Stochastic Maze Solving Under the Geometric Amoebot Model

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Contents

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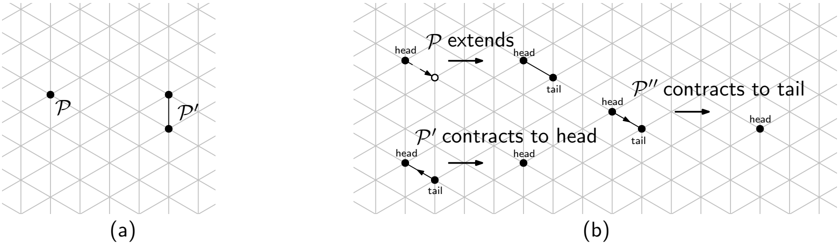
Introduction

The Geometric Amoebot Model [Dolev et al., 2013, Derakhshandeh et al., 2014]

- An abstract model of self-organizing particle systems.
- It consists of individual computational elements (or particles) known as *amoebots* represented as points on an infinite triangular lattice G_{Δ}.
- There is no shared co-ordinate system or a global sense of orientation between particles.



(a) An amoebot may exist in a *contracted* state, like $\mathcal{P}$, or in an *expanded* state wherein the particle occupies two adjacent locations in the lattice like $\mathcal{P}'$.

(b) Particles move via a series of alternating *expansion* and *contraction* steps along the edges.

The Geometric Amoebot Model (contd.)

- Each amoebot must keep track of their *port labels* that uniquely identify the edges surrounding it
- The particles are *anonymous* in that they lack any global identifier, however they can locally identify their *neigbors* by the port labels corresponding to the connecting edge.
- Each particle has a constant amount of local memory and may perform bounded amount of local computation.
- Particles can also communicate with their immediate neighbors on the account of having read and write access to their local memory stores.

Deterministic Algorithms

Deterministic algorithms under the geometric amoebot model comprise a class of carefully designed distributed, local, asynchronous protocols executed by individual amoebots to self-organize into a target configuration.

Example: Shape formation

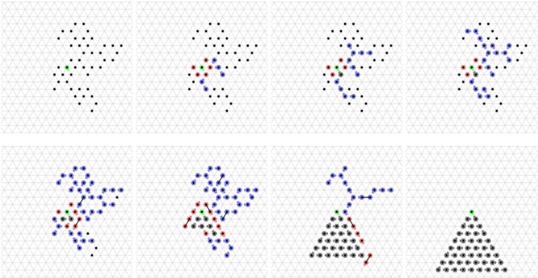


Figure: Much of the early motivation behind the geometric amoebot model can be found in applications involving shape and pattern formation.[1]

[1] Figure source: [Derakhshandeh et al., 2015]

Example: Convex hull formation

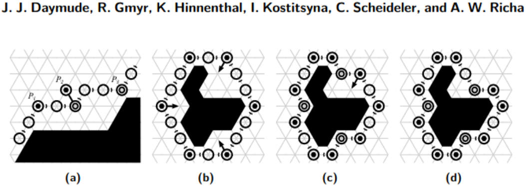


Figure: Figure demonstrates the formation of a convex hull.[2]

[2] Figure source: [Daymude et al., 2020]

Stochastic Algorithms

In contrast to deterministic algorithms, stochastic algorithms under the geometric amoebot model rely on individual particles making probabilistic decisions in order to reach some goal state with respect to a pre-determined objective.

General Design Scheme

Idea

Define a Markov chain over the particle-system configuration and sample with probabilities favouring the desired configuration(s).

Ising model

[Ising, 1925] Consider a regular lattice-graph $([n], E)$ where:

- $[n] = \{0, 1, \ldots, n-1\}$ are the vertices that we call *sites*; and
- E is the collection of pairs of sites $(i, j) \in [n]$ with non-zero *interaction energy*, V_{ij}.

