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The private provision of public goods via dominant assurance contracts *

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Abstract. Many types of public goods can be produced privately by profit seeking entrepreneurs using a modified form of assurance contract, called a dominant assurance contract. I model the dominant assurance contract as a game and show that the pure strategy equilibrium has agents contributing to the public good as a dominant strategy. The game is also modelled under incomplete information as a Bayesian-Nash game.

Economics gives invisible hand explanations for complex phenomena. Economics explains how Paris is fed without the aid of central planning and bureaucratic direction. Many economists argue, however, that central planning and bureaucratic direction are necessary to produce public goods. I show that at least some types of public goods can be produced privately by profit seeking entrepreneurs. Section one introduces the idea of an assurance contract and discusses, without making rigorous, the set of equilibria in an assurance contract game. A new and more powerful form of assurance contract, called a dominant assurance contract, is introduced and analyzed in section two. I show that an entrepreneur can design a contract where the equilibrium has agents contributing to produce the public good as a *dominant* strategy. I introduce incomplete information in section three and solve for a Bayesian equilibrium. Previous work on assurance contracts has restricted attention to complete information settings.[1]

There are two problems involved in the production of public goods, the preference revelation problem and the contribution problem. In this paper I concentrate on the contribution problem, how to get agents to voluntarily contribute to providing the public good. This limits the analysis to goods which naturally come in lumpy quantities or goods for which we can deduce an efficient size. If a bridge or a road or a lighthouse is to built we can probably estimate the efficient size from information about preferences and technology.

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For example, if we know the location and size of the rocks we need to avoid and we know typical weather conditions and other data of this type we can probably limit the agent's choices to no lighthouse or a lighthouse of size X without great loss of efficiency.[2]

1. Public goods and assurance contracts

Consider a group of individuals who must decide independently whether or not to contribute towards providing a public good. For concreteness assume that a plain is flooding and that a dike must be built to hold back the river. It is known with certainty that the flood will be averted if the dike reaches 6ft. If the dike doesn't reach 6ft then the river overflows and the builder's efforts are wasted. Assume that the agent's choice is binary, he can help to build the dike or not. Let there be a number of other agents, then we can write a (highly simplified) form of the game as in Table 1.

Whether or not others contribute, the agent is better off not contributing. If others contribute, the agent's optimal choice is to free ride by not contributing. If the others do not contribute, contributing wastes effort. If all agents reason thus, the public good is not produced and the plain is flooded.

This is the standard analysis of a public good/prisoner's dilemma game but note that the game really consists of two distinct elements (Sen, 1967; Schmidtz, 1991). Consider column one; the problem in this column is not free riding. (The others will not contribute so there is no one to free ride upon). The problem in column one is that the agent doesn't want to waste his effort in building a dike which won't hold the river. Each agent lacks "assurance" that the others will contribute in the event that he contributes.

The problem in column 2 is quite different. In column two each agent assumes that the public good will be provided regardless of his actions. Each agent, therefore, wishes to "free ride" on the others.

That these problems are different can be seen from the following thought experiment. Assume that all agents are anti free-riders. Whether from moral prohibition or for some other reason none of these agents will ever knowingly free ride on another. Assume, however, that each agent is unaware that the others are anti free-riders. If each agent knows the others will contribute, he will too because he doesn't want to free ride. But if each agent believes the others will not contribute, he won't contribute either. Even an anti free-rider doesn't want to waste his effort on a lost cause.

The assurance problem can be overcome via an assurance contract.[3] Assume the town mayor announces that all those willing to help build the dike are to meet at the town hall. And, the mayor adds, work will commence on

Table 1. Averting a flood

Agent i / Others	Do not contribute	Contribute
Do Not Contribute	(0,0)	(950,800)
Contribute	(–100,0)	(900,900)

Table 2. Mayor's game 1

Agent i / Others	Do not contribute	Contribute
Do Not Contribute	(0,0)	(950,800)
Contribute	(0,0)	(900,900)

the dike only if enough workers gather so that the dike can be built to a height of 6ft in time to avert the flood.

The prisoner's dilemma game is now transformed into Table 2.

Each agent now knows that his efforts will not be wasted. He still faces an incentive to cheat his fellow towns-people but if he is altruistic, or he fears their censure, then perhaps he may not wish to do so. If one has an optimistic (naive?) view of human nature the mayor's actions can result in a large improvement in the town's prospects. As given, however, the payoffs indicate that the rational action is still for each agent to free ride and the dike, therefore, not to be built.

The mayor can improve upon his scheme. Let the mayor announce that the dike will be built if and only if everyone in town agrees to contribute. The game is now transformed into Table 3.

