A Cellular Representation of Potts Lattice Gauge Theory

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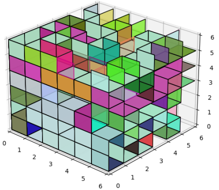
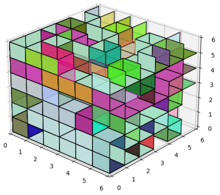
May 7, 2025

The Cubical Complex $\mathbb{Z}^d$



This talk will concern random subcomplexes (and random co-chains) on the cubical complex $\mathbb{Z}^d$: the collection of i-cubes whose vertices are integer lattice points for $0 \leq i \leq 3$. 0-faces are vertices, 1-faces are edges, 2-faces are plaquettes, and 3-faces are cubes, and so on.

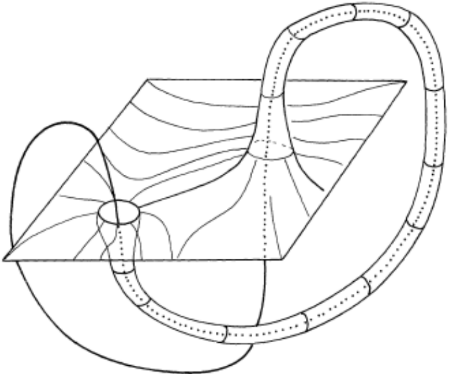
Bernoulli Plaquette Percolation



2-dimensional Bernoulli plaquette percolation on $\mathbb{Z}^3$ with probability p is the random subcomplex of $\mathbb{Z}^3$ including all vertices and edges, where 2-dimensional plaquettes are included independently with probability p.

On a Sharp Transition from Area Law to Perimeter Law in a System of Random Surfaces*

M. Aizenman[1,**], J. T. Chayes[2,***], L. Chayes[2,***], J. Fröhlich[3], and L. Russo[4]



For a 1-cycle γ let W_γ be the event that γ is "bounded by a surface of plaquettes" in two-dimensional plaquette percolation.

On a Sharp Transition from Area Law to Perimeter Law in a System of Random Surfaces*

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Theorem 1.1. *For rectangular* $(N \times M)$ *loops* γ, *in a lattice plane, the quantity* $\langle W_\gamma \rangle_p$ *has the following asymptotic behavior:*

$$\langle W_\gamma \rangle_p \sim \begin{cases} \exp[-\alpha(p)\,\mathrm{Area}(\gamma)] & \text{for} \quad p < 1 - \pi_c \\ \exp[-c(p)\,\mathrm{Per}(\gamma)] & \text{for} \quad p > 1 - \hat{\varrho}_c \end{cases} \tag{1.2}$$

with some $0 < \alpha(p),\ c(p) < \infty$.

The symbol $\langle W_\gamma \rangle \sim e^{-V(\gamma)}$ *means here that* $\lim_{M,N\to\infty} -\log\langle W_\gamma \rangle / V(\gamma) = 1$, *i.e. the constants* $\alpha(p)$ *and* $c(p)$ *are actually well defined.*

We generalize this theorem to a family of dependent plaquette percolation models, as well as to higher dimensional plaquette percolation in co-dimension one.

Commun. Math. Phys. 92, 19–69 (1983)

Of course, our interest in these pure stochastic-geometric effects is also motivated by discussions of "quark confinement" in *gauge models*. There, the quantity for which area, versus perimeter, law is of interest is the expected value of "Wilson loop" variables. It turns out that, at least for the abelian $\mathbb{Z}(2)$ gauge model, such a transition can be traced exactly [8] to a geometric effect of the type discussed here, albeit in a system of *interacting* plaquettes. These, and other, relations with gauge models are described in Sect. 7.

