)\right)\right| \leq -\sum_{i=1}^{d} 2\log\left(1-e^{-b_p L}\left(M\left(r_l\right)\right)^{d-2}\right)$$

Using the assumption that $L(l)=\omega\left(\log\left(M\left(r_l\right)\right)\right)$, we see that the right side is uniformly bounded above for sufficiently large l. Then since $f\left(r_l\right)\to\infty$ as $l\to\infty$,

$$\lim_{l\to\infty}\frac{\log\left(\mu_{\mathbb{Z}^d,p}\left(\Xi_{r_l,L(l)}\right)\right)}{f\left(r_l\right)}=0\,.$$

As such, it suffices to show that

$$\frac{\mu^{\mathbf{w}}_{r_L,p}\left(\overline{C}_{r_l}\right)}{f\left(r_l\right)}\to -c\,.$$

We have that

$$\frac{\log\left(\mu^{\mathbf{w}}_{\tilde{r}_n,p}\left(\overline{C}_{\tilde{r}_l}\right)\right)}{f\left(r_l\right)}\leq\frac{\log\left(\mu^{\mathbf{w}}_{\tilde{r}_l,p}\left(\overline{C}_{r_l}\right)\right)}{f\left(r_l\right)}\leq\frac{\log\left(\mu^{\mathbf{w}}_{r_l,p}\left(\overline{C}_{r_l}\right)\right)}{f\left(r_l\right)}\,.$$

We already showed that term on the right limits to $-c$ as $l\to\infty$. To handle the term on the left, note that $f\left(\tilde{r}_l\right)-f\left(r_l\right)\in o\left(f\left(r_l\right)\right)$ because $L(l)=o\left(m\left(r_l\right)\right)$. As such, the asymptotics remains unchanged if we replace the denominator with $f\left(\tilde{r}_l\right)$. Thus, we may conclude that the middle term also limits to $-c$, which suffices by the logic in the previous paragraph. $\square$

We are now ready to prove Theorem 50.

Proof of Theorem 50. Let $p>p^*\left(p_c(q)\right)$, and let $\gamma_l=\partial\rho_{r_l}$ be a suitable family of $(d-2)$-dimensional rectangular boundaries. Also fix $m\in N$ and set $V^{\mathrm{fin}}_\gamma=V^{\mathrm{fin}}_\gamma(m)$ and $V^{\mathrm{inf}}_\gamma=V^{\mathrm{inf}}_\gamma(m)$.

As noted after the statement, it suffices to show that

$$\liminf_{l\to\infty}\frac{\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{fin}}_{\gamma_l}\right)}{\mathrm{Per}\left(\gamma_l\right)}\geq -c$$

and

$$\liminf_{l\to\infty}\frac{\mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{inf}}_{\gamma_l}\right)}{\mathrm{Per}\left(\gamma_l\right)}\leq -c\,.$$

We begin by showing the second equation. For a fixed l, let $\lambda_1,\ldots,\lambda_{g(n,l)}$ be a maximal set of disjoint cubes of width $2n$ in $\mathbb{Z}^d$ that are centered at points of

∂r_l and are disjoint from the $(d-3)$-faces of r_l. We have that

$$\lim_{l\to\infty}\frac{\mathrm{Per}\left(\gamma_l\right)}{g\left(n,l\right)}=\left(2n\right)^{d-2}\ .$$

The event V_γ implies that events obtained from C_{Λ_n} by rotations or translations occur for each of the cubes λ_i. Thus we can apply Lemma 61 to obtain

$$\mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{inf}}_{\gamma_l}\right)\leq\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)^{g(n,l)}\ .$$

Therefore

$$\begin{aligned}\frac{\log\left(\mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{inf}}_{\gamma_l}\right)\right)}{\mathrm{Per}\left(\gamma_l\right)}&\leq\frac{g\left(n,l\right)\log\left(\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)\right)}{\mathrm{Per}\left(\gamma_l\right)}\\&\xrightarrow{l\to\infty}\frac{\log\left(\mu^{\mathbf{w}}_{\Lambda_n,p}\left(C_{\Lambda_n}\right)\right)}{\left(2n\right)^{d-2}}\\&\xrightarrow{n\to\infty}-c\end{aligned}$$

by Proposition 62.

On the other hand, choose $L\left(l\right)$ so that $L\left(l\right)\in o\left(m\left(r_l\right)^{(d-2)/d}\right)$ and $L\left(l\right)\in\omega\left(\log\left(M\left(r_l\right)\right)\right)$. Set $\tilde{r}_l=\left(s_l\right)^{L(l)}$. Recall from the statement of Proposition 59 that

$$R^{\square}_d\left(y_6\right)\bigcap R^{\square}_d\left(y_7\right)\bigcap\cap_s\overline{C}_{t(s,L)}\bigcap\cap_u E_{u,L}\implies V^{\mathrm{fin}}_\gamma\,,$$

where s and u range over the $(d-2)$ and $(d-3)-$ faces of γ, respectively. All of these events are positive, so by the FKG inequality

$$\begin{aligned}\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(V_{\gamma_l}\right)^{\mathrm{fin}}\right)\geq&\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(R^{\square}_d\left(y_6\right)\right)\right)+\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(R^{\square}_d\left(y_7\right)\right)\right)\\&+\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(\cap_u E_{u,L(l)}\right)\right)+\sum_s\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(\overline{C}_{t(s,L(l))}\right)\ .\end{aligned}$$

We will show that the first three terms grow asymptotically more slowly than $\mathrm{Per}\left(\gamma_l\right)$, and that the third term behaves as $-c\mathrm{Per}\left(\gamma\right)$.

The assumption that $L\in\omega\left(\log\left(M\left(r_l\right)\right)\right)$ yields that

$$\frac{\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(R^{\square}_d\left(y_6\right)\right)\right)}{\mathrm{Per}\left(\gamma_l\right)}=\frac{\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(R^{\square}_d\left(y_7\right)\right)\right)}{\mathrm{Per}\left(\gamma_l\right)}\xrightarrow{l\to\infty}0$$

by Lemma 52. In addition the events $E_{u,L(l)}$ require the activation of adL^d plaquettes where $a = 4(d-1)(d-2)$ is the number of $(d-3)$-faces of γ. Thus

$$\frac{\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(\bigcap_u E_{u,L(l)}\right)\right)}{\operatorname{Per}(\gamma_l)} \leq \frac{\log\left((p/q)^{abL^d}\right)}{\operatorname{Per}(\gamma_l)} \to 0$$

because $L(l) \in o\left(m(r(l))^{(d-2)/d}\right)$. Finally, by Proposition 64,

$$\lim_{l\to\infty} \frac{\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(\overline{C}_{t(s,L)}\right)\right)}{|s|} \to -c$$

as $|s| \to \infty$, where $|s|$ is the total number of $(d-2)$-plaquettes in the face s of r_l. As $\operatorname{Per}(\gamma_l) = \sum_s |s|$, we have that

$$\liminf_{n\to\infty} \frac{\log\left(\mu^{\mathbf{f}}_{\mathbb{Z}^d,p}\left(V^{\text{fin}}_\gamma\right)\right)}{\operatorname{Per}(\gamma)} \geq -c\,,$$

as desired.

