Why Does L5 Exist?

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Recently it occurred to me that I didn't understand why L5 existed. On the World Wide Web, I read detailed descriptions of Lagrange Points, which are special nodes in the orbits of Earth and other planets. And then… I still didn't understand. After more reading and more thought, I found there is a fairly simple explanation for L5. The answer is buried in the rather detailed math of a complete description of Lagrange Points. Here I give the explanation with much less math.

In the first section I describe why Lagrange Points are interesting to me and why they may be interesting to you. The second section talks about some basics of orbital mechanics. Finally, I discuss why there is an L5 and an L4. If you already know about Lagrange Points but can't give a convincing explanation of why L5 has to exist, you may want to go directly to section 3.

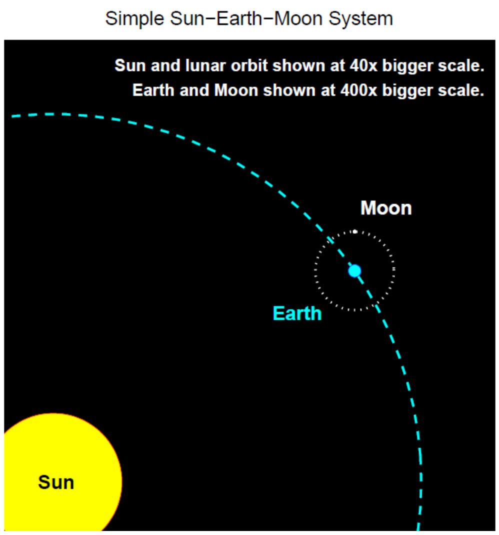
1. L5 and Me

In 1977 Princeton physicist Gerard K. O'Neill published his book *The High Frontier*, in which he proposed creating colonies in outer space. Rather than sitting on the Moon or on other planets, the colonies would be inside enormous free-floating cylinders, kilometers wide, rotating so that centrifugal force would create artificial gravity for the inhabitants. I was twelve years old at the time, which meant that I had been avidly learning everything I could about space travel for seven years already.

01 https://en.wikipedia.org/wiki/O%27Neill_cylinder.

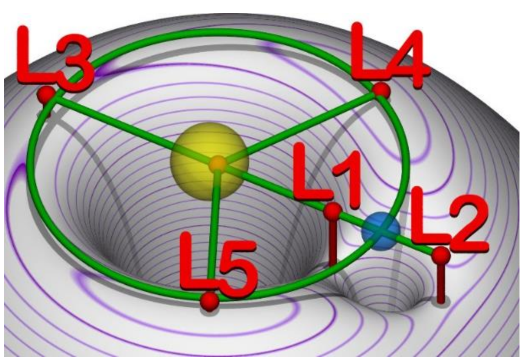
O'Neill proposed putting the colonies at L5. This was a special location in the vicinity of the Earth and Moon that was favorable in some way for satellites to stay for long periods of time without falling on to the Earth or the Moon. I didn't know any physics when I read *The High Frontier* so I only had a vague idea of what "L5" meant.

Soon after that I learned about forces and orbits, and went on to college where I majored in physics and studied astrophysics and space plasmas with professors there. However, my interest in human space travel waned. Every place in the solar system is far more hostile than virtually any environment on the surface of the Earth, and space colonization did not seem to be a good solution to any immediate problems of humanity. Once I learned about circular orbits, the trajectories of objects floating in space did not seem very interesting. My model of the solar system was: the sun, a planet orbiting the sun, and perhaps one or more planets orbiting the planet in pretty much the same way. Repeat for other planets.