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Of course, our interest in these pure stochastic-geometric effects is also motivated by discussions of "quark confinement" in *gauge models*. There, the quantity for which area, versus perimeter, law is of interest is the expected value of "Wilson loop" variables. It turns out that, at least for the abelian $\mathbb{Z}(2)$ gauge model, such a transition can be traced exactly [8] to a geometric effect of the type discussed here, albeit in a system of *interacting* plaquettes. These, and other, relations with gauge models are described in Sect. 7.

Nuclear Physics B235 [FS11] (1984) 1–18

TOPOLOGICAL ANOMALIES IN THE *n* DEPENDENCE OF THE *n*-STATES POTTS LATTICE GAUGE THEORY

Michael AIZENMAN[1] Jurg FRÖHLICH

We will explain these anomalies, and show how to account for them.

Presentation Outline

Co-chains on $\mathbb{Z}^d$

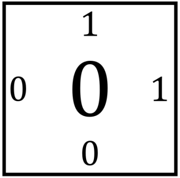
Let $X \subset \mathbb{Z}^d$ be a cubical complex and $0 \leq i \leq d$.

Definition

The co-chain group $C^i(X; \mathbb{Z}_q)$ is the group of $\mathbb{Z}_q$-valued functions on the oriented i-dimensional faces of X so that reversing the orientation of a face inverts its spin.

An element of $C^0(X; \mathbb{Z}_q)$ is an assignment of spins to the vertices of X.

The Coboundary of a Cochain



$$\delta f(\sigma) = f(\partial\sigma)$$
$$= f(e_1) + f(e_2) + f(e_3) + f(e_4)$$
$$= 0 + 0 + 1 + 1 = 0 \pmod 2.$$

The coboundary of an i-cochain f is an $(i+i)$-cochain δf whose value on an $(i+1)$-plaquette σ is the (oriented) sum of its values on the neighboring i-plaquettes in ∂f.

$$\delta f(\sigma) = f(\partial\sigma).$$

The q-state Potts Model

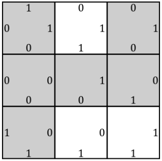
The q-state Potts model is a random 0-cochain $f \in C^0(X; \mathbb{Z}_q)$. Set

$$H(f) = -\sum_{e} I_{\delta f(e)=0} \,.$$

Then the q-**state Potts model** on X is the random 0-cochain $f \in C^0(X, \mathbb{Z}_q)$ so that

$$\mathbb{P}(f = f') \propto e^{-\beta H(f')} \,.$$

Potts Lattice Gauge Theory Hamiltonian



H(f)=-6

For $f \in C^1(X; \mathbb{Z}_q)$ set

$$H(f) = -\sum_{\sigma \in X^{(2)}} I_{\delta f(\sigma)=0}$$

where $X^{(2)}$ is the set of all 2-plaquettes of X.

Potts Lattice Gauge Theory

Definition (Wegner (1971), Kogut et al. (1980))

The q-**state Potts lattice gauge theory** on a finite cell complex X is the random-cochain $f \in C^1(X, \mathbb{Z}_q)$ where

$$\mathbb{P}(f = f') \propto e^{-\beta H(f')}\,.$$

(We will sweep all details about infinite volume measures and boundary conditions under the rug.)

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

Kenneth G. Wilson

Lattice gauge theories were introduced as a discretization of Euclidean Yang–Mills theory. They assign random elements of a complex matrix group G to the edges of a cell complex. When $d = 4$ the cases $G = U(1), G = SU(2)$, and $G = SU(3)$ are models of the electromagnetic, weak nuclear, and strong nuclear forces, respectively.

The case where G is the multiplicative group of second (or third) complex roots of unity is 2(3)-state Potts lattice gauge theory.

The asymptotic behavior of a class of random variables called Wilson loop variables is believed to be related to the phenomenon of quark confinement.

Wilson Loop Variables

The **Wilson loop variable** associated with an 1-cycle γ is the random variable $W_\gamma : C^1(X, \mathbb{Z}_q) \to \mathbb{C}$ given by

$$W_\gamma = (f(\gamma))^{\mathbb{C}},$$

where for $g \in \mathbb{Z}_q$, $g^{\mathbb{C}}$ is the corresponding q-th root of unity in $\mathbb{C}$.