□

Appendix A. Topological Tools

In this section we will review the relevant topological definitions and tools that we use in the rest of the article. We will start with the general theory and then discuss the specific properties of the cubical complex $\mathbb{Z}^d$. A standard reference for the former is [Hat02]. When possible, we use the specifics of our model to simplify the presentation. In particular, all spaces we consider are cubical complexes. While these are not covered in [Hat02], the definitions given there result in equivalent algebraic structures. See [KMM04, Sav16] for a more in depth discussion of the homology and cohomology of cubical complexes.

A.1. **Definitions of Homology and Cohomology.** First we recall the definition of homology. In this paper, we only consider spaces called cubical complexes, which have a combinatorial structure that makes defining and computing homology simpler than for general topological spaces. Our cubical complexes are built from i-dimensional unit cubes, also called i-plaquettes, for $0 \leq i \leq d$. In the case of the $\mathbb{Z}^d$ lattice, the i-dimensional cells are the i-plaquettes with integer corner points.

The key operation is then the map that takes a plaquette to its boundary. As an example, consider an oriented 2-plaquette σ in a cubical complex X. The (point set) topological boundary consists of the union of the four edges (1-plaquettes) between its corners. It turns out to be useful to handle this union as a linear combination. This motivates the definition of the *chain* group $C_i(X;\ G)$ of finite formal linear combinations of i-plaquettes of X with coefficients in an abelian group G. Now to revisit our 2-plaquette example, the boundary operator takes σ to $\partial\sigma$, which is the sum of the boundary edges of σ with orientations consistent with the orientation of σ as in Figure 11. Note that the orientation of each plaquette can be chosen arbitrarily as long as consistency is maintained and the opposite orientation of an i-plaquette σ is $-\sigma \in C_i(X;\ G)$. Then adding the requirement that ∂ be G-linear uniquely defines it on the full space of chains.

It turns out that a similar operator can be defined for plaquettes in every dimension. For concreteness, we give the formal definition here, though a reader unfamiliar with the subject will be comforted to know that it is rarely referred to explicitly in practice. An i-plaquette in $\mathbb{Z}^d$ may be written in the following form. Let $1 \leq k_1 < k_2 < \ldots < k_i \leq d$ and $x_1, \ldots, x_d \in \mathbb{Z}$. Set $\mathcal{I}_j = [x_j, x_j+1]$ for $j \in \{k_1, k_2, \ldots, k_i\}$ and $\mathcal{I}_j = \{x_j\}$ for $j \in [d] \setminus k_1, k_2, \ldots, k_i$. Then $\sigma = \prod_{1\leq j\leq d} \mathcal{I}_j$ is an i-plaquette in $\mathbb{R}^d$, and its boundary $\partial\sigma$ is

$$\sum_{l=0}^{i} (-1)^{l-1} \left(\prod_{1\leq j<k_l} \mathcal{I}_j \times \{x_{k_l}+1\} \times \prod_{k_l<m\leq d} \mathcal{I}_m - \prod_{1\leq j<k_l} \mathcal{I}_j \times \{x_{k_l}\} \times \prod_{k_l<m\leq d} \mathcal{I}_m \right). \tag{13}$$

We pause at this point to note that we can now formally define the event V_γ. We represent the cycle γ as an $(i-1)$-chain consisting of the sum of its $(i-1)$-plaquettes, oriented so that $\partial\gamma = 0$. Then V_γ is the event that there is some element τ of $C_i(P;\ \mathbb{Z}_q)$ so that $\partial\tau = \gamma$. Observe that this depends on q. Our interest in boundaries then leads us to the study of homology.

The chains $\alpha \in C_i(X;\ G)$ satisfying $\partial\alpha = 0$ are called *cycles*, and the space of cycles is written $Z_i(X;\ G)$. The group of chains which are *boundaries* of other chains is denoted $B_i(X;\ G)$. We can then think of V_γ as the event that the cycle γ is also a boundary. A key fact that underlies much of algebraic topology is that the boundary operator satisfies $\partial \circ \partial = 0$, so in particular $B_i(X;\ G) \subseteq Z_i(X;\ G)$. Then the *i-th homology group* is the quotient of the cycles by the boundaries, or

$$H_i(X;\ G) = Z_i(X;\ G)/B_i(X;\ G)\ .$$

We will also use the dual notion of cohomology. Here the basic objects are not chains, but linear functionals on chains. More precisely, the i-th *cochain group* $C^i(X;\ G)$ is defined as the group of G-linear functions from $C_i(X;\ G)$ to G. That is, it is the group of assignments of spins in G to the i-plaquettes of X. Note that when X is infinite, a chain $\sigma \in C_i(X;\ G)$ is by definition supported on finitely many i-plaquettes of X, whereas a cochain may take non-zero values on infinitely many plaquettes. The cochain group comes with an analogous *coboundary operator* $\delta : C^i(X;\ G) \to C^{i+1}(X;\ G)$, which is given by

$$\delta f(\alpha) = f(\partial \alpha)$$

for $f \in C^i(X;\ G), \alpha \in C_{i+1}(X;\ G)$.

We are then interested in similar quantities, now defined with coboundaries instead of boundaries. The i-th cocycle group $Z^i(X;\ G)$ is the group of cochains with zero coboundaries and the i-th coboundary group $B^i(X;\ G)$ is the image of δ applied to $C^{i-1}(X;G)$. Then i-th cohomology group is the quotient

$$H^i(X;\ G) = Z^i(X;\ G) / B^i(X;\ G)\ .$$

The notions of homology and cohomology are closely related in a sense that will be made precise shortly. However, it is useful to have both perspectives available. In the current context, the PLGT is naturally a cochain whereas the event V_γ is best defined in terms of homology. The distinction is also important for algebraic duality theorems, which are often stated in terms of passing from one to the other. These duality theorems are also most naturally stated in terms of variants of homology and cohomology called *reduced homology* and *reduced cohomology*. These are denoted by $\tilde{H}_j(X;\ G)$ and $\tilde{H}^j(X;\ G)$, respectively. The idea is that any nonempty space has at least one connected component, and that it is useful to remove one factor from $H_0(X;\ \mathbb{Z})$ and $H^0(X;\ \mathbb{Z})$ so that a contractible space has zero homology in all dimensions.