All agents contributing is now a Nash equilibrium, even if agents are purely self-interested (Palfrey and Rosenthal, 1984; Bagnoli and Lipman, 1989). Note though that {Do not contribute, Do not contribute} also remains a Nash equilibrium and there are many, many other equilibria in all of which the public good is not produced.[4] In a loose but intuitive sense the {Contribute,

Table 3. An assurance game

Agent i / Others	Do not contribute	Contribute
Do Not Contribute	(0,0)	(0,0)
Contribute	(0,0)	(900,900)

Contribute} equilibrium is weak because it requires that everyone contribute. Let there be but a single non-contributor (perhaps an agent “trembles” and picks the wrong strategy) and the town will be flooded. The equilibrium is actually weaker than this indicates because of the self-fulfilling nature of negative beliefs. If I believe that you will not contribute then it is rational for me not to contribute, but if I do not contribute then it is rational for you not to contribute and so my belief that you will not contribute becomes self-fulfilling. Any failure of the common knowledge assumption can break down this equilibrium.

To avoid the indifference problem we can add a small cost of agreeing to contribute (you have to walk to the town hall), in which case there are only two equilibria, all contribute and none contribute. With a small cost of agreeing to contribute negative beliefs are especially powerful. If I believe that you will not contribute then I have an incentive not to contribute (to avoid the loss). But if I do not contribute then you have an incentive not to contribute and my initial belief becomes self-fulfilling.

Assurance contracts appear to be a useful means of overcoming the assurance problem but the fact that the no-contribute equilibria are not eliminated suggests that these games are not robust to large numbers or to deviations from perfect knowledge.

A new type of contract is now introduced which can overcome the problems of the assurance contract. In particular, in the new contract there is only one equilibrium, it is in dominant strategies, and the public good is always provided.

2. Dominant assurance contracts

Consider the following two stage game called the public good game: In the first stage the entrepreneur ends the game immediately or offers to each of N agents a contract denoted by the triple (\$F, \$S, K agents), (the meaning of these terms is described below). In the second stage each of the N agents can accept or reject the contract. If an agent accepts the contract, she receives a payoff which is conditional on the total number of agents who accept. If fewer than K agents accept, the contract is said to fail and each accepting agent receives from the entrepreneur a payoff F (for Fail). If K or more agents accept, the contract is said to succeed and each accepting agent must pay the entrepreneur S (for Succeed). If the contract succeeds, the entrepreneur produces a public good which is worth $V_i = V$ to each agent and costs C in total to produce. Letting X be the number of agents who accept, the contract and payoffs can be written as in Table 4.

Table 4. Agent payoffs

If $X < K$ each accepting agent receives a payment from the entrepreneur
of F. Payoff = $F > 0$
If $X \geq K$ each accepting agent must pay the entrepreneur S and the
entrepreneur produces a public good worth V to each agent.
Payoff $= V - S$

Payoffs to the entrepreneur are therefore $(-XF)$ if the contract fails, $(XS - C)$ if the contract succeeds, and zero if the entrepreneur decides not to offer a contract. Payoffs to non-accepting agents are zero if the contract fails and V if it succeeds. The entrepreneur and the agents are both assumed to be risk neutral. The game is voluntary and the entrepreneur cannot non-contractually impose costs on the agents, thus, $F \geq 0$.[5] In order to recover its costs the firm must charge $S > 0$ for $C > 0$.

To solve this game begin at the second stage and look for Nash equilibria to the subgame. Note first that if $K < N$ then neither all "accept" nor all "reject" is a NE. If all agents accept then a deviator earns $V > (V - S)$. If all agents reject then a deviator earns $F > 0$.

There are two types of pure-strategy sub-game equilibria; one in which the contract succeeds and the other in which it fails. If $V - S \geq 0$ then the following is a pure strategy sub-game equilibria; K agents accept and the remainder reject. A rejecter cannot increase his payoff by accepting since $V > V - S$. An accepter cannot increase his payoff by rejecting since $V - S \geq 0$. There are $\binom{N}{K}$ of these equilibria, one for each possible set of acceptors. Note that in every one of these sub-games the public good is provided.

If $V - S < 0$ then the following are pure-strategy sub-game equilibria; $K - 1$ agents accept and the remainder reject. A rejecter cannot increase his payoff by accepting since if he accepts the contract succeeds and $0 > V - S$. An accepter cannot increase his payoff by rejecting since $F > 0$. There are $\binom{N}{K-1}$ of these equilibria.

These are the only pure strategy sub-game equilibria (see the Appendix).[6]

To solve the full game we now turn to stage one of the game. The entrepreneur can always earn a zero payoff by exiting the game in stage one. Thus, we know that the entrepreneur will always set $V > S$ so that the contract will succeed in all the pure strategy equilibria of the full game and the entrepreneur will earn positive profits.