Sharpness for Potts Lattice Gauge Theory?

Conjecture

There exists a $\beta_c(q) > 0$ and constants

$$0 < c_1(\beta, q), c_2(\beta, q) < \infty$$

so that, for rectangular γ in $\mathbb{Z}^d$,

$$\mathbb{E}(W_\gamma) \sim \begin{cases} \exp(-c_1(\beta, q)\mathrm{Area}(\gamma)) & \beta < \beta_c(q) \\ \exp(-c_2(\beta, q)\mathrm{Perimeter}(\gamma)) & \beta > \beta_c(q) \end{cases}.$$

Sharpness for Potts Lattice Gauge Theory?

This is the $i = 1$ case of a more general conjecture for which the $i = 0$ case is sharpness for the Potts model in $\mathbb{Z}^d$.

It's easy to show that area law and perimeter law phases exist when the inverse temperature β is sufficiently low/high.

A proof of the area law for $q = 2$, $d = 3$ follows from a theorem of Lebowitz and Pfister (1981) by arguments of Bricmont, Lebowitz, and Pfister (1980).

Laanait, Messager, Ruiz showed that the conjecture holds for sufficiently large q when $d = 4$.

Potts Lattice Gauge Theory on $\mathbb{Z}^3$

Theorem (Duncan and S., 2023)

Consider Potts lattice gauge theory on $\mathbb{Z}^3$. For rectangular boundaries γ

$$\mathbb{E}(W_\gamma) \sim \begin{cases} \exp(-c_1(\beta, q)\mathrm{Area}(\gamma)) & \beta < \beta^*(\beta_{\mathrm{slab}}(q)) \\ \exp(-c_2(\beta, q)\mathrm{Perimeter}(\gamma)) & \beta > \beta^*(\beta_c(q)) \end{cases} .$$

where

$$\beta^*(\beta) = \log\left(\frac{e^\beta + q - 1}{e^\beta - 1}\right).$$

Presentation Outline

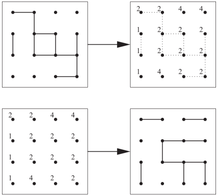
The Random Cluster Model

The **random cluster model** with parameters $p \in [0,1]\,, q \geq 0$ on a finite graph X is the random subgraph P so that

$$\mathbb{P}_{p,q}(P = \hat{P}) \propto p^{\#\text{edges}}(1-p)^{\#\text{non}-\text{edges}}q^{b_0(\hat{P})}$$

where $b_0(P)$ is the number of connected components.

The Coupling



We can couple the random cluster model P with parameters $p = 1 - e^{-\beta}$ and $q \in \mathbb{N}_{\geq 2}$ with Potts model f with parameters β and q so that:

- The conditional measure of P given f is Bernoulli percolation with probability p on edges where f is constant.
- The conditional measure of f given P assigns independent spins to each component (that is, f is a random uniform co-cycle in $Z^0(P; \mathbb{Z}_q) = \mathrm{Ker}\,\delta$).

Connection/Correlation Theorem

The spin correlation function for the q-state Potts model is

$$\tau_{\beta,q}(x,y) = \mathbb{P}(f(x) = f(y)) - \frac{1}{q}\,.$$

Theorem

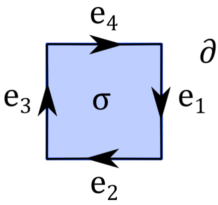
$$\tau_{\beta,q}(x,y) = \frac{1}{q}\mathbb{P}_{p,q}(x \leftrightarrow y)$$

where the probability is taken with respect to the random cluster model on X with parameters $p = 1 - e^{-\beta}$ and q.