A configuration, $\sigma = \{\sigma_i\}_{i \in [n]}$ is, then, an assignment of positive/up $(\sigma_i = +1)$ and negative/down $(\sigma_i = -1)$ spins to all the sites on the graph. Further, the *energy* of a configuration under an external field B is given by its Hamiltonian

$$H(\sigma) = -\sum_{\{i,j\} \in E} V_{ij} \sigma_i \sigma_j - B \sum_{k \in [n]} \sigma_k$$

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Ising model: Partition Function

The probability that a system is in a state (or configuration) σ at equilibrium is $\exp(-\beta H(\sigma))/Z$ where $\beta > 0$ is inversely proportional to the temperature and

$$Z = Z(V_{ij}, B, \beta) = \sum_{\sigma \in \{-1,+1\}^n} \exp(-\beta H(\sigma))$$

is a normalizing factor known as the *partition function*.

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Compression in the Amoebot model

Problem definition

Let $p(\sigma)$ denote the perimeter of a system configuration σ, then $p(\sigma)$ is the sum of the lengths of all boundaries (including those surrounding holes) of σ.

$$p_{\text{min}} := \min_{\{\sigma \in \Omega : |\sigma| = n\}} p(\sigma)$$

is the minimum possible perimeter among all systems of size n. A *simply connected* configuration σ is said to be α-compressed if $p_{\text{min}} < \alpha \cdot p(\sigma)$ for any $\alpha > 1$.

Markov chain for Compression

Consider a Markov chain $\mathcal{M}$, with stationary distribution π over the state space Ω.

For any $\omega, \nu \in \Omega$, state-transitions of the form $\omega \to \nu$ correspond to a single particle move that occurs with probability $p_{\omega,\nu} = \min\left\{1, \frac{\pi_\nu}{\pi_\omega}\right\}$.

These transition probabilities can be calculated locally by each activating particle by counting only neighbouring edges!

Stochastic Algorithm for Compression

Let $\varepsilon(\sigma)$ be total number of edges in a configuration $\sigma \in \Omega$

Then the Hamiltonian, $H(\sigma) = -\varepsilon(\sigma)$; accordingly each configuration is sampled in proportion to its weight $w(\sigma) = e^{-H(\sigma)/\tau} = \lambda^{\varepsilon(\sigma)}$.

The transition probability $p_{\omega,\nu}$ is

$$\min\left\{1, \frac{\pi_\nu}{\pi_\omega}\right\} = \min\left\{1, \lambda^{\varepsilon(\nu)-\varepsilon(\omega)}\right\}$$

Where $\varepsilon(\nu) - \varepsilon(\omega)$ can be calculated locally by counting the change in the number of edges around the moving particle.

Phototaxing in the Amoebot model

The setup

- Assume all edges of the lattice graph G_Δ are of unit length.
- A fixed, continuous line of point sources on the vertices of G_Δ broadcasts rays along the vertical lattice line towards the system.
- The y co-ordinate of the center of mass of a particle system starting in some initial configuration, σ_0, is referred to as its height, y_0, with respect to the fixed jagged line of point sources itself.
- A particle may be *occluded* from the light. Particles are sensitive to the signal and can locally determine whether they are in an occluded or unoccluded state.

Algorithm for Phototaxing

Algorithm 1 [Savoie et al., 2018] Phototaxing subroutine of each particle $\mathcal{P}$.

1: **if** $\mathcal{P}$ is *unoccluded* **then**
2: Perform compression.
3: **else**
4: Perform compression with probability $1/4$.

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Phototaxing in the Amoebot model

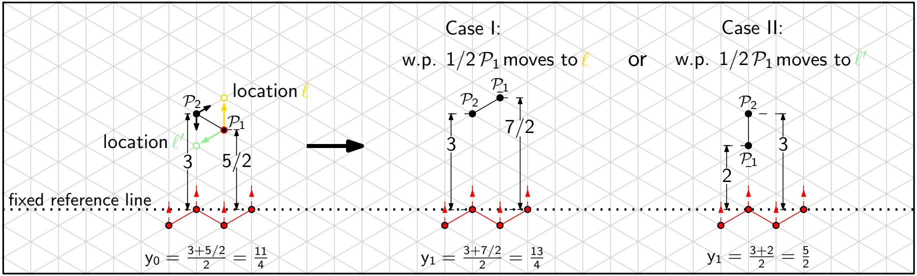

Figure: An activating particle $\mathcal{P}_1$, in some configuration at starting height y_0 (with respect to the fixed reference line), moves to one of the two locations ℓ or ℓ', resulting in the height of the system respectively increasing or decreasing to y_1.