2 Schematic of simple Sun-Earth-Moon system

Since then, I've grown to appreciate a richer view of the solar system. When you include gravitational fields, space is no longer a featureless void between remote objects, but a crowded medium where every location has its own unique flavor. Suppose there was only the Sun, and consider a circle with the Sun at its center. Any location on the circle is at the same "height" above the Sun: its potential energy is the same anywhere on that circle. Another way of saying that is that the entire circle is at the same **potential**. Then add a planet orbiting the Sun along that circle. Now the space around the sun has a complex topography, something like this:

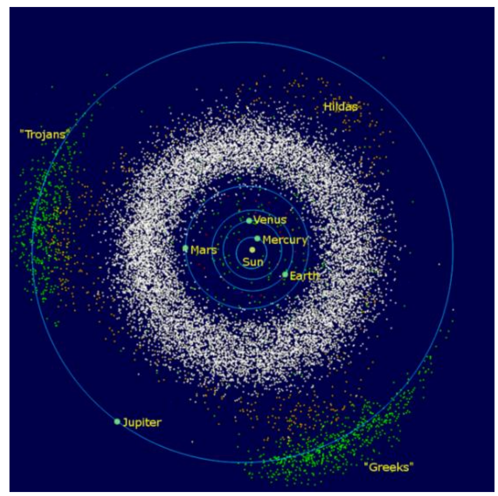


3 https://space.stackexchange.com/questions/4050/is-there-a-lot-of-space-trash-at-the-earth-moon-lagrange-points

As the **Appendix** below describes, the surface represents the force field due to gravity. You may notice that there are five points where the topography flattens out. These are the **Lagrange Points**, L1 through L5, named after the mathematician who discovered two of them in the 1700s. Where the topography is flat, there is no net force on an object to move it from the location.

Lagrange Points have been in the news recently. NASA just launched a satellite to be parked at L2 of the Earth-Sun system. The satellite is the James Webb Space Telescope, which will be kept at an especially low temperature in order to take accurate measurements of infrared light. To stay cold enough, it will have a kind of space parasol to block out the burning light of the Sun as well as dimmer infrared light radiated from the dark side of the Earth. To do this, the Earth and Sun must be in about the same direction (from the satellite) all the time, which is the case for L2.

Other planets can have Lagrange Points. The most prominent ones belong to Jupiter, which has over 300 times the mass of the Earth. Its strong gravity has kept large numbers of asteroids at L5 and L4. The asteroids are named after Homeric heroes, Trojans at L5 and Greeks at L4. Some of the largest asteroids there are 624 Hektor (225 km diameter), 911 Agamemnon (131 km) and 1143 Odysseus (115 km). Just this Fall, NASA launched a space probe that will tour the Jupiter Trojans.



4 https://en.wikipedia.org/wiki/Jupiter_trojan

Earth also may have some "Trojans" – asteroids that stay in the vicinity of the Earth-Sun L4 and L5. The Earth-Moon system also has Lagrange Points, defined by the Moon's orbit around Earth. It is the Earth-Moon L5 that Gerard O'Neill proposed for Cities in Space. A colony there could stay in about the same location relative to both the Earth and the Moon in order to stay in contact with both. Its solar panels would also keep a view of the sun except for an occasional eclipse.

This got me thinking about Lagrange Points. Why do they exist?