Presentation Outline

Chains

- Given a cubical complex X, define the chain group $C_i(X; \mathbb{Z}_q)$ to be the group of formal $\mathbb{Z}_q$ linear combinations of i-dimensional faces of X.
- Given an i-face α, define $\partial_i(\alpha)$ to be a signed sum of the $(i-1)$-faces contained in α, and extend ∂_i linearly to give a linear function $\partial_i : C_i(X; \mathbb{Z}_q) \to C_{i-1}(X; \mathbb{Z}_q)$.



$$\partial\sigma = e_1 + e_2 + e_3 + e_4$$

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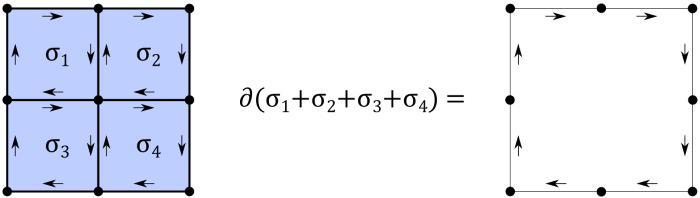
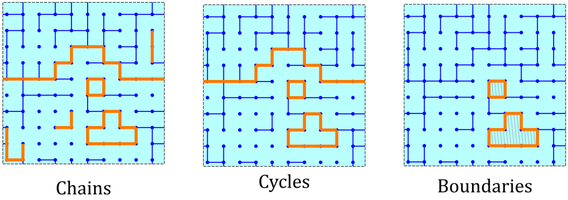


Cycles and Boundaries



Let $Z_i(X;\mathbb{Z}_q) = \ker \partial_i$ be the group of **cycles**, and $B_i(X;\mathbb{Z}_q) = \operatorname{im} \partial_{i+1}$ be the group of i-chains that are **boundaries** of an $(i+1)$-chain.

Definition

Define the i^{th} **homology group** as the quotient $H_i(X;G) = Z_i(X;\mathbb{Z}_q)/B_i(X;\mathbb{Z}_q)$.

Cohomology

Recall that the coboundary map $\delta^i : C^i(X;\mathbb{Z}_q) \to C^{i+1}(X;\mathbb{Z}_q)$ is defined by

$$\delta^i f(\sigma) = f(\partial\sigma) .$$

Let $Z^i(X;\mathbb{Z}_q) = \ker \delta^i$ be the group of **cocycles**, and $B^i(X;\mathbb{Z}_q) = \operatorname{im} \delta^{i-1}$ be the group of i-cochains that are **coboundaries** of an $i+1$-cochain.

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Presentation Outline

The Idea

We'd like to couple Potts lattice gauge theory with a dependent 2-dimensional plaquette percolation $P(p, q)$ so that Wilson loop expectations equal the probability that the loop is bounded by a "surface of plaquettes."

Attempts towards this end in the 80s were stymied by the discovery of so-called topological anomalies.

Nuclear Physıcs B235 [FS11] (1984) 1–18

TOPOLOGICAL ANOMALIES IN THE n DEPENDENCE OF THE n-STATES POTTS LATTICE GAUGE THEORY

Mıchael AIZENMAN[1]   Jurg FRÖHLICH

While $b_0(P) = \operatorname{rank} H_0(P; G)$ does not depend on G, the "number of independent loops" $b_1(P) = \operatorname{rank} H_1(P; G)$ and the ''number of independent closed surfaces" $b_2(P) = \operatorname{rank} H_2(P; G)$ do!

Maritan and Omero (1982) defined a random two complex weighted by the "number of independent closed surfaces of plaquettes." The coupling failed because by not accounting for the dependence on G. A different attempt by Ginsparg, Goldschmidt, and Zuber (1980) ran into similar difficulties.

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The Plaquette Random Cluster Model

A second (more minor) topological anomaly is that when q is not prime, $\left|H^1(X;\mathbb{Z}_q)\right| \neq q^{\mathrm{rank}(H_1(X;\mathbb{Z}_q))}$.

