Week 13 — Exam Review

MATH:113, Recitations 304 and 305

Implicit Differentiation, Related Rates

Names: SOLUTIONS

For each of the following, find $\frac{dy}{dx}$ first by solving directly for y and differentiating directly; then, use implicit differentiation to find $\frac{dy}{dx}$. Check that the derivatives you get are the same.

$$x/y^3 = 1$$

$x = y^3 \Rightarrow \sqrt[3]{x} = y$

$\frac{dy}{dx} = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

$1 = x \cdot y^{-3}$

$\Rightarrow 0 = x \cdot -3y^{-4}\frac{dy}{dx} + y^{-3}$

$\Rightarrow \frac{dy}{dx} = \frac{y^{-3}}{3xy^{-4}} = \frac{y}{3x}$

plug in to check!

$$x^2 + y^3 = 4$$

$y^3 = 4 - x^2$

$\Rightarrow y = \sqrt[3]{4-x^2}$

$\frac{dy}{dx} = -\frac{2}{3}x(4-x^2)^{-2/3}$,

$\frac{dy}{dx} = -\frac{2x}{3y^2}$

plug in to check!

$$x^2 + y^2 = 2$$

$y = \pm\sqrt{2-x^2}$

$$\frac{dy}{dx} = \begin{cases} -x(2-x^2)^{-\frac{1}{2}} & y>0 \\ x(2-x^2)^{-\frac{1}{2}} & y<0 \end{cases}$$

$0 = 2x + 2y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

plug in to check!

Implicitly determine the derivatives $\frac{dy}{dx}$ for each of the following expressions. If a point is provided, evaluate $\frac{dy}{dx}$ at that point.

$$e^x - \cos(y) = x$$

$\frac{d}{dx}\; e^x - \cos(y) = e^x + \frac{dy}{dx}\sin(y)$

$\Rightarrow e^x + \frac{dy}{dx}\sin(y) = 1$

$\Rightarrow \frac{dy}{dx} = \boxed{\frac{1-e^x}{\sin(y)}}$

chain rule

product rule

$$\tan(x^2y^4) = 3x + y^2 \quad \textit{(warning: very gross)}$$

$3 + 2y\frac{dy}{dx} = \sec^2(x^2y^4) \cdot \frac{d}{dx}(x^2 \cdot y^4)$

$= \text{" "} \cdot (2xy^4 + x^2 \cdot 4y^3\frac{dy}{dx})$

simplifying gives

$\frac{dy}{dx} = \boxed{\frac{3 - 2x \cdot y^4 \cdot \sec^2(x^2y^4)}{x^2 \cdot 4y^3 \cdot \sec^2(x^2y^4) - 2y}}$

$$x^2 + y^2 = 4 \text{ at } (0,2)$$

Since this is a circle, we can guess that the derivative is 0 @ (0,2). we also get

$0 = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

$\frac{-(0)}{2} \boxed{= 0}$

it works!

[redacted]

$x^4 + y^2 = 3$ @ $(1, -\sqrt{2})$

$0 = 4x^3 + \frac{dy}{dx} \cdot 2y$

$\Rightarrow \frac{dy}{dx} = -\frac{2x^3}{y}$

plug in

$\Rightarrow \frac{-2(1)^3}{-\sqrt{2}} \boxed{= \sqrt{2}}$

Complete the following related rates problems.

1. The length of one edge of a cube is increasing at a rate of 8cm/s. How quickly is the volume of the cube growing when this edge is 6cm long?

$x(t)$ = length of edge, volume $= (x(t))^3 = x^3(t)$

$$\frac{dx}{dt} = 8 \implies \frac{d}{dt}(x(t))^3 = 3\cdot x(t)^2 \cdot \frac{dx}{dt} = 3\cdot(6)^2\cdot 8 \quad \boxed{= 864 \ \text{cm}^3/\text{s}}$$

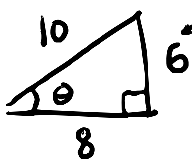
2. A dinghy carrying a horse (a *horse boat*, patent Dwight Schrute) is pulled toward a dock by a rope. The dock is 6 feet higher than the top of the boat.

   (a) If we pull in the rope at 2 ft/s, how quickly is the boat approaching the dock if 10 feet of rope are out?

   $\theta(t)$

   (b) How quickly is the *angle from the rope to the boat* changing when 10 feet of rope are out?

This question is directly from your practice exam, save for a small change. Make sure you're comfortable with the concepts, setup, and computation strategies used in this problem.



(notice that it's a 3-4-5 triangle!)

(a) let $R(t)$ be the amount of rope; $R(t)=10$ for some t. now, we also know

$$\frac{dR}{dt} = 2$$

and that the speed $s(t)$ of the boat should follow the Pythagorean thm. thus,

$$R(t)^2 = s(t)^2 + 6^2$$

want this!!!

$$\implies 2R(t)\cdot\frac{dR}{dt} = 2s(t)\cdot\frac{ds}{dt}$$

we know these!

Plugging in known values, we get

$$2(10)\cdot 2 = 2(8)\frac{ds}{dt}$$

$$\implies \frac{ds}{dt} = \frac{20}{8} \boxed{= \frac{5}{2} \ \text{ft/s}}$$

(b) treat the angle θ as a f'n of time: $\theta(t)$ = "angle @ time t". then, we know that

$$\sin(\theta(t)) = \frac{6}{R(t)}$$

so

want this! we know these!

$$-\cos(\theta(t))\cdot\frac{d}{dt}(\theta(t)) = -\frac{6}{R(t)^2}\,\frac{dR}{dt}$$

plugging in, we get

$$\left(\frac{8}{10}\right)\cdot\frac{d\theta}{dt} = \frac{-6}{10^2}\cdot 2 \implies \frac{d\theta}{dt} \boxed{= -\frac{12}{80}}$$

(radians/s)

3. Air is escaping from a perfectly (and impossibly) spherical balloon at a rate of 2 m^3/min. When the radius of the balloon is 1m, how quickly is the *surface area* of the balloon shrinking?

$V(t)$ = volume $= \frac{4}{3}\pi\cdot r(t)^3$, $r(t)$ = radius, $S(t)$ = surface area $= 4\pi\cdot r(t)^2$

what we know: $\frac{dV}{dt} = 2$, $r(t)=1$ (for some t)

what we <u>need</u> to know: $\frac{dr}{dt}$, $\frac{dS}{dt}$

we can use the volume derivative to find $\frac{dr}{dt}$, then solve for $\frac{dS}{dt}$!

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finding $\frac{dr}{dt}$:

$$\frac{dV}{dt} = \frac{4}{3}\pi\cdot 3r(t)^2\cdot\frac{dr}{dt}$$

$$\implies 2 = \frac{4}{3}\pi\cdot 3(1)^2\cdot\frac{dr}{dt} = 4\pi\cdot\frac{dr}{dt}$$

$$\boxed{\frac{1}{2\pi}} = \frac{dr}{dt}$$

finding $\frac{dS}{dt}$:

$$\frac{dS}{dt} = 4\pi\cdot 2r(t)\cdot\frac{dr}{dt} = 4\pi\cdot 2(1)\cdot\frac{1}{2\pi}$$

$$\boxed{= 4 \ \text{m}^2/\text{s}}$$