This is accomplished by defining a boundary operator on $C_0(X;\ \mathbb{Z})$ with the map $\epsilon : C_0(X;\ \mathbb{Z}) \to \mathbb{Z}$ that sends each 0-plaquette (or vertex) to 1. The reduced cohomology groups are obtained similarly with the dual map to ϵ. The resulting groups coincide with the usual j-th homology and cohomology groups except in the case $j = 0$, where

$$\tilde{H}_0(X;\ \mathbb{Z}) \oplus \mathbb{Z} \cong H_0(X;\ \mathbb{Z}) \qquad \text{and} \qquad \tilde{H}^0(X;\ \mathbb{Z}) \oplus \mathbb{Z} \cong H^0(X;\ \mathbb{Z})\ .$$

for nonempty spaces X. To reduce notational complexity, *we always write H_j and H^j to denote the reduced homology and cohomology groups in the remainder of the article.* Notice that this does not change the random-cluster measure, since it changes the partition function by a constant factor.

A.2. **Some Properties of Homology and Cohomology.** In this section we review a number of basic properties of homology and cohomology groups of percolation subcomplexes. From these statements we will see that there is a straightforward relationship between $H^{d-2}(P;\ G)$ for a general coefficient group G and $H^{d-2}(P;\ \mathbb{Q})$.

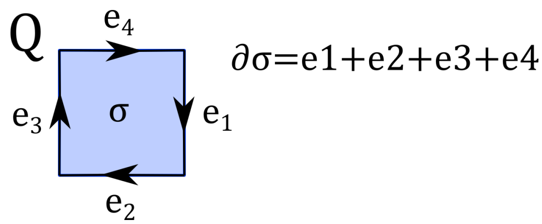


FIGURE 11. The boundary map for a two dimensional plaquette, reproduced from [DS22].

Let $0 \leq j \leq d-1$. The homology of P with coefficients in the integers $\mathbb{Z}$ is a finitely generated abelian group so

$$H_j\left(P;\ \mathbb{Z}\right) \cong \mathbb{Z}^{\mathbf{b}_j(P;\,\mathbb{Z})} \oplus T_j\left(P\right)\ , \tag{14}$$

where $\mathbf{b}_j\left(P;\ \mathbb{Z}\right)$ is the rank of the $\mathbb{Z}$-module $H_j\left(P;\ \mathbb{Z}\right)$. The second summand, $T_i\left(P\right)$, consists of all elements of $H_j\left(P;\ \mathbb{Z}\right)$ of finite order and is referred to as the *torsion subgroup* of $H_j\left(P;\ \mathbb{Z}\right)$. When $T_i\left(P\right) = 0$,, we call $H_j\left(P;\ \mathbb{Z}\right)$ torsion-free.

It turns out that $(d-1)$-dimensional percolation complexes are torsion-free in dimensions $(d-1)$ and $(d-2)$. This can be be shown using Alexander duality, which relates the homology of a subset of Euclidean space with that of its complement. It is most naturally stated for subcomplexes of the d-dimensional sphere S^d; we can embed our complexes into we can embed our percolation subcomplexes in S^d by adding a point at infinity. Here is where it is important to note that we are using reduced homology and cohomology. The following statement is not true otherwise when $i = 0$ or $i = d-1$.

Theorem 65 (Alexander Duality)**.** *Let K be a nonempty, locally contractible, proper subspace of the d-dimensional sphere S^d. Then there is an isomorphism*

$$\hat{\mathcal{I}} : H_j\left(S^d \setminus K;\ \mathbb{Z}\right) \cong H^{d-j-1}\left(K;\ \mathbb{Z}\right)\ .$$

We have the following corollary.

Proposition 66 (Corollary 3.46 of [Hat02])**.** *Let P be a percolation subcomplex of a box in $\mathbb{Z}^d$. Then $T_{d-1}\left(P;\ \mathbb{Z}\right) = T_{d-2}\left(P;\ \mathbb{Z}\right) = 0$.*

It is not hard to prove this statement for the $(d-1)$-dimensional percolation subcomplexes using the tools developed below in Section A.3. In particular, the complement of P has the same cohomology as the dual graph, and graphs do not have torsion in their homology or cohomology.

Next, the *Universal Coefficients Theorem for Homology* (Theorem 3A.3 in [Hat02]) allows us to compute the homology of P with coefficients in abelian group G in terms of the homology with coefficients in $\mathbb{Z}$.

In particular, it implies that

$$H_j(P;G) \cong (H_j(P;\ \mathbb{Z}) \otimes G) \oplus \operatorname{Tor}(H_{j-1}(P;\ \mathbb{Z}), G)\ . \tag{15}$$

For a definition of the Tor functor, please refer to Section 3A of [Hat02]. Here, we only require two facts. First, $\operatorname{Tor}(0, G)$ vanishes for any abelian group G. Second, $\operatorname{Tor}(H_{j-1}(X;\ \mathbb{Z}), \mathbb{Q}) = 0$ for any topological space X. It follows that $\mathbf{b}_j(X;\ \mathbb{Z}) = \mathbf{b}_j(X;\ \mathbb{Q})$, the dimension of the vector space $H_{j-1}(X;\ \mathbb{Q})$. In particular, we may rewrite (14) as

$$H_j(P;\ \mathbb{Z}) \cong \mathbb{Z}^{\mathbf{b}_j(P;\,\mathbb{Q})} \oplus T_j(P;\ \mathbb{Z})\ . \tag{16}$$

There is also a *Universal Coefficients Theorem for Cohomology* (Theorem 3.2 of [Hat02]) that relates the homology and cohomology groups of a topological space. We do not need the full statement here, only the following corollary.

Corollary 67. *If $H_{j-2}(P;\ G)$ vanishes (or, more generally, is a free G-module) then*

$$H^{j-1}(P;\ G) \cong \operatorname{Hom}(H_{j-1}(P;\ G), G)\ .$$

In particular, when $G = \mathbb{Z}_q$ we have that

$$H^{j-1}(P;\ \mathbb{Z}_q) \cong H_{j-1}(P;\ \mathbb{Z}_q)\ . \tag{17}$$

We can now compare the $(d-2)$-homology of P across different coefficients.

Proposition 68. *Let P be a $(d-1)$-dimensional percolation subcomplex of a box $r \subset \mathbb{Z}^d$. Then*

$$H^{d-2}(P;\ \mathbb{Z}_q) \cong \mathbb{Z}_q^{\mathbf{b}_{d-2}(P;\,\mathbb{Q})}\ .$$

Proof. First,

$$H_{d-2}(P;\ \mathbb{Z}) \cong \mathbb{Z}^{\mathbf{b}_{d-2}(P;\,\mathbb{Q})}$$

by (16) and Proposition 66. It follows from (15) and the fact that $H_{d-3}(P;\ \mathbb{Z}) = 0$ that

$$H_{d-2}(P;\ \mathbb{Z}_q) \cong \mathbb{Z}_q^{\mathbf{b}_{d-2}(P;\,\mathbb{Q})}\ .$$

Finally, by Corollary 67,

$$H^j(P;\ \mathbb{Z}_q) \cong \operatorname{Hom}\left(\mathbb{Z}_q^{\mathbf{b}_{d-2}(P;\,\mathbb{Q})}, \mathbb{Z}_q\right) \cong \mathbb{Z}_q^{\mathbf{b}_{d-2}(P;\,\mathbb{Q})}\ .$$

∎

We will use this statement to work with $\mathbb{Q}$ coefficients instead of $\mathbb{Z}_q$ coefficients in codimension one. See Proposition 19.