Phototaxing in Expectation

Theorem [Savoie et al., 2018]

For systems of two and three particles, phototaxing occurs in expectation.

Proof sketch. Separately calculate the expected change in height in t-steps, $h_t(\cdot)$, for each configuration in an iterative manner until $h_N(\sigma_0) > 0$ for some positive integer N and for every starting configuration $\sigma_0 \in \Omega$. Then observe that in the case of two and three particles where the expected change in height in one step is either positive or zero for all configurations.

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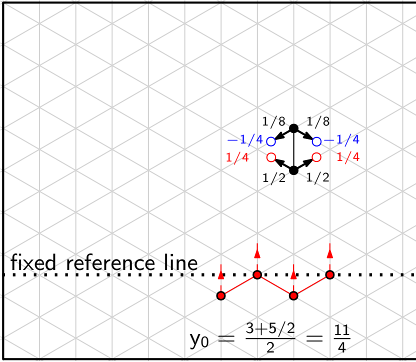
Calculating the expected change in height



$$y_0 = \frac{3+5/2}{2} = \frac{11}{4}$$

$$h_1(\sigma_0) = \sum_{\sigma_1}(y_1 - y_0)\Pr[\sigma_0 \to \sigma_1]$$
$$= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2} - \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{8} - \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{8}$$
$$= \frac{3}{32}$$

Figure: The one step expected change in height calculated for a two particle case.

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Phototaxing Verification

Goal

- Generalise the proof technique of [Savoie et al., 2018]; and
- give an algorithm for verifying that phototaxing occurs in expectation.

Phototaxing Verification

For all valid configurations[3] $\omega_i \in \Omega$ for $i = 1, 2, \ldots, |\Omega|$, the transition matrix is

$$\mathbf{P} = \begin{bmatrix} p_{\omega_1,\omega_1} & p_{\omega_1,\omega_2} & \cdots & p_{\omega_1,\omega_{|\Omega|}} \\ p_{\omega_2,\omega_1} & p_{\omega_2,\omega_2} & \cdots & p_{\omega_2,\omega_{|\Omega|}} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\omega_{|\Omega|},\omega_1} & p_{\omega_{|\Omega|},\omega_2} & \cdots & p_{\omega_{|\Omega|},\omega_{|\Omega|}} \end{bmatrix}$$

where each $p_{\omega_i,\omega_j} = \Pr[\omega_i \to \omega_j]$.

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[3] The ordering is arbitrary but must be maintained once fixed.

Phototaxing Verification

Similarly define the one-step change in height matrix

$$\mathbf{Y} = \begin{bmatrix} 0 & \delta_{\omega_1,\omega_2} & \cdots & \delta_{\omega_1,\omega_{|\Omega|}} \\ \delta_{\omega_2,\omega_1} & 0 & \cdots & \delta_{\omega_2,\omega_{|\Omega|}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\omega_{|\Omega|},\omega_1} & \delta_{\omega_{|\Omega|},\omega_2} & \cdots & 0 \end{bmatrix}$$

where

$$\delta_{\omega_i,\omega_j} = \begin{cases} y_{(\omega_j)} - y_{(\omega_i)} & \text{if } \Pr[\omega_i \to \omega_j] > 0 \ \forall \omega_i, \omega_j \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Phototaxing Verification

The vector of expected change in height in one step

$$\mathbf{h}_1 = \begin{bmatrix} h_1(\omega_1) \\ h_1(\omega_2) \\ \vdots \\ h_1(\omega_{|\Omega|}) \end{bmatrix}$$

is then given by the vector made of diagonal elements of $\mathbf{P} \cdot \mathbf{Y}^\top$.