The entrepreneur thus wishes to maximize profits subject to the condition that $V > S$.

$$
\begin{aligned}
&E\Pi = SK - C > 0 \\
&\text{s.t. } V - S \geq 0
\end{aligned}
$$

It is clear that maximum profits are reached when the entrepreneur sets $K = N$ and $S = V - \text{epsilon}$. Maximum profits are then given by $\simeq VN - C$. Note that VN is the total value of the public good and C the total cost. The entrepreneur's profit maximizing decision, therefore, implies that a necessary and sufficient condition for the entrepreneur to produce the public good is that it be efficient to do so, i.e. that $VN > C$.[7]

To review; in the first stage of the game the entrepreneur sets $K = N$ and $S = V - \text{epsilon}$. In the second stage all agents accept the contract. The game where $K = N$ is similar to an assurance contract but there are important differences. In the assurance game, all accept is only one equilibrium among many and all but one of the equilibria have the public good not being provided. The all accept equilibrium in the public good game is the unique subgame perfect Nash equilibrium.[8] Furthermore, accept is a dominant strategy. Every agent has an incentive to accept the contract regardless of what he believes others will do. Dominance makes the all accept equilibrium very strong.[9]

Negative beliefs are not self-fulfilling in the dominant assurance contract equilibrium because regardless of what he believes other agents will do agent i has an incentive to accept the contract. Even in the case where $K < N$ negative beliefs are not self-fulfilling because if agent i believes the contract will fail then agent i wishes to accept in order to earn F.

An interesting special case of the above model occurs when the public good is excludable. If the public good is excludable then accept is a dominant strategy for each agent regardless of K. Consider a group of N agents and an excludable public good, like a bridge, which costs C to produce. Let the entrepreneur offer the agents an (F, S, K) contract with the additional proviso that only accepting agents can consume the public good if it is produced. In equilibrium every agent will accept the contract because non-accepting agents earn zero while accepting agents earn either $V - S > 0$ or $F > 0$.

In the model given above, the entrepreneur earns all the consumer surplus from the public good. Offering a contract, however, is a very low cost activity and thus, following Demsetz (1968), contract provision should be a contestable market. Competition will push S down to C/N so that public good provision will be efficient and will benefit consumers.[10]

Table 5. Agent payoffs and decision rule: Imperfect information game

Expected payoff from reject = $P_r * V_i$
Expected payoff from accept = $(1 - P_a)F + P_a(V_i - S)$
Decision Rule: Accept if: $(1 - P_a)F + P_a(V_i - S) \geq P_r V_i$

3. Incomplete information

A good criticism of assurance contracts and the dominant assurance contract discussed above is that they assume complete information and homogenous preferences. In this section it is assumed that preferences are distributed according to the continuous density function g[V] with distribution function G[V] and support $[V_L, V_H]$. It is assumed that $0 \leq V_L < V_H$ so we may assume G has support $[0, 1]$ without loss of generality. We will look for a symmetric Bayesian equilibrium.

The game is modelled as follows: Nature chooses independently for each agent a value of the public good V_i according to G[V], this information is revealed to agent i only. G[V] is common knowledge. A strategy is a choice by an agent to accept or reject the contract as a function of the agent's type V_i. It is well known that in models of this type the optimal strategy for the agent is a decision rule of the form accept if $V_i \geq V^*$ and reject otherwise. V^* will be a function of N, K, F, and S. In the first stage of the game the entrepreneur chooses F, S, and K to maximize profit knowing N and the agent's decision rule $V^* = V^*[N, K, F, S]$. Agent payoffs are given in Table 5.

In Table 5 P_a is the probability the contract succeeds given the agent accepts, this is the probability that $K - 1$ or more agents out of $N - 1$ accept the contract. P_r is the probability the contract succeeds given the agent rejects, or equivalently the probability that K or more out of $N - 1$ agents accept the contract. We wish to solve the decision rule for V^*, the minimum V such that accept is optimal. Once we know the exact form of the agent's decision rule, accept if $V_i \geq V^*[N, K, F, S]$, we can find the entrepreneur's expected profit function and deduce his optimal choices. It will be useful to write the decision rule condition in "long form":

$$\begin{aligned} & V^* \binom{N-1}{K-1} (1 - G(V^*))^{K-1} G(V^*)^{N-K} \\ = \; & (F + S) \sum_{x=K-1}^{N-1} \binom{N-1}{x} (1 - G(V^*))^x G(V^*)^{N-1-x} - F \end{aligned} \tag{3.1}$$