A.3. **Duality and Percolation Subcomplexes.** Let P be a percolation subcomplex of $\mathbb{Z}^d$ or of a box contained therein, and let Q be the dual complex. The topological properties of Q are closely related to those of the complement $\mathbb{R}^d \setminus P$. Before providing a precise statement, we review the definition of a deformation retraction from a topological space X to a subset $Y \subset X$. We say that Y deformation retracts to X if there is a continuous function $I : X \times [0,1] \to Y$ so that $I(x,0) = x$ for all $x \in X$, $I(X,1) = Y$ and $I(y,t) = y$ for all $y \in Y$ and all $t \in [0,1]$. It is a standard fact that a deformation retraction is a homotopy equivalence and therefore induces isomorphisms on homology and cohomology groups (see chapter 2.1 of [Hat02]).

Lemma 69. *Let P be an i-dimensional percolation subcomplex of $X \subset \mathbb{Z}^d$ with dual Q. Also, let r be a box in $\mathbb{Z}^d$.*

(1) *If $X = \mathbb{Z}^d$, $\mathbb{R}^d \setminus P$ deformation retracts to Q.*

(2) *If $X = \overline{r}$, $\mathbb{R}^d \setminus P$ deformation retracts to $Q \cup \partial r^\bullet$.*

(3) *If $X = r$, P deformation retracts to a subcomplex P' of $\overline{r^{-1}}$ whose dual complex Q' satisfies $Q' \cup \partial \overline{r^\bullet} = (Q \cap \overline{r^\bullet}) \cup \partial \overline{r^\bullet}$.*

Proof. In the first case, the same construction as in the proof of Lemma 8 of [DKS20] yields a deformation retraction $I : \mathbb{R}^d \setminus P \to Q$. In the second case, we combine two deformation retractions: the deformation retraction $I : \mathbb{R}^d \setminus \hat{P} \to Q$ defined for the subcomplex $\hat{P}$ of $\mathbb{Z}^d$ obtained by adding the $(i-1)$-skeleton of $\mathbb{Z}^d$ to P, and the straight line deformation retraction $J : \mathbb{R}^d \setminus r^\bullet \times [0,1] \to \partial r^\bullet$. Then, we can define a deformation retraction $I' : \mathbb{R}^d \setminus P \times [0,1] \to Q$ by

$$I'(x,t) = \begin{cases} I(x,t) & x \in r^\bullet \\ J(x,t) & x \in \mathbb{R}^d \setminus r^\bullet \end{cases} .$$

See Figure 12. In the third case, all i-dimensional cells that share an $(i-1)$-face with ∂r are orthogonal to a face of ∂r and can be deformation retracted to ∂r^{-1} via straight lines. The claim then follows from the second case. □

Proof of Proposition 11. First, let P be a percolation subcomplex of $\overline{r}$. Compactify $\mathbb{R}^d$ to S^d by adding a point at infinity. Since $i > 0$ and P is bounded, applying Alexander duality (Theorem 65) gives

$$H_i(P;\, \mathbb{Z}) \cong H_i(P \cup \{\infty\};\, \mathbb{Z}) \cong H^{d-i-1}\left(\mathbb{R}^d \setminus P;\, \mathbb{Z}\right) .$$

The cohomology of $\mathbb{R}^d \setminus P$ is isomorphic to that of $Q \cup \partial r^\bullet$ by the previous proposition.

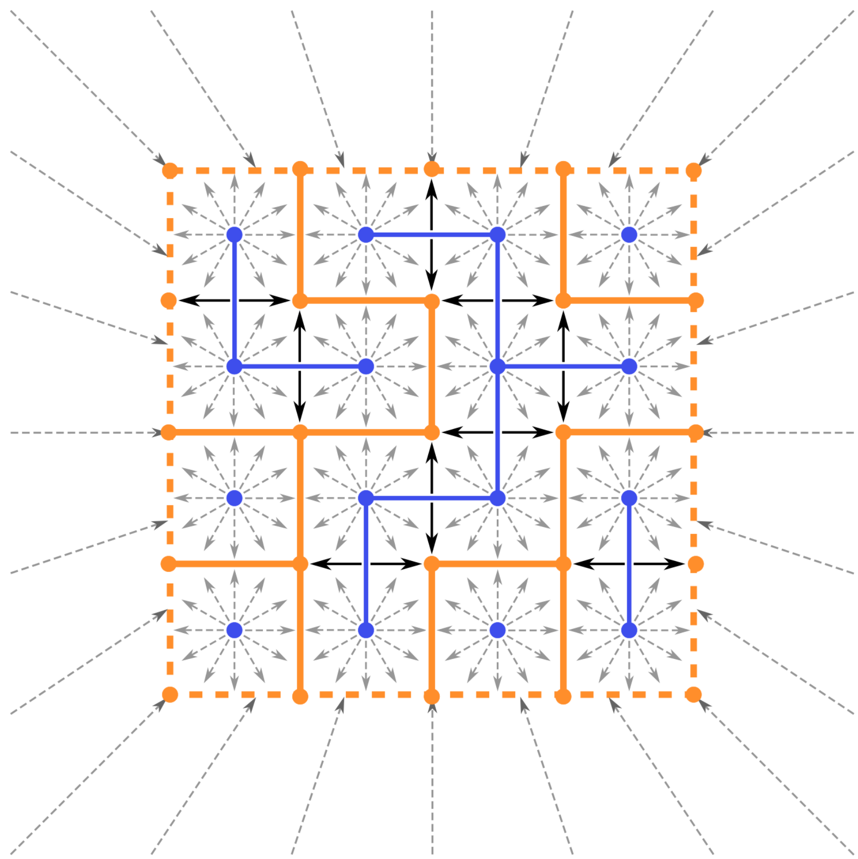


FIGURE 12. The deformation retract I' defined in the proof of Lemma 69. P is shown in blue, Q in orange, and $\partial r^{\bullet}$ by a dashed orange line. The deformation retract proceeds first along the gray arrows and then along the black ones. Adapted from [DKS20], which includes details on the construction inside the inner square. Observe that in the case $i = 1$, the addition of the boundary of $r^{\bullet} \setminus r$ to Q has the effect of merging all vertices in the boundary of the square. As we will see below, this is related to the duality between random cluster models with free and wired boundary conditions.

When P is a subcomplex of r, the existence of such an isomorphism follows because of the third item in the previous proposition. $\square$

APPENDIX B. WIRED BOUNDARY CONDITIONS FOR PLGT

Here we detail the adjustments required to adapt the arguments of Section 4 to infinite complexes constructed using wired boundary conditions.