Phototaxing Verification

Lemma

Let π *be stationary distribution of the Markov chain for phototaxing. There exists an* $N \in \mathbb{N}$ *such that* $h_{m+1} - h_m \succ 0$ *for all integers* $m \geqslant N$ *if and only if* $\pi \cdot h_1 \succ 0$.

Phototaxing Verification

Lemma

Let π *be stationary distribution of the Markov chain for phototaxing. There exists an* $N \in \mathbb{N}$ *such that* $h_{m+1} - h_m \succ 0$ *for all integers* $m \geqslant N$ *if and only if* $\pi \cdot h_1 \succ 0$.

Corollary

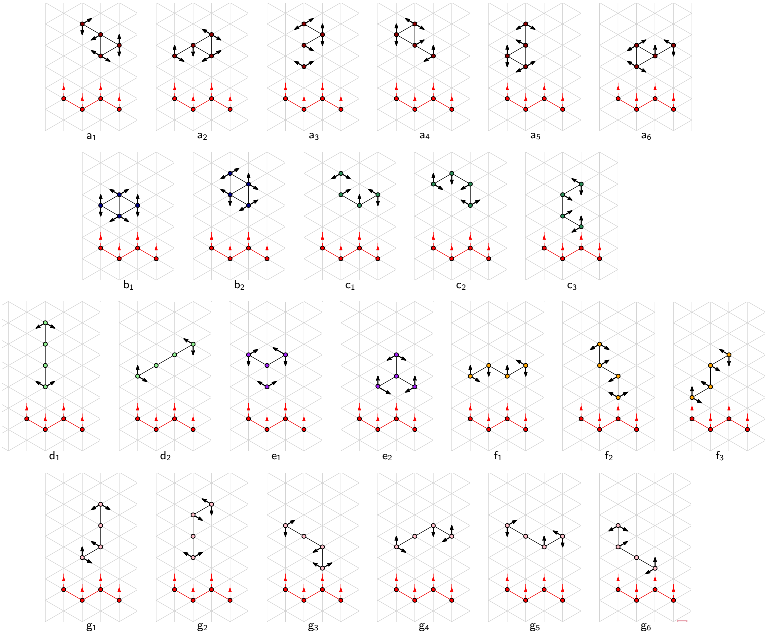
If $\pi \cdot \mathbf{h_1} \succ 0$*, then starting in any configurations* $\sigma_0 \in \Omega$, $\lim_{t \to \infty} h_t(\sigma_0)$.

Algorithm for Phototaxing Verification

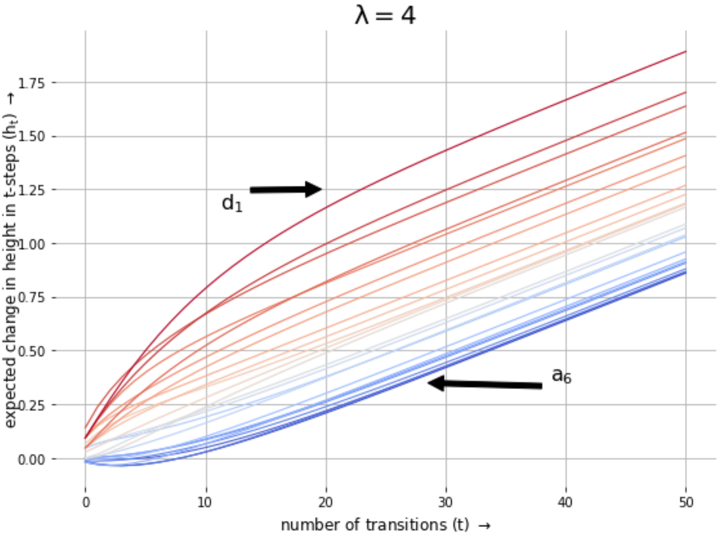
- Obtain a "reasonable" estimate $\tilde{\mathbf{P}}$ for the staionary distribution π.
- Verify whether $\tilde{\mathbf{P}} \cdot \mathbf{h}_1 = \tilde{\mathbf{P}} \cdot \mathsf{diag}(\mathbf{P} \cdot \mathbf{Y}^\mathsf{T}) \succ 0$.
- If the answer is yes, then from the previous Corollary we know that phototaxing must occur in expectation.