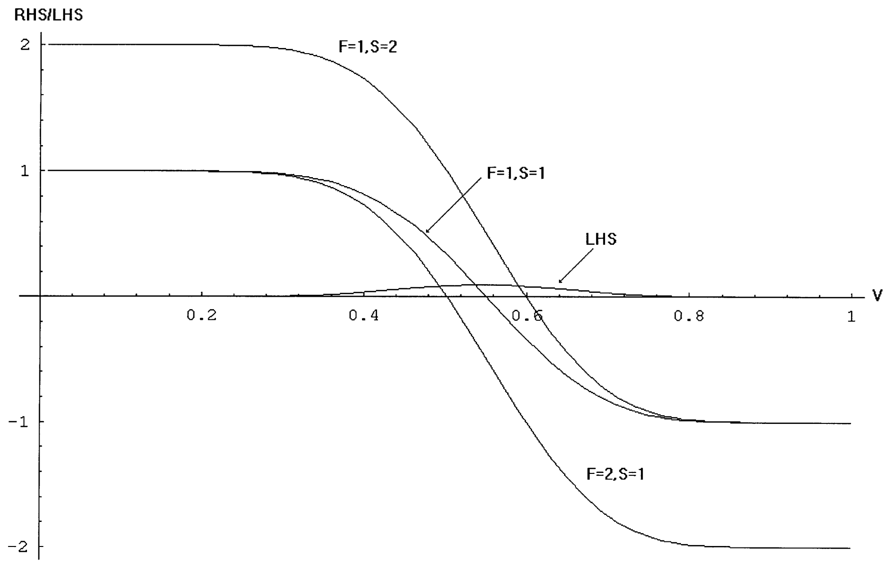

Figure 1. Bayesian equilibrium for agents: Equation 3.1.

In the Appendix it is shown that an equilibrium to this system always exists.

The summation term on the RHS is a probability between zero and one, thus the RHS of 3.1 goes from S to −F as V^* goes from 0 to 1. Note that the LHS does not depend on F or S and an increase in F decreases the RHS and an increase in S increases the RHS of 3.1 for all values of V. Figure 1 provides the intuition by plotting the LHS and RHS of equation 3.1 against V.[11] The equilibrium point, V^*, is found at the intersection of the LHS and respective RHS curves. An increase in S increases V^* and an increase in F decreases V^*. It should be clear from equation 3.1 that for any V^* any number of F, S pairs exist which satisfy equation 3.1.

The entrepreneur chooses F, S, and K to maximize profit, which can be written:

$$\underset{F,S,K}{\text{Max}}\ E\Pi = P_e\, S\, E(x \mid x \geq K) - (1 - P_e)\, F\, E(x \mid x < K) - P_e\, C \tag{3.2}$$

Where P_e is the probability the contract succeeds which is the probability that K or more out of N agents accept the contract. Equation 3.2 says that expected profit is equal to the probability the contract succeeds times the per agent payoff to the entrepreneur (S) times the expected number of agents who accept the contract conditional on the contract succeeding minus a similarly interpreted term for when the contract fails minus the expected cost of the public good. We can simplify this equation to the following (see Appendix for proof):

$$\underset{V^*,K}{\text{Max}}\ E\Pi = V^*K\binom{N}{K}(1-G(V^*))^K(G(V^*))^{N-K} - P_eC \tag{3.3}$$

To gain some intuition about equation 3.3 note that $\binom{N}{K}(1 - G(V^*))^K (G(V^*))^{N-K}$ is the probability that exactly K agents accept the contract. But if exactly K agents accept the contract then every agent is "pivotal," in the sense that if any agent were to reject, the contract would fail. Now consider a situation in which every agent knows that he is pivotal. In this situation, no free riding is possible. Every agent knows that to get the public good he must pay the price, thus, as far as the individual agents are concerned, the public good is a private good. Since every agent treats the public good as if it were a private good there is no public good problem. The pivot point is special because at the pivot point, and only at the pivot point, the public good problem disappears.[12] All of the entrepreneur's profits arise because at the pivot point agents behave as if the public good were a private good and, given a finite number of agents, there is always a positive probability that the pivot point will occur. Now recall that V^* is the maximum amount the marginal agent is willing to pay for the public good. The entrepreneur doesn't have enough information to price discriminate, so V^*K is the maximum amount of revenue the entrepreneur can expect to earn when the marginal agent has value V^*. Equation 3.3 can thus be read as expected profit is equal to expected revenue minus expected cost.

Equation 3.3 does not directly depend on F or S, instead these are subsumed in the more fundamental variable V^*. From above, we know that for any N, K pair, F and S determine V^*. This is a useful property because a) we can find the optimal V^* and then work backward to find an implied F, S pair rather than maximize over F, S directly and b) we can choose F and S such that V^* is held constant when K or N changes and this will help in deriving comparative statics results.

Setting $C = 0$, profit maximization leads to the following first order condition for V^*:

$$V^* = \frac{-(1-G(V^*))G(V^*)}{g(V^*)(N(1-G(V^*))-K)} \tag{3.4}$$

The first order condition for K is not continuous but reasoning by analogy with the continuous case we require:

$$\frac{\Delta\Pi}{\Delta K} = V^*(K+1)\binom{N}{K+1}(1-G(V^*))^{K+1}(G(V^*))^{N-(K+1)} - V^*K\binom{N}{K}(1-G(V^*))^K(G(V^*))^{N-K} \simeq 0$$

In the Appendix it is shown that this first order condition reduces to:[13]

$$N(1 - G(V^*)) \simeq K \tag{3.5}$$

Equation 3.5 tells us that the optimal K must be near the mean of the distribution (since $N(1 - G(V^*))$ is the mean of the binomial). The solution is intuitive if we recall that the entrepreneur maximizes profits when there is a high probability that an agent is pivotal. Keeping K close to the mean maximizes the probability that an agent is pivotal.[14] We can specify the optimum slightly more accurately by noting that the first order condition for V^* requires $N(1 - G(V^*)) < K$ for $V^* > 0$. It follows that K must be slightly greater than the mean of the distribution.