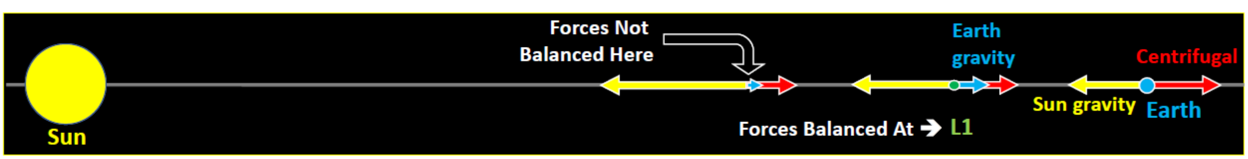
2. Euler's Lagrange Line

Three of the Lagrange Points are easy to understand. First, think about a smaller body of mass m (such as the Earth) that has a circular orbit around a larger body of mass M such as the Sun. Since the same reasoning applies to different pairs of bodies (Sun-Earth, Earth-Moon, Sun-Jupiter, etc.), let's just refer to the objects as M and m. Any object moving in a circle has its velocity changing direction at a constant rate all the time. The rate of change of velocity is known as the **acceleration**. If an object takes a time T to go around a circle of radius R, the acceleration is $a_c = (2\pi/T)^2 R$. The acceleration caused by gravity from M is given by $a_G = GM/R^2$, where G is a constant of nature. If you want the object to have an orbit of radius R, you have to adjust the orbital period (by adjusting its speed) so that the $a_c = a_G$. Or if you want it to have an orbital period T, then you have to adjust R.

Now suppose you add a third object and try to make it orbit M with the same R and T of m. Gravity from M will make it accelerate at just the right rate to maintain the orbit, but a tug from the smaller mass m will slowly move it from its original path. The simple orbit doesn't work for the third object, which will take some other trajectory, the shape of which is more complicated than a circular orbit and harder to figure out.

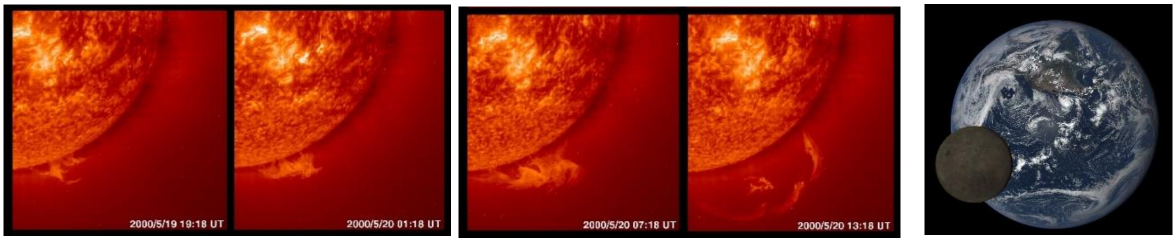
If an object is circling M, by definition it is accelerating. There is an alternative way of looking at the object's motion that is often convenient. Consider everything from a frame of reference which itself is rotating around M with the same period as m. In this frame of reference, m is not moving, and of course not accelerating, nor is anything else taking a circular path with period T around M. There is an apparent contradiction in this reference frame because M's gravity is pulling m inward, yet it does not accelerate towards M. The solution is to pretend that there is an outward **centrifugal force** associated with the moving frame. The acceleration caused by the centrifugal force is exactly the same magnitude as the acceleration shown above for circular motion: $a_c = (2\pi/T)^2 r$ – here I use r to mean any distance from M, not just the orbital distance of m. At a distance $r = R$ from M, the force is exactly equal to the gravitational force. The centrifugal force is weaker than M's gravity if you are closer than m (that is, $r < R$) and greater than M's gravity if you are further.

Suppose we put a satellite between M and m. Because it is closer to M than m is, M will exert a stronger force on it, which necessitates a faster (shorter period) orbit. But remember, m is tugging on it in the opposite direction. If you add the two opposing forces, the total force towards M may be weaker, or (if the object is close enough to m) may even be pointing away from M. There is a location between M and m where the centrifugal force balances gravity. That location is Lagrange Point #1, L1.



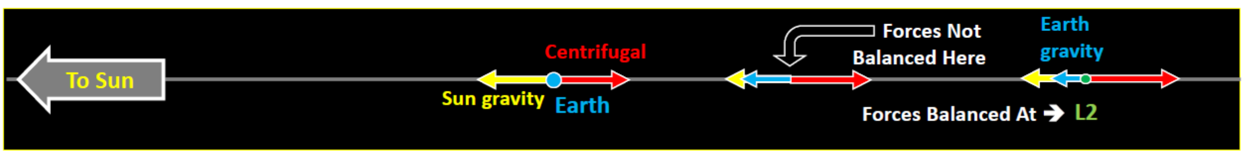
5 Examples of force balance for locations between m and M, in rotating frame of reference. Distances and force vectors are not to scale.