B.1. **Finite Volume Measures.** The relationship between the finite volume PRCM with wired boundary conditions and the finite volume PLGT with wired or η boundary conditions is analogous to the one discussed Section 4.1. We recall the definitions. First, the PRCM with wired boundary conditions on a box r is the measure

$$\tilde{\mu}^{\mathbf{w}}_{r,p,G,i}\left(P\right) \propto p^{|P|}\left(1-p\right)^{\left|r^{(i)}\right|-|P|}\left|H^{i-1}\left(P^{\mathbf{w}};\ G\right)\right|\,,$$

where $P^{\mathbf{w}}$ is the percolation subcomplex of $\overline{r}$ obtained by adding all i-plaquettes in ∂r to P.

Let

$$\psi : C^{i-1}\left(r;\ \mathbb{Z}_q\right) \to C^{i-1}\left(\partial r;\ \mathbb{Z}_q\right)$$

be the map which restricts a cochain on r to one on ∂r, and let

$$D_\eta\left(r;\ \mathbb{Z}_q\right) = \psi^{-1}\left(\eta\right)\,.$$

Also, set

$$D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right) = \ker \delta \circ \psi\,.$$

Then $\nu^{\mathbf{w}}_{r,\beta,q,i-1}$ $\nu^{\eta}_{r,\beta,q,i-1}$ are respectively the Gibbs measures on $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)$ and $D_\eta\left(r;\ \mathbb{Z}_q\right)$ induced by the Hamiltonian (1).

We demonstrate that the expectations of gauge invariant random variables coincide for wired and η boundary conditions. Note that the sets $D_\eta\left(r;\ \mathbb{Z}_q\right)$ are the cosets of $\ker\psi'$ in $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)$, where ψ' is the restriction of ψ to $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)$,

The gauge action of $C^{k-2}\left(r;\ \mathbb{Z}_q\right)$ on $C^{k-1}\left(r;\ \mathbb{Z}_q\right)$ is defined by setting $\phi_h :$ $C^{k-1}\left(r;\ \mathbb{Z}_q\right) \to C^{k-1}\left(r;\ \mathbb{Z}_q\right)$ to be the map

$$\phi_h\left(f\right) = f + \delta h$$

for $f \in C^{k-1}\left(r;\ \mathbb{Z}_q\right)$ and $h \in C^{k-2}\left(r;\ \mathbb{Z}_q\right)$.

Lemma 70. *The gauge action descends to a transitive group action on* $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)/\ker\psi'$.

Proof. First, we show that the induced action on $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)/\ker\psi'$ is well-defined. Let $\eta \in Z^{i-1}\left(\partial r;\ \mathbb{Z}_q\right)$, $h \in C^{k-2}\left(r;\ \mathbb{Z}_q\right)$, and $f_1, f_2 \in D_\eta\left(r;\ \mathbb{Z}_q\right)$. Then

$$\psi'\left(\phi_h\left(f_1\right)\right) = \psi'\left(\phi_h\left(f_2\right)\right) = \eta + \psi'\left(\delta h\right)$$

and

$$\delta\left(\eta + \psi'\left(\delta h\right)\right) = \delta\eta + \psi'\left(\delta \circ \delta h\right) = 0$$

so $\eta + \psi'\left(\delta h\right) \in Z^{i-1}\left(\partial r;\ \mathbb{Z}_q\right)$ and ϕ_h maps $D_\eta\left(r;\ \mathbb{Z}_q\right)$ to $C^{i-1}_{\eta+\psi(\delta h)}\left(r;\ \mathbb{Z}_q\right)$ bijectively. Thus the action of h on $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)/\ker\psi'$ is well-defined.

Next, we prove transitivity. Let $\eta_1, \eta_2 \in Z^{i-1}(\partial r;\ \mathbb{Z}_q)$ and set $\eta = \eta_2 - \eta_1$. As ∂r is homeomorphic to the $(d-1)$-dimensional sphere and $i < d$,

$$H^{i-1}(\partial r;\ \mathbb{Z}_q) = Z^{i-1}(\partial r;\ \mathbb{Z}_q) / B^{i-1}(\partial r;\ \mathbb{Z}_q) = 0$$

so $\eta = \delta h$ for some $h \in C^{i-2}(\partial r;\ \mathbb{Z}_q)$. We can extend h to an element $\hat{h}$ of $C^{i-2}(r;\ \mathbb{Z}_q)$ (by, for example, requiring it to vanish on all other $(i-2)$-faces). Then

$$C^{i-1}_{\eta_2}(r;\ \mathbb{Z}_q) = \phi_{\hat{h}}\left(C^{i-1}_{\eta_1}(r;\ \mathbb{Z}_q)\right) ,$$

as desired. □

Corollary 71. *For $\eta_1, \eta_2 \in Z^{i-1}(\partial r;\ \mathbb{Z}_q)$ there exists an $h \in C^{i-2}(r;\ \mathbb{Z}_q)$ so that*

$$(\phi_h)_* \left(\nu^{\eta_1}_{r,\beta,q,i-1}\right) = \nu^{\eta_2}_{r,\beta,q,i-1}.$$

In particular, if $\gamma \in Z_{i-1}(r;\ \mathbb{Z}_q)$ then

$$\mathbb{E}_{\nu^{\eta_1}_{r,\beta,q,i-1}}(W_\gamma) = \mathbb{E}_{\nu^{\eta_2}_{r,\beta,q,i-1}}(W_\gamma) = \mathbb{E}_{\nu^{\mathbf{w}}_{r,\beta,q,i-1}}(W_\gamma) .$$

Proof. By transitivity, there exists an h so that

$$\phi_h\left(C^{i-1}_{\eta_1}(r;\ \mathbb{Z}_q)\right) = C^{i-1}_{\eta_2}(r;\ \mathbb{Z}_q) .$$

The first claim follows because gauge transformations preserve the Hamiltonian. The gauge-invariance of Wilson loop variables and the law of total conditional expectation implies the second as

$$\mathbb{E}_{\nu^{\mathbf{w}}_{r,\beta,q,i-1}}(W_\gamma) = \sum_{\eta \in Z^{i-1}(\partial r;\ \mathbb{Z}_q)} \mathbb{E}_{\nu^{\eta}_{r,\beta,q,i-1}}(W_\gamma)\, \nu^{\mathbf{w}}_{r,\beta,q,i-1}\left(D_\eta(r;\ \mathbb{Z}_q)\right) .$$

□

Next, we couple $\nu^{\eta}_{r,\beta,q,i-1}$ with the wired PRCM measure. We perform a preliminary computation.