Definition (S. and Duncan (2023))

The 2-**Random Cluster Model** $P(p,q)$ on a finite cell complex X is the random set of 2-plaquettes so that

$$\mathbb{P}_{p,q}(P(p,q)=P) \propto p^{|P|}(1-p)^{|X|-|P|}\left|H^1(X;\mathbb{Z}_q)\right| .$$

The definition for prime q was discovered by Hiraoka and Shirai (2016). The current definition (rather, one equivalent to it) was suggested in our 2022 paper, and details of the coupling were worked our 2023 paper and independently by Shklarov (2023).

The Coupling, Generalized

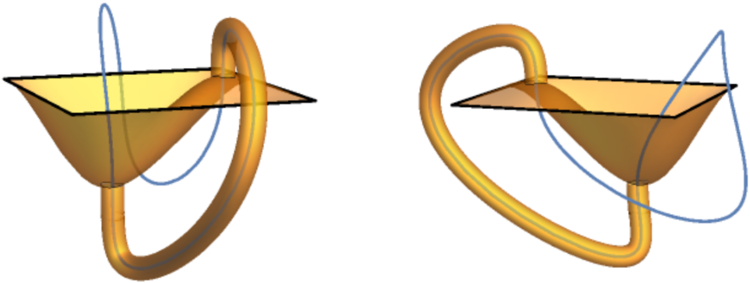
Theorem (Duncan and S., 2023)

Let $q \geq 2$, $\beta > 0$, and $p = 1 - e^{-\beta}$. The 2-dimensional plaquette random cluster model $P = P(p, q, \mathbb{Z}_q)$ can be coupled with q-state Potts lattice gauge theory with inverse temperature β so that

- *The conditional measure of P given f is Bernoulli plaquette percolation with probability p on the 2-plaquettes satisfying $\delta f = 0$.*
- *The conditional measure of f given P is the uniform measure on $Z^1(P; \mathbb{Z}_q)$.*

The special case of prime q is due to to Hiraoka and Shirai (2016). The case of general q was considered by Duncan and S. (2023) and Shklarov (2023).

V_γ



Let P be a 2-dimensional cubical complex. For a 1-cycle in $\mathbb{Z}^d$, define $V_\gamma = V_\gamma(q)$ to be the event that $[\gamma] = 0$ in $H_1(P; \mathbb{Z}_q)$ (that is, that γ is the boundary of a 2-chain.)

Wilson Loops and Homology

Theorem (Duncan and S. (2022, 2023))

Let γ be an 1-cycle in $\mathbb{Z}^d$. Then

$$\mathbb{E}_{\beta,q}(W_\gamma) = \mathbb{P}_{p,q}(V_\gamma(q)),$$

where the expectation in the right is taken with respect to q-state Potts lattice gauge theory and the probability on the right is evaluated for the corresponding plaquette random cluster model.

This is false if we take homology in a group other than $\mathbb{Z}_q$. This was the third topological anomaly observed by Aizenman and Fröhlich.

Theorem

There exist constants and $0 < \tau(p,q), \upsilon(p,q) < \infty$ *so that for any suitable family of rectangular boundaries* $\{\gamma_l\}$,

$$\mathbb{P}_{p,q}(V_{\gamma_l}) \sim \begin{cases} \exp(-c_1(p,q)\mathrm{Area}(\gamma)) & p < p^*(p_{\mathrm{slab}}(q)) \\ \exp(-c_2(p,q)\mathrm{Perimeter}(\gamma)) & p > p^*(p_c(q)) \end{cases},$$

where $p_c(q)$ *and* $p_{\mathrm{slab}}(q)$ *are the critical thresholds for the classical (one-dimensional) random-cluster model on* $\mathbb{Z}^3$ *and in slabs in* $\mathbb{Z}^3$*, respectively, and*

$$p^* = p^*(p) = \frac{(1-p)q}{(1-p)q+p}.$$

The ACCFR theorem is the case $q = 1$. Also, this implies a phase transition for Wilson loop variables in the corresponding PLGT.