For over twenty-five years, a NASA satellite named **SOHO** (Solar and Heliospheric Observatory) has been sitting at the Earth-Sun L1 (https://soho.nascom.nasa.gov/home.html). This is an ideal location for a perpetually unobstructed view of the Sun (see figure below). Since 2015, NOAA's **DSCOVR** (Deep Space Climate Observatory) has been taking advantage of L1's continuous view of the daylight side of the Earth (http://www.nesdis.noaa.gov/news/noaas-dscovr-satellite-celebrates-its-sixth-launch-iversary), including times when the Moon passes in front of the Earth (see figure).



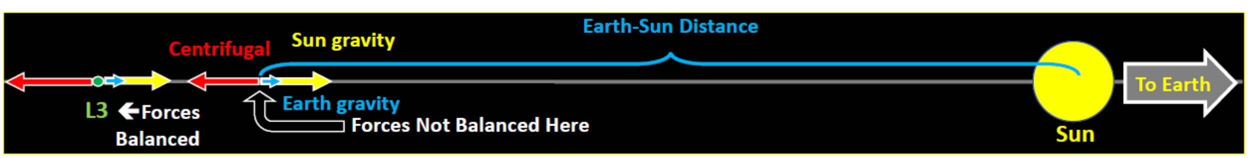
6 Four pictures of the Sun taken over 2 days (https://soho.nascom.nasa.gov/gallery/images/eit003.html) and an image of the Earth and the far side (but not the dark side) of the moon, both taken from the vicinity of the Earth-Sun L1.

For the next Lagrange Point, go along the line from M to m till m is between M and you. In other words, your distance r from M is greater than m's distance. M and m are both tugging you in the same direction, so the combined force is greater than that of M alone. You can adjust r so that the same T as m's period gives you just the right acceleration to balance the force from M and m. That is L2.



7 Schematic of L2.

Finally, there is the mysterious L3, the "mirror Earth" location on the opposite side of m. As with L2, an object at L3 is tugged by both M and m, but in this case m is much further away and so makes a much smaller contribution to the force. For this reason, L3's orbital radius R only has to be slightly larger than m's orbital radius in order to have the same T.



8 Schematic of L3.

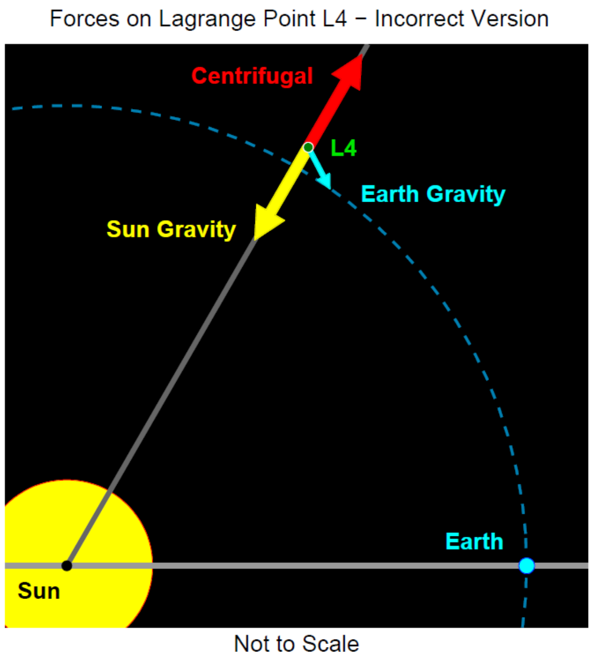
The 3 Lagrange points on the same line as M and m were all discovered by Leonard Euler. Lagrange studied them afterwards, but Euler already has so many things named after him (including the number $e = 2.718281828 \ldots$) that some of the things he studied are named after the next person that wrote about them.