The key tradeoff in the model is between V^* and K. The entrepreneur would like both to be high but raising V^* reduces the probability that an agent accepts, and the lower the probability an agent accepts, the lower is the optimal K. The terms of the tradeoff are governed by $G(V^*)$. If (at a given point) $G(V^*)$ is such that a small decrease in V^* allows for a large increase in the probability of acceptance then it will pay to reduce V^* and raise K. Since the optimum depends on the entire shape of the distribution function more specific results require that we specify the distribution function, $G(V^*)$. For example, let Nature draw the agent's valuations from a uniform distribution on [0, 1]. Equation 3.4 then becomes:

$$V^* = \frac{-(1-V^*)V^*}{(N(1-V^*)-K)}$$
$$\Rightarrow V^* = \frac{N+1-K}{N+1}$$

Substituting the optimal V^* into the profit function and maximizing for K we find that for N odd, $K = \frac{N+1}{2}$and for N even profit is maximized at either $K = \frac{N}{2}$ or $K = \frac{N}{2} + 1$ (see the Appendix).[15] The optimal V^* is equal to $\frac{1}{2}$ for N odd and approaches $\frac{1}{2}$ as N increases for N even.

The probability that the public good is provided is $\frac{1}{2}$. This can be found as follows: for large N the entrepreneur requires that half the agents accept and the probability each agent accepts is $\frac{1}{2}$. The probability the contract succeeds, therefore, is $\frac{1}{2}$.

Setting V^* to $\frac{1}{2}$ and K to $\frac{N}{2}$ it is easy to check that expected profit is proportional to $\frac{N}{2}$ which is increasing in N. We prove below that profit increases in N in the general case. To find the F, S pairs which sustain the above equilibrium note that as N increases the LHS side of equation 3.1 goes to zero and the probability on the RHS goes to $\frac{1}{2}$ so any F, S pair such that $F \simeq S$ will generate the required V^*.[16]

Other examples can be calculated numerically. Figure 2 shows the expected profit function when $N = 100$ and $V_i \sim N_{ormal}(\frac{1}{2}, \frac{1}{12})$.[17] Expected profit

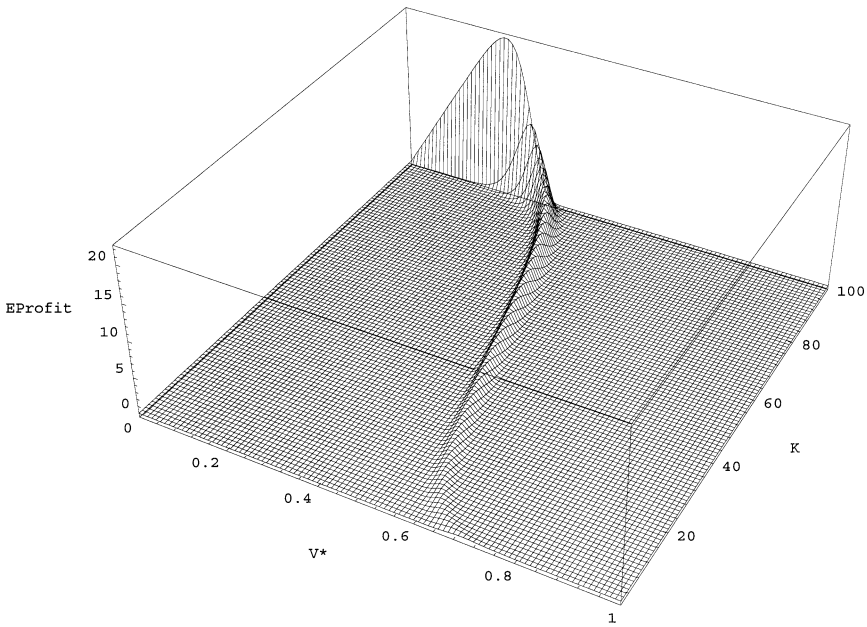

Figure 2. Expected profits as a function of V* and K.

is maximized at $K = 100$ and $V^* = 0.243$. For purposes of comparison, note that the mean and variance of the normal are the same as for the uniform distribution for which the optimum is $K = 50$ and $V^* = 0.505$. The normal distribution has more of its mass clustered around the mean than the uniform distribution. If the V_i are normally distributed, therefore, a small decrease in V^* from the mean will increase the probability an agent accepts more than if the V_i are uniformly distributed. When the probability that an agent accepts increases, the entrepreneur increases K, the required number of acceptors. The probability the contract succeeds is, in this example, over 99%. With such a low variance it is not surprising that the incomplete information case approaches the complete information case.