Caveat: the set of valid free configurations and their equivalence classes with respect to the signal source must be known.

Phototaxing in the four-particle system



Phototaxing in the four-particle system



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Demo: Phototaxing in the four-particle system

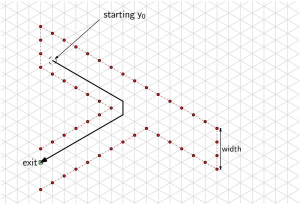
Maze Solving Phenomenon

- To navigate unfamiliar topologies, ants create a network of fading chemical trails that lead to the destination through gradual reinforcement of the optimal path.
- Slime molds – often considered model organisms for studying biological self-organization – are known to optimally solve mazes by spreading their mass across the maze and then pruning any extensions that do not lead to an exit.

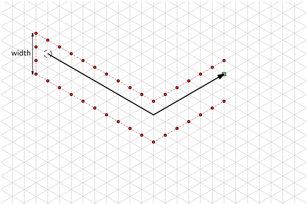
Stochastic Maze Solving

- We consider mazes to be made up of a collection of vertices forming an open-polygon on the triangular lattice. We call such vertices, *walls* of the maze.
- A particle may not expand onto a wall.
- A maze is solved when an agent, starting at any location in the maze, is able to find a path to the exit created by the open face of the polygon.

Three classes of mazes



(a)



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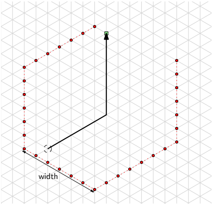
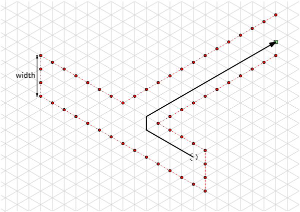
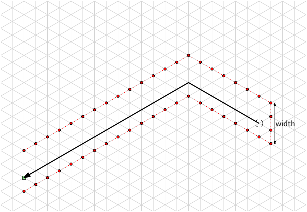


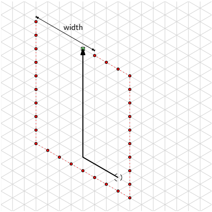
(e)



(b)



(d)



(f)

An Algorithm for Maze Solving

Idea

Perform a random-walk through the maze guided by the walls such that a system far away from a wall expands ($\lambda \leqslant 1$) and a system close to the wall compresses ($\lambda > 1$).

An Algorithm for Maze Solving

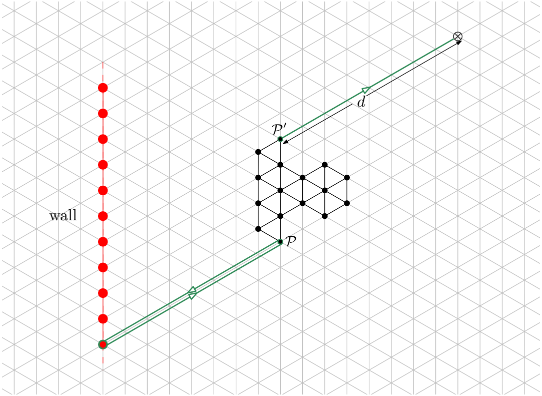


Figure: Signalling system for the maze solving algorithm. Particles $\mathcal{P}$ and $\mathcal{P}'$ send a ray in a randomly chosen direction and calculate their probability of movement according to their distance from the wall).

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An algorithm for maze-solving

Algorithm 2 Maze-solving subroutine of each particle $\mathcal{P}$.