All these balances are easy to picture, but how can the forces balance if M, m, and our third object are not all in a straight line? These are the ones that Lagrange actually discovered.

3. The Lagrange Triangles

When we come to L4 and L5 and try the game of balancing the forces so that they are just right, we run into a little problem: it doesn't work.

Any object standing still in the frame of reference rotating around M has its centrifugal force pointing in the opposite direction to the force from M's gravity. If the object is not somewhere on the line that includes M and m, then the gravitational force from m does not line up with either of the other forces (see Figure), no matter how far from M you stay. That's a problem, because there is no way for forces to cancel each other if they are not parallel.



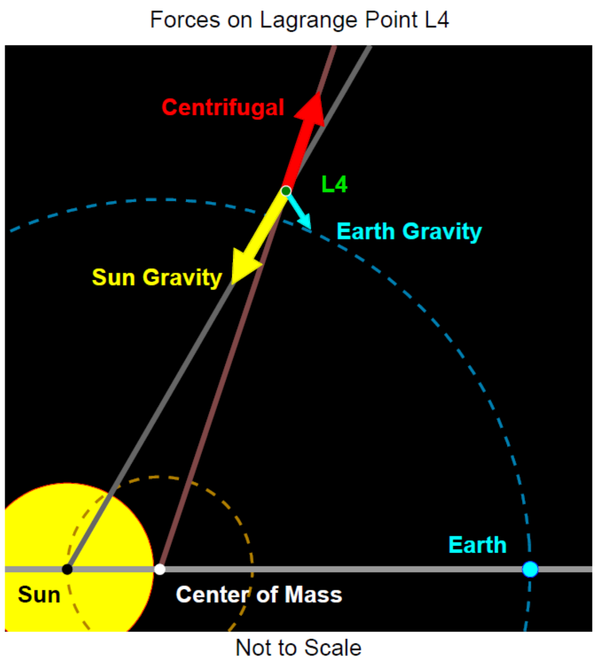
9 Schematic showing apparent contradiction in defining L4.

This is quite a conundrum. The force diagram is simple and the problem is obvious, and yet astronomers insist there really is an L4 and an L5. If the astronomers are right about L4 and L5, there must be an error in least one of our assumptions. What is it?

The incorrect assumption is that the Earth orbits the Sun.

That assumption is almost correct. It does go around the Sun. But, m does not orbit the <u>center</u> of M, but the <u>center of mass</u> (**barycenter**) of the system. In fact, both objects orbit the barycenter. The line between the centers of M and m always passes through the barycenter, which is always between the centers of the two objects. Where is the center of mass? If we call the distance between each mass and the barycenter r_M and r_m, then $Mr_M = mr_m$, and so $r_M/r_m = m/M$. If M is much more massive then m, then M is much closer to the center of mass than m is. In fact, for the Earth-Moon system, and the Earth-Sun system, the barycenter is inside M. The Sun's "orbit" around the Sun-Earth barycenter looks like a kind of wobble.

Separating the center of the Sun from the center of mass fixes our problem with L4/L5. In order to show the effect more clearly, the figure exaggerates the distance to the barycenter and puts it outside the Sun. Because the orbits are around the barycenter, the centrifugal force points away from the barycenter rather than from the center of the Sun. The Sun's gravitational pull on the object still points to the Sun center. This slight difference in alignment is enough that there are two additional locations around the Earth's orbit (and slightly outside of it) where the pull of the Earth balances the force from the slight mismatch in centrifugal force and the Sun's gravity. The two locations are L4 and L5. L5 trails behind m in its orbit, and L4 stays ahead.



10 Schematic showing importance of centroid in establishing L4 and L5.

4. Lesson: Don't Ignore the Wobbles

To understand many things about orbits we can ignore the wobble - the relatively small motion of the heavier object at the [near] center of the system - but L4/L5 depend critically on this feature. The wobbling of the central mass creates two cradles in which to let other objects rest. This contrasts with the "straight line" Lagrange Points which would exist even without the wobble.