4. Comparative statics

An important comparative statics result is that profit increases in N – the more valuable the public good the greater the incentive to produce it. Proof: From the envelope theorem we know that:

$$\mathrm{SGN}\frac{\partial \mathrm{EX}\Pi}{\partial N} = \mathrm{SGN}\,\frac{\partial \binom{N}{K} G(V^*)^{N-K}}{\partial N} \tag{4.1}$$

Where $G(V^*)$ is treated as a constant and a number of other constants have been dropped. By analogy with the continuous case we are interested in the sign of:[18]

$$\begin{aligned}\frac{\Delta EX\Pi}{\Delta N} &= \binom{N}{K} G(V^*)^{N-K} - \binom{N-1}{K} G(V^*)^{N-1-K} \\ &= \frac{N!}{(N-K)!K!} G(V^*) - \frac{(N-1)!}{(N-1-K)!K!}\end{aligned}$$

Cancelling the factorial terms and rearranging we have:

$$SGN\frac{\Delta EX\Pi}{\Delta N} > 0 \text{ if } N(1 - G(V^*)) < K$$

The last condition is a necessary condition for $V^* > 0$ and so must hold.

Intuition suggests (correctly) that expected profit will increase the more likely it is that agents place a high value on the public good. Consider $G_1(V)$ such that $G_1(V)$ first order stochastically dominates $G(V)$. That is, $G_1(V) < G(V)$ for all V. Now consider V_1, V such that $G_1(V_1) = G(V)$ it follows that $V_1 > V$. Looking at equation 3.3 note that if we replace $G(V)$ with $G_1(V_1)$ the only thing which changes is the very first term of the profit function which increases from V to V_1. Thus, expected profit increases with an increase in the probability that an agent values the public good highly.

The introduction of $C > 0$ modifies the conclusions only slightly. An increase in V always causes a decrease in $P_e * C$ which increases expected profit. Thus, the optimal V^* will be larger the larger is C. An increase in K also reduces $P_e * C$ thus the optimal K increases as C increases.

5. Conclusions

Many public goods problems are contribution problems rather than revelation problems. Dominant assurance contracts are a powerful method for solving contribution problems. The Nash equilibrium for the dominant assurance contract has all agents contributing to the public good as a dominant strategy. The Bayesian equilibrium give entrepreneurs a very flexible means of eliciting contributions to public goods.

In future work it will be interesting to extend the Bayesian game to multiple rounds. A negative outcome in the first round of the Bayesian game reveals a considerable amount of information. If the first round contract fails then it must be that fewer than K agents have valuations greater than V^*. Moreover, if the contract fails in the first round the entrepreneur will know exactly which agents value the public good more than V^* (those who accepted the contract) and which less. Further research will be necessary to ascertain whether the entrepreneur can use this information to improve the second round contract and what affect this will have on the first round equilibrium.

Appendix

A.1. *The pure strategy equilibria*

For each possible relationship between $V - S, F$, and 0 we check for pure strategy equilibria. The necessary and sufficient conditions for pure strategy Nash equilibria are that neither acceptors nor rejecters can improve their payoff by switching to the other strategy. The table illustrates whether the conditions hold under each of the parameter specifications. For example, in the first table acceptors earn $V - S$ when accepting and zero when rejecting thus $V - S > 0$ is a necessary and sufficient condition for K agents to accept.

A.1.1. *Equilibrium: K agents accept, N-K reject. Contract succeeds*

Strategy	Nec. Condition	$V - S > F > 0$	$F > V - S > 0$	$F > 0 > V - S$
Acceptors	$V - S > 0$	Yes	Yes	No
Rejectors	$V > V - S$	Yes	Yes	Yes

A.1.2. *Equilibrium: K-1 accept. N-K+1 reject. Contract fails.*

Strategy	Nec. Condition	$V - S > F > 0$	$F > V - S > 0$	$F > 0 > V - S$
Acceptors	$F > 0$	Yes	Yes	Yes
Rejectors	$V - S < 0$	No	No	Yes

A.2. Simplification of equation 3.2 to equation 3.3

Note that $E(x \mid x \geq K) = \left(\sum_{x=K}^{N} x \binom{N}{x} (1 - G(V))^x G(V)^{N-x}\right) / P_e$, using this expression and the related expression for $E(x \mid x < K)$ write equation 3.2 as:

$$S \sum_{x=K}^{N} x \binom{N}{x} (1 - G(V))^x G(V)^{N-x} - F \sum_{x=0}^{K-1} x \binom{N}{x} (1 - G(V))^x G(V)^{N-x}$$

(We have dropped the $P_e C$ term as it is simply carried through for the entire proof.)