Lemma 72. *Let P be a percolation subcomplex of r and $\eta \in Z^{i-1}(\partial r;\ \mathbb{Z}_q)$. Then*

$$\left|Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q) \cap D_\eta(r;\ \mathbb{Z}_q)\right| = \frac{\left|Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)\right|}{\left|Z^{i-1}(\partial r;\ \mathbb{Z}_q)\right|} .$$

Proof. By construction, $Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q) \subset D_{\mathbf{w}}(r;\ \mathbb{Z}_q)$. The action of gauge transformations sends $Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)$ to itself and it acts transitively on the sets $D_{\eta'}(r;\ \mathbb{Z}_q)$, so we may find an $f \in Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)$ so that $\psi(f) = \eta$. In other words, the restriction of ψ to $Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)$ surjects onto $Z^{i-1}(\partial r;\ \mathbb{Z}_q)$. Thus,

$$\frac{Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)}{Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)\cap\ker\psi}\cong Z^{i-1}(\partial r;\ \mathbb{Z}_q)\ ,$$

and the statement follows. □

Proposition 73. *Let $r\subset\mathbb{Z}^d$ be a box, $q\in\mathbb{N}+1$, $\beta\in[0,\infty)$, and $p=1-e^{-\beta}$. Let $\#=\mathbf{w}$ or $\#=\eta$ for $\eta\in Z^{i-1}(\partial r;\ \mathbb{Z}_q)$. Define a coupling on $C^{i-1}_{\#}(r;\ \mathbb{Z}_q)\times\{0,1\}^{r^{(i)}}$ by*

$$\kappa^{\#}(f,P)\propto\prod_{\sigma\in r^{(i)}}\left[(1-p)\,I_{\{\sigma\notin P\}}+pI_{\{\sigma\in P,\delta f(\sigma)=0\}}\right]\ .$$

Then $\kappa^{\#}$ has the following marginals.

- *The first marginal is $\tilde{\mu}^{\mathbf{w}}_{X,p,\mathbb{Z}_q,i}$.*
- *The second marginal is $\nu^{\#}_{r,\beta,q,i-1}$*

Proof. The computation of the first marginal is no different than before [HS16].

The derivation of the second marginal is nearly identical as that in the proof of Proposition 12. For $\#=\mathbf{w}$, the only differences are that the terms $Z^{i-1}(P;\ \mathbb{Z}_q), B^{i-1}(P;\ \mathbb{Z}_q)$, and $H^{i-1}(P;\ \mathbb{Z}_q)$ for P are replaced by the corresponding quantities for $P^{\mathbf{w}}$. For $\#=\eta$, on the fourth line one obtains the quantity $|Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)\cap D_\eta(r;\ \mathbb{Z}_q)|$ in place of $|Z^{i-1}(P;\ \mathbb{Z}_q)|$, but this is proportional to $|Z^{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)|$ by Lemma 72.

□

Proposition 74. *Let r be a box in $\mathbb{Z}^d$, $0<i<d-1$, $q\in\mathbb{N}+1$, $\eta\in Z^{i-1}(\partial r;\ \mathbb{Z}_q)$, and $\gamma\in Z_{i-1}(r;\ \mathbb{Z}_q)$. Denote by $V^{\mathbf{w}}_\gamma$ the event that $[\gamma]=0$ in $H_{d-2}(P\cup\partial r;\ \mathbb{Z}_q)$. Then*

$$\mu\left(V^{\mathbf{w}}_\gamma\right)=\mathbb{E}_{\nu^{\mathbf{w}}}(W_\gamma)=\mathbb{E}_{\nu^\eta}(W_\gamma)$$

where $\nu^\eta=\nu^{\#}_{r,\beta,q,i-1,d}$ is the PLGT with boundary conditions η, $\tilde{\mu}=\tilde{\mu}^{\mathbf{w}}_{r,1-e^{-\beta},\mathbb{Z}_q,i}$ is the corresponding PRCM, and $V^{\mathbf{w}}_\gamma$ is the event that $[\gamma]=0$ in $H_{i-1}(P^{\mathbf{w}};\ \mathbb{Z}_q)$.

Proof. The first equality follows from the argument in the proof of Proposition 14 applied to the complex $P\cup\partial r$, where P is distributed according to $\tilde{\mu}$. The second is Corollary 71. □

B.2. **The Infinite Volume Limit.** The proofs in this section are similar to that in Section 4.2, but the technical details are different. Just like the measure $\nu_r^{\mathbf{w}}$ is defined on a subgroup $D_{\mathbf{w}}\left(r;\ \mathbb{Z}_q\right)$ of $Z^{i-1}\left(r;\ \mathbb{Z}_q\right)$, the conditional measure $\nu_{\mathbb{Z}^d}^{\mathbf{w}}$ given P will be defined on a subgroup of $D_{\mathbf{w}}\left(P;\ \mathbb{Z}_q\right)$ of $Z^{i-1}\left(P;\ \mathbb{Z}_q\right)$. We would like to impose an additional constraint to require that cocycles vanish on cycles which are "homologous to ∞." Recalling the definition $C^{i-1}\left(P;\ \mathbb{Z}_q\right) = \operatorname{Hom}\left(C_{i-1}\left(P;\ \mathbb{Z}_q\right), \mathbb{Z}_q\right)$, note that we could equivalently define $Z^{i-1}\left(P;\ \mathbb{Z}_q\right)$ to be the kernel of the restriction map

$$C^{i-1}\left(P;\ \mathbb{Z}_q\right) \to \operatorname{Hom}\left(B_{i-1}\left(P;\ \mathbb{Z}_q\right), \mathbb{Z}_q\right) .$$

We will enlarge $B_{i-1}\left(P;\ \mathbb{Z}_q\right)$ to include finite $(i-1)$-cycles that are the boundaries of infinite i-cycles. Toward that end, let $C_j^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right)$ denote the group of (possibly infinite) formal sums of plaquettes in P with coefficients in $\mathbb{Z}_q$ (here lf stands for "locally finite"). Set

$$B_j^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right) = \ker\left(\partial : C_{j+1}^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right) \to C_{j+1}^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right)\right) ,$$

let

$$\chi : C^{i-1}\left(P;\ \mathbb{Z}_q\right) \to \operatorname{Hom}\left(C_{i-1}\left(P;\ \mathbb{Z}_q\right) \cap B_{i-1}^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right), \mathbb{Z}_q\right)$$

be the natural map, and set

$$D_{\mathbf{w}}\left(P;\ \mathbb{Z}_q\right) = \ker \chi .$$

For percolation subcomplex P of $\mathbb{Z}^d$ let $P_N^{\mathbf{w}}$ be the percolation subcomplex of $\overline{\Lambda_N}$ obtained by adding all i-plaquettes in $\partial\Lambda_N$ to P_N. Now, we can replace the maps $\phi_{N,n}$ and $\phi_{\infty,n}$ with corresponding maps $\hat{\phi}_{N,n} : Z^{i-1}\left(P_N^{\mathbf{w}};\ \mathbb{Z}_q\right) \to Z^{i-1}\left(P_n;\ \mathbb{Z}_q\right)$ and $\hat{\phi}_{\infty,n} : D_{\mathbf{w}}\left(P;\ \mathbb{Z}_q\right) \to Z^{i-1}\left(P_n;\ \mathbb{Z}_q\right)$. Set $\hat{Y}_{N,n} = \operatorname{im}\hat{\phi}_{N,n}$ and $\hat{Y}_{\infty,n} = \operatorname{im}\hat{\phi}_{\infty,n}$. Before constructing the infinite volume coupling, we provide analogues for Lemmas 15 and 16.