Presentation Outline

Proof Overview

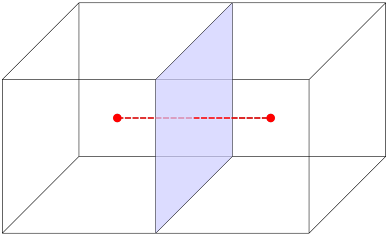
- With the PRCM in hand and a couple results about its duality properties, the proof isn't difficult.
- The perimeter law argument is nearly identical to ACCFR. Showing there is a well-defined constant requires a bit more work.
- The area law argument is similar to ones of ACCFR and Bricmont, Lebowitz, and Pfister.

Proof Ingredient 0: Trivial Bounds

Let γ be the boundary of a rectangle r in $\mathbb{Z}^3$. V_γ occurs if all plaquettes in r are occupied, yielding an area law lower bound on $\mathbb{P}_{p,q}(V_\gamma)$.

On the other hand, the absence of all plaquettes adjacent to γ is incompatible with V_γ. This leads to a perimeter law upper bound.

Proof Ingredient 2: Duality

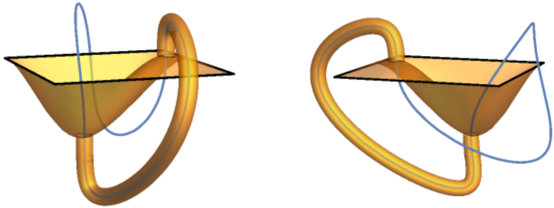


Proposition

The dual of the (free) 2-random cluster model with parameters q and p is the (wired) $(3-2)$-random cluster model with parameters q and $p^*(p)$.

This follows from Alexander duality and the Euler–Poincaré formula

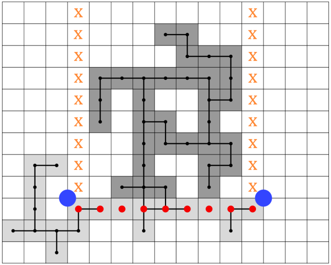
Proof Ingredient 4: Duality and V_γ



Proposition

V_γ does not occur if and there is a loop of dual edges whose linking number with γ is non-zero modulo q.

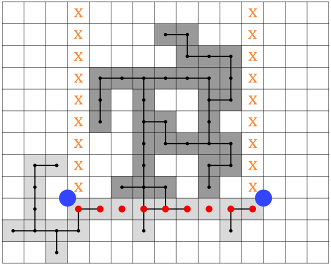
Proof Sketch: Perimeter Law



The ACCFR argument is easily modified. Let $p > p^*(p_c(q))$ so the dual random cluster model is subcritical.

Let C be the connected component of all vertices "below" the rectangle (shown in gray), and let $C' \subset C$ include all vertices contained in the cylinder above the rectangle (dark gray). With positive probability, we can block C' from leaving the cylinder (orange x's).

Proof Sketch: Perimeter Law



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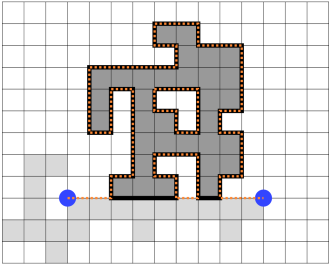


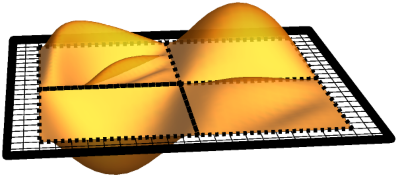
Let $\tau_1, \dots, \tau_j$ be the cubes centered at the dual vertices of C. We can write