The heavy object's motion is important in another phenomenon, more familiar than Lagrange Points: the tides. The Earth experiences tidal forces as it orbits the Sun. The Sun's gravity balances the centrifugal force of Earth's orbit at the center of the Earth, but the noon and midnight locations on the Earth's surface, being closer and further (respectively) from the sun, changes the balance and creates a tidal force. The strongest tidal force however, does not come from the Sun but from the Moon. It makes no sense to speak of balances between the Moon's gravity and the centrifugal force of the Earth's orbit around the Moon. But the Earth does orbit around the Earth-Moon barycenter, which is inside the Earth about ¾ of an Earth radius from the Earth's center.

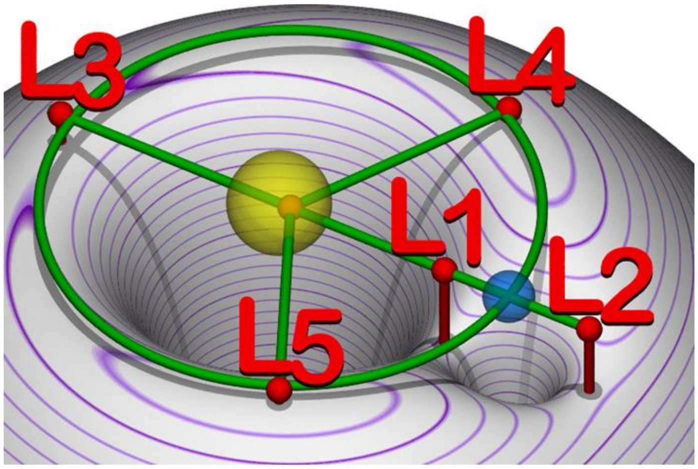
The slight motion of a star due to the tug of a planet, which is much smaller and dimmer and hence impossible to see directly with current astronomical tools, allows us to infer the existence of planets orbiting other stars.

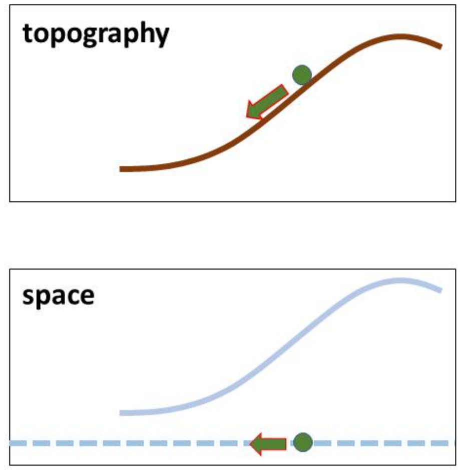
There is much more to say about Lagrange Points themselves. If you park your favorite spacecraft at L1, L2, or L3, you have a stability problem. Orbital perturbations, no matter how small, can make the object drift far from the placidity of the Lagrange Point into the chaos of other orbits. The perturbation could be as small as placing the spaceship a short distance from the exact Lagrange Point. There are special orbits around these Lagrange Points that will keep an object close to them, which is what the L1 and L2 satellites mentioned in Section 3 do. Rockets near these points need to expend a small amount of fuel to maintain their orbit. In contrast, what starts near L4 and L5 tends to stay near L4 and L5, which is why scientists think the Jupiter Trojans have been there for millions or even billions of years. Why all this occurs is a bit more complicated than what I've demonstrated here.