Lemma 75. *Let $f \in Z^{i-1}\left(P_n;\ \mathbb{Z}_q\right)$ and $N > n$. Then $f \notin \operatorname{im}\hat{\phi}_{N,n}$ if and only if there exists a cycle $\sigma \in Z_{i-1}\left(P_n;\ \mathbb{Z}_q\right)$ so that $f(\sigma) \neq 0$ and $\sigma \in B_{i-1}\left(P_N^{\mathbf{w}};\ \mathbb{Z}_q\right)$.*

Similarly $f \notin \operatorname{im}\hat{\phi}_{\infty,n}$ if and only if there exists a cycle $\sigma \in Z_{i-1}\left(P_n;\ \mathbb{Z}_q\right)$ so that $f(\sigma) \neq 0$ and $\sigma \in B_{i-1}^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right)$.

Proof. The proof of the first claim is exactly the same as that of Lemma 15. For the second, one direction is immediate. For the other, it suffices to show that if $f \in Z^{i-1}\left(P_n;\ \mathbb{Z}_q\right)$ satisfies $f(\sigma) = 0$ for all $\sigma \in Z_{i-1}\left(P_n;\ \mathbb{Z}_q\right) \cap B_{i-1}^{\mathrm{lf}}\left(P;\ \mathbb{Z}_q\right)$ then it can be extended to obtain an element of $D_{\mathbf{w}}\left(P;\ \mathbb{Z}_q\right)$. In this case, there exists an $N > n$ so that no element in $Z_{i-1}\left(P_n;\ \mathbb{Z}_q\right)$ is homologous to a cycle supported on $\partial\Lambda_N$. By the first claim, we may find an $f' \in Z^{i-1}\left(P_N^{\mathbf{w}};\ \mathbb{Z}_q\right)$ so that $\hat{\phi}_{N,n}\left(f'\right) = f$. As $H^{i-1}\left(\partial\Lambda_N;\ \mathbb{Z}_q\right) = 0$, there exists an $h' \in C^{i-2}\left(\partial\Lambda_N;\ \mathbb{Z}_q\right)$

so that $\delta h' = f' |_{\partial \Lambda_N}$. We can extend h' to a cochain $h \in C^{i-2}\left(\mathbb{Z}^d;\ \mathbb{Z}_q\right)$ by setting it to equal 0 on all $(i-2)$-plaquettes outside of $\partial\Lambda_N$. Let $g \in C^{i-1}\left(P;\ \mathbb{Z}_q\right)$ be the cochain

$$g\left(\sigma\right) = \begin{cases} f'\left(\sigma\right) & \sigma \in \Lambda_N \\ \delta h\left(\sigma\right) & \sigma \notin \Lambda_N \end{cases},$$

Then g is a cocycle and $\phi_{\infty,n}\left(g\right) = f$. ☐

Lemma 76. *Fix n.*

- $\hat{Y}_{N,n} = \hat{Y}_{\infty,n}$ *for all sufficiently large N.*
- *For N sufficiently large as in the previous statement, the pushforward by $\hat{\phi}_{N,n}$ of the uniform measure on $Z^{i-1}\left(P_N^{\mathbf{w}};\ \mathbb{Z}_q\right)$ is the uniform measure on $\hat{Y}_{\infty,n}$.*
- *For all $N > n$, $\phi_{N,n}\left(\hat{Y}_{\infty,N}\right) = \hat{Y}_{\infty,n}$ and the pushforward of the uniform measure on $\hat{Y}_{\infty,N}$ by $\phi_{N,n}$ is the uniform measure on $\hat{Y}_{\infty,n}$.*

Note that the third item is for the map $\phi_{N,n}$ rather than $\hat{\phi}_{N,n}$.

Proof. The proof is exactly the same as that of Lemma 16. ☐

We can now construct the infinite volume measures with wired boundary conditions, and prove Theorem 8 for wired boundary conditions.

Proposition 77. *Let $0 < i < d-1$, $q \in \mathbb{N}+1$, $\beta \in (0,\infty)$ and $p = 1 - e^{-\beta}$. The weak limits*

$$\mu^{\mathbf{w}}_{\mathbb{Z}^d,p} = \lim_{N\to\infty} \mu^{\mathbf{w}}_{\Lambda_N,p,q,d-1}$$

and

$$\nu^{\mathbf{w}}_{\mathbb{Z}^d} = \lim_{n\to\infty} \nu^{\mathbf{w}}_{\Lambda_N,\beta,q,d-1}$$

exist and are translation invariant. Moreover, if γ is a $(i-1)$-cycle in $\mathbb{Z}^d$ then

$$\mathbb{E}_{\nu^{\mathbf{w}}_{\mathbb{Z}^d}}\left(W_\gamma\right) = \mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(V_\gamma^{\inf}\right).$$

Proof. The weak limit of $\mu^{\mathbf{w}}_{\Lambda_N,p,q,d-1}$ exists by a standard monotonicity argument. In fact, we may couple the PRCMs with $\overline{P}\left(1\right) \supset \overline{P}\left(2\right) \supset \ldots$ where $P\left(N\right) \sim \mu^{\mathbf{w}}_{\Lambda_n,p,q,d-1}$, $\overline{P}\left(N\right)$ is the percolation subcomplex of $\mathbb{Z}^d$ so that $\overline{P}\left(N\right) \cap \Lambda_N = P\left(N\right)$ and $\overline{P}\left(N\right) \cap \Lambda_n^c$ contains all possible i-plaquettes, and $P = \cap_N P\left(N\right) \sim \mu^{\mathbf{w}}_{\mathbb{Z}^d,p}$.

We will construct a coupling on $\Sigma \times \Omega$ whose first marginal is $\mu^{\mathbf{w}}_{\mathbb{Z}^d,p}$ and whose second marginal is the weak limit of $\nu^{\mathbf{w}}_{\Lambda_N,\beta,q,d-1}$ as $N \to \infty$. As before, set:

$$\kappa^{\mathbf{w}}_{\beta,q}\left(\mathcal{K}\left(P_n\right) \times \mathcal{L}\left(f_n\right)\right) = \sum_{H \subseteq Z^{i-1}(P_n;\, \mathbb{Z}_q)} \frac{I_{\{f_N \in H\}}}{|H|} \mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(\left\{\hat{Y}_{\infty,n} = H\right\} \cap \mathcal{K}\left(P_n\right)\right) .$$

The proof that this extends to a translation-invariant measure on $\Omega \times \Sigma$ proceeds exactly the same as that for free boundary conditions, and the argument as to why the marginals are as claimed is nearly identical. The only difference is the justification for why we can choose N large enough so that $P\left(N\right) \cap \Lambda_n = P_n$: it is because $\overline{P}\left(N\right) \searrow P$ and $\overline{P}\left(N\right) \cap \Lambda_n = P\left(n\right)$ for $N > n$.