$$\partial \sum_{i=1}^{j} \tau_j = \alpha_0 + \alpha_1$$

where α_1 consists of all boundary plaquettes "above" the rectangle. Then

$$\partial \alpha_1 = (-1)^{d-1} \gamma \, .$$

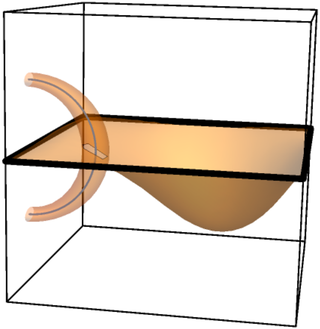
Proof Sketch: Area Law



$$\lim_{l\to\infty} -\frac{\log(\mathbb{P}_{p,q}(V_{\gamma_l}))}{\mathrm{Area}(\gamma_l)}$$

exists and is independent of the sequence $\{\gamma_l\}$. This is shown by tiling γ_l with m translates of $\gamma'_k := [0,K]^2 \times \{0\}$, and comparing the probability of the events V_{γ_l} with that of m copies of $V_{\gamma'_k}$.

Proof Sketch: Area Law



Next, if p is such that the PRCM has a unique Gibbs measure, the area law coefficient is the same if we take γ'_k to be the "equator" of $\Lambda = [0, K]^2 \times [-K, K]$. Duality relates this to the decay of the probability of a dual connection between $\partial\Lambda \cap \{\vec{e}_d > 0\}$ and $\partial\Lambda \cap \{\vec{e}_d < 0\}$; Bodineau proved this notion of surface tension for the RCM is non-vanishing above the slab percolation threshold.

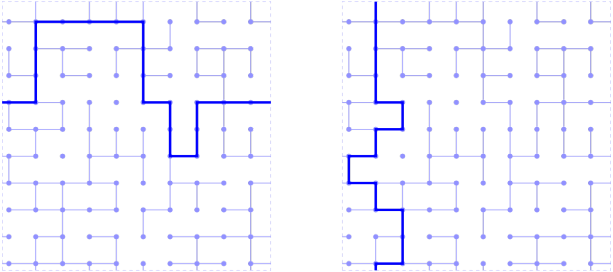
Presentation Outline

What happens in four dimensions?

The dual of Potts lattice gauge theory on $\mathbb{Z}^4$ is Potts lattice gauge theory. Conjecture: the self-dual point is the critical point for Wilson loop expectations.

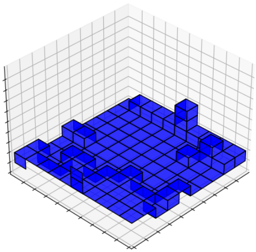
We can't show this, but we can show that a different transition happens at the same point. Let $\mathbb{T}_N^4$ be the four torus obtained by identifying opposite faces of $[0, N]^4$.

Homological Percolation



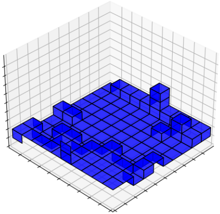
- A path that wraps around $\mathbb{T}_N^d$ for large N is a natural analogue of an infinite path in $\mathbb{Z}^d$.
- This was generalized by Bobrowski and Skraba to a notion of higher dimensional **homological percolation**.

Homological Percolation



- A path that wraps around $\mathbb{T}_N^d$ for large N is a natural analogue of an infinite path in $\mathbb{Z}^d$.
- This was generalized by Bobrowski and Skraba to a notion of higher dimensional **homological percolation**.

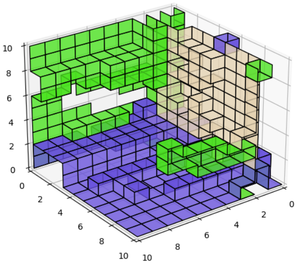
Giant Cycles



Definition (Bobrowski and Skraba)

i-dimensional homological percolation occurs if P contains an i-cycle which is non-trivial as an element of the homology of $\mathbb{T}_N^d$.