Lagrange Points and the orbit of a body pulled by a much smaller orbiting body are just the beginning of the complications which lie beyond simple circular orbits around an unmoving central mass. The planets' orbits are circular, but not exactly circular: they are slightly elliptical. They are generally in about the same plane, but the plane of each differs by a little bit as well. The Moon's own orbit around the Earth is not exactly in the same plane as the Earth's orbit around the Sun, which is why there isn't an eclipse once a month. The Earth-Moon tides are causing the Earth to shove the Moon, slowly but inexorably, into a bigger orbit. Gravity from other planets makes the Earth's orbit cycle through changes over thousands of years, which makes giant changes in climate. Jupiter and Saturn have their own mini-solar systems of moons orbiting them. And all that is before we start looking at the trajectories of space ships we toss out into the solar system, or start imagining what other planetary systems may look like.

Appendix: Gravitational Potential, Rotating Reference Frame

What is actually going on in Figure 3? The surface shown in the figure (reproduced below) is an abstraction representing the **gravitational potential** around the Sun and Earth in the imaginary case that there are no moons or other planets. To understand that abstraction, first imagine that it represents something else: an actual topographic surface on the Earth, maybe 50 meters wide, with a couple of holes. Call the height of the surface at each location h. As everyone knows, if the surface is smooth and you place a marble on it, Earth's gravity will tend to pull the marble down the slope. As the marble rolls down, it picks up speed V in a particular way: the quantity $½mV^2$ (m is the marble's mass) <u>increases</u> by exactly the same amount as the quantity mgh <u>decreases</u> (g is the acceleration caused by gravity, 9.8 m/s^2 at the surface of the Earth). The first quantity is the kinetic energy, the second is the potential energy, and since the motion due to gravity of any object doesn't depend on the object's mass, it is convenient to talk about the **gravitational potential** gh of each point on the Earth's surface.[1]





Let's go back to outer space. Now we are only talking about the horizontal plane that contains the Earth's orbit. The vertical coordinate shown in the figure is not height above this plane, it's the strength of the gravitational potential due to the Earth and the Sun. If the surface were an actual solid surface on Earth, objects would be pushed downward and in some horizontal direction as they roll down (upper right panel of figure). But if the surface just represents potential in space, the force is only in the horizontal direction and there is no force pushing objects "downward" below the plane (lower right panel). The Earth is in the center of a steep bowl representing the pull towards the Earth and the Sun is in the center of an even steeper bowl due to its stronger gravity. In each case, the bowl tells you that an object placed there would be pulled towards the center of the bowl.

The formula gh we use for gravitational potential for objects on Earth is just an approximation that is accurate if h is much smaller than the distance to the center of the Earth. In general, the potential, at each point, from each object depends on the distance r to the object in a more complicated way. If gravity is due to more than one object (such as the Sun and Earth in this example), you just calculate the potential from each object and add them all up.

If the potential represents the effect of gravity, why is there a downward slope away from the sun outside Earth's orbit? It's because we are viewing the arrangement as static: The Earth appears to be stationary even though we know it is orbiting. That is because we are viewing this in a rotating frame, as if we were sitting above the Sun and turning around so that we are always facing the Earth. When we use a rotating frame to describe the motion, we have to

[1] I am ignoring some details such as the kinetic energy associated with the spin of the marble as it rolls.

include **centrifugal force** (see **Section 2**). It turns out there is also a potential due to this force, and we just add that to the gravitational potential of the Earth and Sun. The centrifugal potential has the shape of a hill with its peak in the center of the orbit (at the Sun). It's a hill because that way "rolling down the hill" pushes an object away from the Sun, which is the direction the centrifugal force points. Inside the Earth's orbit, the centrifugal hill is "steeper" than the gravitational bowl due to the Sun, so the net effect is still a bowl (objects are pulled towards the Sun). Outside the orbit, the hill is steeper than the bowl, and so the net force is away from the Sun – an object travelling fast enough to go once around the Sun in a single Earth year will be pushed away from the Sun by centrifugal force.

Potential surfaces are a cool way to illustrate the Lagrange points. Simple rules explain why the Sun and the Earth are each sitting in a gravity bowl imbedded in a centrifugal hill. But these rules don't explain why there are separate gravity hills around L4 and L5. For that you need the extra factor that **Section 5** discusses.