Change the index of the second term to run from 1 to $K - 1$ and then write out the binomial terms in factorial form.

$$S\sum_{x=K}^{N} x\frac{N!}{x!(N-x)!}(1-G(V))^{x}G(V)^{N-x}-$$

$$F\sum_{x=1}^{K-1} x\frac{N!}{x!(N-x)!}(1-G(V))^{x}G(V)^{N-x}$$

Now cancel the x term (with the first factorial term) change the index variable to $y = x - 1$ and rewrite the above.

$$S\sum_{y=K-1}^{N-1} \frac{N!}{y!(N-1-y)!}(1-G(V))^{y+1}G(V)^{N-1-y}-$$

$$F\sum_{y=0}^{K-2} \frac{N!}{y!(N-1-y)!}(1-G(V))^{y+1}G(V)^{N-1-y}$$

Pull out an $N(1 - G(V))$ term from each expression and rewrite the factorials:

$$N(1-G(V))\begin{pmatrix} S\sum_{y=K-1}^{N-1}\binom{N-1}{y}(1-G(V))^{y}G(V)^{N-1-y}- \\ \\ F\sum_{y=0}^{K-2}\binom{N-1}{y}(1-G(V))^{y}G(V)^{N-1-y} \end{pmatrix}$$

The above can be simply rewritten as $N(1 - G(V))[SP_a^* - F(1 - P_a^*)]$. Where P_a^* is the equilibrium P_a which can be found by rearranging equation 3.1. Substitute this value into the above:

$$VN(1-G(V))\left(\binom{N-1}{K-1}(1-G(V))^{K-1}G(V)^{N-K}\right)$$

Push through the $(1-G(V))$ term, multiply by $\frac{K}{N}$, adjust the binomial term appropriately, and we arrive at equation 3.3.

A.3. *Existence and uniqueness of a solution to the Bayesian problem*

Write equation 3.1 as follows:

$$V^* = \frac{(F+S)\sum_{x=K-1}^{N-1}\binom{N-1}{x}(1-G(V^*))^{x}G(V^*)^{N-1-x} - F}{\binom{N-1}{K-1}(1-G(V^*))^{K-1}G(V^*)^{N-K}} \qquad (A.1)$$

The LHS of equation A.1 is always non-negative but the RHS goes from a large positive number (S/denominator) to a large negative number (−F/denominator) as V^* goes from 0 to 1. Since G[V] is continuous the RHS is continuous and therefore an equilibrium exists.

For large N the equilibrium is unique regardless of the distribution function. Consider equation 3.1 and note that for large N the LHS becomes diffuse, that is it goes to zero for all V^*. Since the RHS is always negatively sloped the equilibrium is unique. Figure 1 illustrates.

A.4. *First order condition for K*

The first order condition is (where $\Delta K = 1$):

$$\begin{aligned}\frac{\Delta \Pi}{\Delta K} = {} & V^*(K+1)\binom{N}{K+1}(1-G(V^*))^{K+1}(G(V^*))^{N-(K+1)} - \\ & V^*K\binom{N}{K}(1-G(V^*))^{K}(G(V^*))^{N-K} \simeq 0\end{aligned}$$

Cancelling terms we have:

$$(K+1)\binom{N}{K+1}(1-G(V^*)) \simeq K\binom{N}{K}G(V^*)$$

Writing the binomial formula in long form and cancelling the $(K+1)$ terms on the LHS, we have:

$$\frac{N!}{(N-K-1)!K!}(1-G(V^*)) \simeq K\frac{N!}{(N-K)!K!}G(V^*)$$

Further cancelling the factorial terms leaves us with:

$$\begin{aligned}(1-G(V^*)) & \simeq \frac{K}{N-K}G(V^*) \\ \text{or } N(1-G(V^*)) & \simeq K\end{aligned}$$

A.5. *Proof that for V distributed uniformly on [0,1] and N odd K*=(N+1)/2.*

To proof the supposition we show that profit at $K = \frac{N}{2}$ equals profit at $K = \frac{N+2}{2}$. Since the binomial distribution is unimodal we have that profit at $K = \frac{N+1}{2}$ is maximized.

Recall that the optimal $V^* = \frac{N+1-K}{N+1}$. Let $K_m = \frac{N}{2}$, then $V_m = (\frac{N/2+1}{N+1})$ and $1 - V_m = \frac{N/2}{N+1}$. Let $K_p = \frac{N+2}{2}$, then $V_p = \frac{N/2}{N+1}$ and $1 - V_p = \frac{N/2+1}{N+1}$. Substituting these into the profit functions at the respective K's we have:

$$\frac{N/2+1}{N+1}N/2\binom{N}{N/2}\left(\frac{N/2}{N+1}\right)^{N/2}\left(\frac{N/2+1}{N+1}\right)^{N/2}$$
$$=\frac{N/2}{N+1}(N/2+1)\binom{N}{N/2+1}\left(\frac{N/2+1}{N+1}\right)^{N/2+1}\left(\frac{N/2}{N+1}\right)^{N/2-1}$$

The $(N+1)$ terms in the denominator all cancel, pushing through some of the terms in the numerator and cancelling we have:

$$N/2\binom{N}{N/2}=(N/2+1)\binom{N}{N/2+1}$$

Which can easily be shown to be true by converting the binomial terms to factorial style and then cancelling.

