Week 13 — Exam Review

MATH:113, Recitations 304 and 305

Curves

Names: SOLUTIONS

Write the expression for the linearization of a function $f(x)$ at $x = a$. What does each term represent?

For a function f differentiable at a point a, the linear approximation $L(x)$ of f is

$$L(x) = \underbrace{f(a)}_{\text{value of } f \text{ at } a} + \overbrace{f'(a)}^{\text{"slope" at } a}(x-a)$$

What does the *extreme value theorem* say? What assumptions do we make?

For a function f continuous on the closed interval $[a,b]$, there are numbers $c, d \in [a,b]$ such that $f(c) \le f(x) \le f(d)$ for all $x \in [a,b]$.

What does the *mean value theorem* say? What assumptions do we make?

For a function f continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), there is a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Write the definitions for *critical points* and *inflection points*.

For a function f twice differentiable on (a,b) and cont's on $[a,b]$, a critical point is a point $x^* \in (a,b)$ such that $f'(x) = 0$.

An inflection point is a critical point for $f'(x)$.

Problem 1. Consider the function $f(x) = \ln(4 - x^2)$. Find:

(a) the domain.

(b) the range (or *image*).

(c) and simplify $f'(x)$ and $f''(x)$.

(d) all critical points.

(e) all inflection points.

(f) intervals where the function is increasing and decreasing.

(g) intervals where the function is concave-up or concave-down.

Using the information above, sketch $f(x)$.

This question is directly from your practice exam, save for a small change. Make sure you're comfortable with the concepts, setup, and computation strategies used in this problem.

(a) $4-x^2=0$ @ $2,-2$ so domain is $(-2,2)$.

(b) max is at $x=0 \Rightarrow \ln(4)$, min is $-\infty$, so image is $(-\infty, \ln(4))$

(c) $f'(x) = \frac{-2x}{4-x^2}$, $f''(x) = \frac{-2(4+x^2)}{(4-x^2)^2}$

(d),(e) critical points @ 0, no inflection points

(f) increasing on $(-2,0)$, decreasing on $(0,2)$

(g) always concave-down

Problem 2. Find two positive numbers x and y whose sum is 300 and whose product xy is as large as possible.

$$\left.\begin{array}{l} x+y=300, \\ y=300-x \end{array}\right\} \quad xy = x(300-x) = \underline{300x - x^2}$$

maximize!

$$\frac{d}{dx}(300x-x^2) = \underline{300-2x},$$

$$\underline{0 = 300-2x} \Rightarrow \boxed{x=150} \Rightarrow \boxed{y=150}$$

Problem 3. Because we're cheap, we'd like to construct a box that has no top, a square bottom, and has volume 216 in^3. What should the width and height of the box be to *minimize its surface area?*

let x be width of base, y be height of box. then,

Surface area $= A(x) = 4xy + x^2$, Volume $= \underline{x^2 y = 216 \text{in}^3}$.

now we have $y = 216/x^2$, so

$$A(x) = 4x\left(\frac{216}{x^2}\right) + x^2 = 864/x + x^2.$$

$A'(x) = -864/x^2 + 2x$, but x can't $=0$, so x must be $\sqrt[3]{432} = \boxed{6\sqrt[3]{2}}$. This is the only defined critical point — plugging in for $S(x)$, we get $\boxed{y = 3\sqrt[3]{2}}$.