We now prove the statement relating Wilson loop variables to V^{inf}_γ. Fix $\gamma \in Z_{i-1}\left(\mathbb{Z}^d;\, \mathbb{Z}_q\right)$. Let $V'\left(n\right)$ be the event that $[\gamma] = 0$ in $H_{i-1}\left(P\left(N\right) \cup \partial\Lambda_N;\, \mathbb{Z}_q\right)$ and let $V''\left(N\right)$ be the event that $[\gamma] = 0$ in $H_{i-1}\left(\overline{P}\left(n\right);\, \mathbb{Z}_q\right)$. Note that if γ is supported on Λ_N, then $V''\left(N\right) \iff V'\left(N\right)$. This, together with the fact that

$$V^{\mathrm{inf}}_\gamma = \cap_{N \in \mathbb{N}} V''\left(N\right)$$

gives the first inequality below:

$$\mu^{\mathbf{w}}_{\mathbb{Z}^d,p}\left(V^{\mathrm{inf}}_\gamma\right) = \lim_{N \to \infty} \mu^{\mathbf{w}}_{\Lambda_N}\left(V'\left(N\right)\right) = \lim_{N \to \infty} \mathbb{E}_{\nu^{\mathbf{w}}_{\Lambda_N}}\left(W_\gamma\right) = \mathbb{E}_{\nu^{\mathbf{w}}_{\mathbb{Z}^d}}\left(W_\gamma\right) ,$$

where the second equality is Proposition 74 and the third follows by weak convergence.

□

Index of Notation

Notation	Description	Section
$\beta, \beta_c, \beta_{\mathrm{slab}}$	Inverse temperature, critical points in the lattice and slab	2
γ	Loop or $(i-1)$-chain	2
∂	Boundary operator	A.1
δ	Coboundary operator	A.1
$\mu^{\xi}_{X,p,q,i}$	i-dimensional PRCM on a complex X with parameters p, q coefficients in $\mathbb{Q}$, and boundary conditions ξ	2
$\mu^{\bullet}_{X,p^*}, \mu^{\bullet,\mathbf{w}}_{X,p^*}$	Dual classical RCM on a complex X with parameter p^* with free or wired boundary conditions	2
$\tilde{\mu}_{X,p,G,i}$	PRCM on X with coefficients in G and parameter p	2

$\nu^{\xi}_{X,\beta,q,i-1}$	$(i-1)$-dimensional PLGT on a complex X with states in $\mathbb{Z}_q$, inverse temperature β, and boundary conditions ξ	2
ρ_Y	Chain associated with a subspace Y	8
$\Gamma(v,w), \Gamma_{\mathbb{V}}(\mathbb{S},\mathbb{S}')$	Event that two vertices of $\mathbb{Z}^2$ are separated by a loop	7.2
Λ_n	Cube $[-n,n]^d$	4.2
$\Xi_{r,L}$	Event that r is separated from ∂r^L by a hypersurface of plaquettes	9.1
$\mathbf{b}_k$	k-th Betti number	2
$\mathbb{S}$	Renormalized site	7.1
$\mathbb{V}$	Renormalized lattice	7.1
$\mathbb{Z}(q)$	Multiplicative group of q-th roots of unity	2
$\mathcal{F}(\mathbb{S})$	Event that $\mathbb{S}$ is “full,” or has a connected set of bonds containing crossings in both directions	7.1
$\mathcal{H}(A)$	Hamming distance from an event A	7.1
$\mathcal{O}(\mathbb{S})$	Event that $\mathbb{S}$ is open, meaning that it is full and connected to its full neighbors	7.1
$\mathcal{S}_M$	Slab of thickness M	7
$B_i(X;\ G), B^i(X;\ G)$	Boundary and coboundary groups	A.1
$\mathcal{C}_v$	Connected component of the vertex v	8
$C_i(X,G), C^i(X,G)$	Chain and cochain groups	A.1
C_t	Event that s is homologous to ∂t in P	9.2
D_t	Event that there are plaquette crossings forming a “tube” around s	9.2
$\overline{C}_t$	$C_t \cap D_t$	9.2
$E_{u,L}$	Event that all plaquettes within distance L from a box u are open	9.2
F_h	Event that a ray is separated from an orthogonal plane in the dual bond percolation	9.2
G	Abelian coefficient group	2
$H_i(X,G), H^i(X,G)$	Homology and cohomology groups	A.1
I_A	Indicator function for the event A	4.1

M	Slab thickness	7
$M(r), m(r)$	Maximum and minimum nonzero dimensions of a box r	2
N	Renormalized site side length	7.1
O	Set of open sites of the renormalized system	7.1
P	PRCM plaquette subcomplex	2
Q	Dual RCM bond set	2
$R(r), R_j(r)$	Bond crossing events for r in the longest direction or the j-th direction	7.1
$R_j^{\square}(r)$	$(d-1)$-plaquette crossing events for r orthogonal to the $j-$th direction	9
$V_\gamma^{\text{fin}}, V_\gamma^{\text{inf}}$	Events for γ being a boundary of plaquettes	2
W_γ	Wilson loop variable	2
X	General cell complex	A.1
$X^{(k)}$	k-skeleton of X	A.1
$\lvert X\rvert$	Number of top dimensional cells of a complex X	2
$Z_i(X,G), Z^i(X,G)$	Cycle and cocycle groups	A.1
d	Dimension of the lattice $\mathbb{Z}^d$	1
f	PLGT state or $(i-1)$-cochain	2
$l(\gamma_1, \gamma_2)$	Linking number of γ_1 and γ_2	6
p, p_c, p_{slab}	Percolation parameter, critical points in the lattice and slab	2
$p^*(p)$	Dual parameter to p with respect to q	2
q	PRCM/PLGT parameter	2
r	Box in $\mathbb{Z}^d$ without boundary cells	2
$\overline{r}$	Union of r with its boundary cells	2
$\hat{r}$	Shift of r into the upper slab	7.1
$\tilde{r}$	Union $r \cup \hat{r}$	7.1
r^L	Thickening of the box r by L	7.1
s	$(d-2)-$face of the support of γ	9.2
$t(s, L)$	Thickening of s by L in the remaining 2 directions	9.2

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Email address: `paul.duncan@mail.huji.ac.il`

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem 91904, Israel

Email address: `bschwei@gmu.edu`

Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA