Week 14 Recitation Problems

MATH:113, Recitations 304 and 305

Solutions

Show that, if $f'(x) = g'(x)$, then $f(x) = g(x) + C$.

Let $f'(x) = g'(x)$. Then, we have

$$\int f'(x)\, \mathrm{d}x = \int g'(x)\, \mathrm{d}x.$$

Going by our definition of the indefinite integral, we then have

$$\int f'(x)\, \mathrm{d}x = f(x) + C_f \quad \text{and} \quad \int g'(x)\, \mathrm{d}x = g(x) + C_g,$$

where C_f and C_g are constants. But because we know that the integrals are the same, we must have that

$$f(x) + C_f = g(x) + C_g,$$

so

$$f(x) = g(x) + C_g - C_f.$$

If we let $C = C_g - C_f$, then we have

$$f(x) = g(x) + (C_g - C_f) = g(x) + C \implies \boxed{f(x) = g(x) + C}$$

as desired.

What function did we differentiate to get $f(x) = x^4 + 3x - 9$?

Using our definition of the indefinite integral, we get

$$\int x^4 + 3x - 9\, \mathrm{d}x = \boxed{\frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + C}$$

where C is a constant.

$\int k \cdot f(x)\, \mathrm{d}x = k \cdot \int f(x)\, \mathrm{d}x$ for k a real number.

$$\begin{aligned}
\boxed{\int k \cdot f(x)\, \mathrm{d}x} &= \int k \cdot 3x^2 + k \cdot 8x + k \cdot 6\, \mathrm{d}x \\
&= k \cdot x^3 + k \cdot 4x^2 + k \cdot 6x + C \\
&= k(x^3 + 4x^2 + 6x + {}^{C}\!/\!_{k}) \\
&= k \cdot \int 3x^2 + 8x + 6\, \mathrm{d}x \\
&= \boxed{k \cdot \int f(x)\, \mathrm{d}x}
\end{aligned}$$

$\int f(x) + g(x)\, \mathrm{d}x = \left(\int f(x)\, \mathrm{d}x\right) + \left(\int g(x)\, \mathrm{d}x\right)$.

$$\begin{aligned}
\boxed{\int f(x) + g(x)\, \mathrm{d}x} &= \int 3x^2 + 8x + 6 + e^x\, \mathrm{d}x \\
&= x^3 + 4x^2 + 6x + e^x + C \\
&= (x^3 + 4x^2 + 6x + C_f) + (e^x + C_g) \\
&= \int 3x^2 + 8x + 6\, \mathrm{d}x + \int e^x\, \mathrm{d}x \\
&= \boxed{\int f(x)\, \mathrm{d}x + \int g(x)\, \mathrm{d}x}
\end{aligned}$$

Give two functions $f(x)$ and $g(x)$ such that $\int f(x) \cdot g(x)\ \mathrm{d}x \neq \int f(x)\ \mathrm{d}x \cdot \int g(x)\ \mathrm{d}x$.

Suppose $f(x) = x$ and $g(x) = x^2$. Then, we have

$$\begin{aligned} \int f(x) \cdot g(x)\ \mathrm{d}x &= \int x \cdot x^2\ \mathrm{d}x \\ &= \int x^3\ \mathrm{d}x \\ &= \boxed{\frac{1}{4}x^4 + C} \end{aligned}$$

but

$$\begin{aligned} \int f(x)\ \mathrm{d}x \cdot \int g(x)\ \mathrm{d}x &= \int x\ \mathrm{d}x \cdot \int x^2\ \mathrm{d}x \\ &= \left(\frac{1}{2}x^2 + C_f\right) \cdot \left(\frac{1}{3}x^3 + C_g\right) \\ &= \boxed{\frac{1}{6}x^5 + C_f \cdot \frac{1}{3}x^3 + C_g \cdot \frac{1}{2}x^2 + C_f C_g} \end{aligned}$$

These improper integrals are polynomials of differing degree, and so they cannot be the same.

Verify that $y(x) = 2e^{2x}$ is a solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y(x)$.

$$\begin{aligned} \boxed{\frac{\mathrm{d}}{\mathrm{d}x} y(x)} &= \frac{\mathrm{d}}{\mathrm{d}x} 2e^{2x} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}2} x\right) \cdot 2e^{2x} \\ &= 2 \cdot 2e^{2x} \\ &= \boxed{2 \cdot y(x)} \end{aligned}$$

Find a solution for the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}t} = -3y(t)$ where $y(t_0) = -3$ for $t_0 = 0$.

What function $y(t)$ has first derivative -3 times itself? The function $y(t) = -3e^{-3t} + C$ works, as

$$\begin{aligned} \boxed{\frac{\mathrm{d}}{\mathrm{d}t} y(t)} &= \frac{\mathrm{d}}{\mathrm{d}t} - 3 \cdot e^{-3t} + C \\ &= -3 \cdot \frac{\mathrm{d}}{\mathrm{d}t} e^{-3t} + C \\ &= -3 \cdot -3e^{-3t} \\ &= \boxed{-3y(t)} \end{aligned}$$

We also have to satisfy the initial value condition. Right now, we have $y(t) = -3e^{-3t} + C$, so

$$y(t_0) = -3e^{-3 \cdot 0} + C = -3$$

and, solving for C, we get

$$-3 = -3 + C \implies 0 = C,$$

so our solution is

$$\boxed{y(t) = -3e^{-3t}}$$

Find a solution for the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^{-2}$ where $y(x_0) = 1$ for $x_0 = 1$.

What function $y(x)$ has first derivative $-3/x^2$? If we use our inverting-the-power-rule strategy, we find that

$$\int \frac{-3}{x^2}\,\mathrm{d}x = \frac{3}{x} + C,$$

which (as you can check by taking the first derivative with respect to x) satisfies the differential equation given. However, we need to satisfy the initial condition as well: right now, we have

$$y(x_0) = \frac{3}{1} + C = 1$$

and, solving for C, we get

$$1 = 3 + C \implies -2 = C,$$

so our solution is

$$\boxed{y(x) = \frac{3}{x} - 2}$$