Given parameters d, τ_0, τ **and** γ

1: $\mathcal{P}$ picks a direction uniformly at random.
2: It then sends a ray up to a maximum distance d in the chosen direction.
3: **if** ray returns to $\mathcal{P}$ after travelling a total distance of $2k$ **then**
4: $\quad$ $\mathcal{P}$ sets $\lambda \leftarrow \exp\left(\frac{1}{\tau+\tau_0\gamma^{-k}}\right)$.
5: **else** $\quad$ ▷ ray does not return to $\mathcal{P}$
6: $\quad$ $\mathcal{P}$ sets $\lambda \leftarrow 1$.
7: $\mathcal{P}$ performs compression with parameter value λ.

Demo: Stochastic Maze Solving

Diagnosing Deficiencies in the Model

- The random-walk is effectively trapped in regions where all walls are roughly at the same distance, and the effect gets more pronounced as the exit distance is increased.
- By its very nature, the algorithm maze-solving is memory-less in that a system has no way of knowing if it has been at a certain location before.

Concluding remarks

Contributions:

- We propose a stochastic algorithm for collective maze-solving under the *geometric amoebot model* of self-organizing particle systems; and
- give an algorithm to verify *phototaxing* in arbitrarily large systems **given the set of possible configurations** is known.

Future direction:

Recently [Li et al., 2020] proposed another Markov chain based algorithm for compression/expansion that does away with the connectivity constraints of the original algorithm. Combining their algorithm with the ant-inspired approach is a possible future direction of this work.

Open problems

- The mixing-time of the Markov chain for the Ising model – and by extension for the compression algorithm – is believed to be polynomial but a proof is not known. Current best upper-bound is quasipolynomial [Martinelli and Toninelli, 2010].

- It is not known whether phototaxing occurs in a general system of n-particles.

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Compression without connectivity

To address these issues, we define a modified aggregation and dispersion algorithm $\mathcal{M}_{\text{AGG}}$ where particles can disconnect and moves rely only on the current state. Here, particles occupy nodes of a finite region of the triangular lattice, again moving stochastically and favoring configurations with more pairs of neighboring particles. Each particle has its own Poisson clock and, when activated, chooses a random adjacent lattice node. If that node is unoccupied, the particle moves there with probability λ^{-n}, where n is the number of current neighbors of the particle, for bias parameter $\lambda > 0$. Thus, rather than biasing particles towards nodes with more neighbors, we instead discourage moves away from nodes with more neighbors, with larger λ corresponding to a stronger ferromagnetic attraction between particles (Figure 1A). This new chain $\mathcal{M}_{\text{AGG}}$ converges to the same Boltzmann distribution $\pi(\sigma) \propto \lambda^{E(\sigma)}$ over particle system configurations σ as the original SOPS algorithm. Details of the proofs can be found in the Materials and Methods.

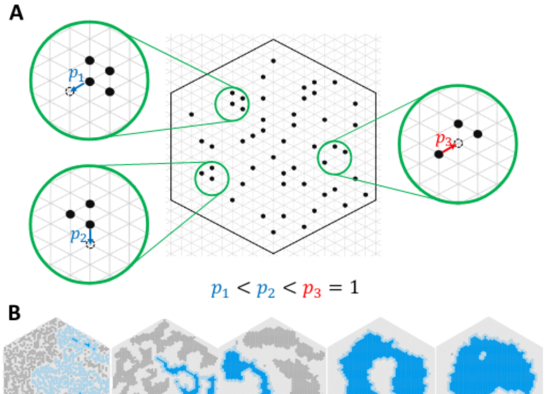

Figure 1: The theoretical self-organizing particle system (SOPS). (**A**) A particle moves away from a node where it has n neighbors with probability λ^{-n}, where $\lambda > 0$. Thus, moves from locations with more neighbors are made with smaller probability than those with fewer (e.g., in the insets, $p_1 = \lambda^{-3} < p_2 = \lambda^{-2} < p_3 = 1$). (**B**) Time evolution of a simulated SOPS with 1377 particles for $\lambda = 7.5$ showing progressive aggregation (Movie S1). The bulk of the largest connected component is shown in blue and its periphery is shown in light blue. (**C**) Time evolution of N_{AGG}, the size of the

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