The Homology of the Torus



A basis for $H_i(\mathbb{T}^d)$ is given by the i-dimensional planes in the coordinate directions.

$$D := \operatorname{rank} H_i(\mathbb{T}^d) = \binom{d}{i}$$

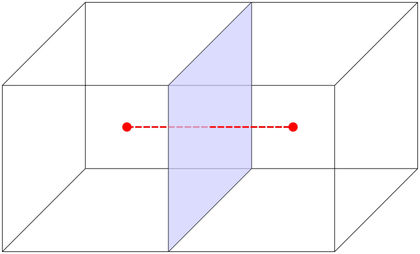
Phase Transition for Homological Percolation

Theorem (Duncan and S., 2022)

Consider the i-dimensional plaquette random cluster model on $\mathbb{T}_N^d$*. If* $d = 2i$, $\mathrm{char}(F) \neq 2$, *and* $p_{sd} = \frac{\sqrt{q}}{1+\sqrt{q}}$ *then as* $N \to \infty$

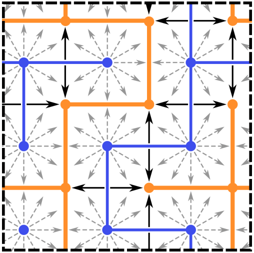
$$\begin{cases} \mathbb{P}_{p,q}(\textit{no giant cycles}) \to 0 & p < p_{sd} \\ \mathbb{P}_{p,q}(\textit{all giant cycles}) \to 1 & p > p_{sd} \end{cases}.$$

Plaquette Duality



- Consider the dual lattice shifted by 1/2 in each coordinate direction.
- Each i-plaquette intersects exactly one dual $(d-i)$-plaquette. Let $P^\bullet$ be the set of dual $(d-i)$ plaquettes which do not meet P.
- If P has parameters p and q then $P^\bullet$ is "almost" an RCM with parameters $p^* = \frac{(1-p)q}{(1-p)q+p}$ and q.
- $\mathbb{T}_N^d \setminus P$ deformation retracts to $P^\bullet$.

Plaquette Duality



- Consider the dual lattice shifted by $1/2$ in each coordinate direction.
- Each i-plaquette intersects exactly one dual $(d-i)$-plaquette. Let $P^\bullet$ be the set of dual $(d-i)$ plaquettes which do not meet P.
- If P has parameters p and q then $P^\bullet$ is "almost" an RCM with parameters $p^* = \frac{(1-p)q}{(1-p)q+p}$ and q.
- $\mathbb{T}_N^d \setminus P$ deformation retracts to $P^\bullet$.

Plaquette Duality

Lemma (Duality Lemma)

Let $D = \operatorname{rank} H_i(\mathbb{T}^d) = \binom{d}{i}$, $\phi_* : H_i(P) \to H_i(\mathbb{T}^d)$, $\psi_* : H_{d-i}(P^\bullet) \to H_i(\mathbb{T}^d)$. *Then*

$$\operatorname{rank} \phi_* + \operatorname{rank} \psi_* = D.$$

Let A be the event that P has at least one giant cycle and S be the event that the giant cycles of P span $H_i(\mathbb{T}^d)$.

Corollary

At least one of the events A *and* $A^\bullet$ *occurs,* $A^c \iff S^\bullet$, *and* $S \iff (A^\bullet)^c$.

Proof Sketch

1. **Duality**:

$$\text{rank}\,\phi_* + \text{rank}\,\psi_* = D \implies \mathbb{P}_{p_{sd}}(A) \geq 1/2\,.$$

2. **Global Symmetry**: $H_i(\mathbb{T}^d;\mathbb{F})$ is an irreducible representation of the point symmetry group of $\mathbb{T}^d_N$ if $\mathbb{F}$ does not have characteristic 2: yields $\mathbb{P}_{p_{sd}}(S) > c_0 > 0$.
3. **Local Symmetry**: Sharp threshold theorem of Graham–Grimmett $\mathbb{P}_{p_{sd}+\epsilon}(S) \to 1$.
4. Another application of duality gives $\mathbb{P}_{p_{sd}-\epsilon}(A) \to 0$.

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