Notes

1. See the papers in note 3.
2. The binary assumption is common in the literature, see among many others the papers in note 3.
3. Sen (1967) drew attention to the dual nature of the prisoner's dilemma problem. Brubaker (1975) is an early proponent of using assurance contracts to solve public goods problems. Schmidtz (1991) provides a good overview of this literature. A more formal literature on assurance contracts has arisen lately. The classic paper is by Palfrey and Rosenthal (1984), Bagnoli and Lipman (1989) allow contributions to be continuous. Nitzan and Romano (1990) examine the case where the public good has uncertain costs. Gradstein and Nitzan (1990) let the public good be provided in continuous amounts.
4. There are $\sum_{i=0}^{N}\binom{N}{i}$ equilibria in this game. One in which all donate, one in which none donate, N in which all but one donate and so forth. For a game with 25 agents there are over 33 million equilibria.
5. If there is a cost of agreeing to contribute denoted ε then the net payoff in the event of failure must be positive which requires $F>\varepsilon$. We show below that there is an equilibrium for any F so the exact size of F is not material.
6. There is also a mixed strategy equilibrium to the sub-game. The mixed strategy sub-game never occurs in the full game, however, and so is not discussed.
7. It is worth noting that F may be epsilon small. Thus, a small probability of agent deviation will not be worrisome to the entrepreneur.
8. As is often the case in extensive form games, there is an odd Nash equilibrium in this game where $N+1-K$ or more agents reject and the entrepreneur does not offer the contract. This equilibrium is not sub-game perfect. Moreover, unlike games in which subgame perfection rules out an "incredible" threat, in this game the odd equilibrium is not utility maximizing for *either* the entrepreneur or the agents.

   It is sometimes argued that the entrepreneur has no incentive to pay the agents if the contract fails. We are modelling, however, an assurance *contract* which we assume can be enforced. The entrepreneur pays the fee for the same reason that lotteries pay out the prizes once the tickets have been sold.

9. It may seem puzzling that in equilibrium the contract always succeeds and the entrepreneur never pays out F. How can the promise of a payment that is never made affect the equilibrium? The intuition is similar to that of Diamond and Dybvig's (1983) model of bank runs in which a promise to fully pay depositors, should a bank run occur, precludes bank runs ever occurring.
10. Competition in contract provision requires that the consumers engage in some limited coordination to choose the best contract provider who then approaches each agent independently.
11. Figure 1 plots the specific case where the V_i are distributed uniformly with $N = 20$, and $K = 10$.
12. The "pivot principle" as used here was first deduced in a related argument by Bagnoli and Lipman (1988); see also Holmstrom and Nalebuff (1992), and in a different context Kreps's (1990: 704–713) discussion of mechanism design and the Gibbard-Satterthwaite theorem.
13. The approximation sign is necessary since K is integer valued but $(1 - G(V^*))$ is not.
14. The statement in the text is slightly inaccurate. The probability an agent is pivotal is, $Pr_P = \binom{N}{K}(1 - G(V^*))^K(G(V^*))^{N-K}$, but the entrepreneur wishes to maximize $V^*K * Pr_P$. To maximize the probability an agent is pivotal, however, requires $K \simeq N(1 - G(V^*)) - G(V^*)$ which is almost identical to what is required to maximize $V^*K * Pr_P$.
15. The reader may wonder, as I did, why we can't use the first order conditions to solve for an equilibrium directly. The reason is that the first order condition for K is only approximate since K is discontinuous. We can write $K = N(1 - G(V^*)) + \varepsilon$ which is exact for some $\varepsilon \in (0, 1)$. If we solve for the uniform case we find $K = \varepsilon(N + 1)$, which allows for any value of K.
16. It is easy to solve equation 3.1 for an exact F, S pair for any N, K, V. The limiting case is presented for convenience. For the uniform distribution, for example, and $N = 101$ we have $K^* = 51$ and $V^* = 1/2$. This equilibrium can be sustained by an F, S pair such that $F = -0.0865 + 1.1723 * S$.
17. Strictly speaking the model assumes that $V_i \in [0, 1]$ whereas the Normal distribution allows for $V_i \in (-\infty, \infty)$. Given the specified mean and variance, however, the tails of the distribution outside [0, 1] are negligible.
18. This and some of the other results in the paper can also be proved using Stirling's formula to create a continuous approximation to $\binom{N}{K}$.

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