# Week 12 Recitation Problems MATH:113, Recitations 304 and 305 

Names:

## Critical points and curve sketching

1. Concepts. Discuss the following with your group.
(i) What is a critical point?
(ii) What is an inflection point?
(iii) What information do critical and inflection points give us?
(iv) Given a differentiable function $f: \mathbb{R} \rightarrow \mathbb{R}$, two theorems give us guarantees about certain values of $f$ or its derivative $f^{\prime}$ on a given interval. What are these theorems? What properties of $f$ or the interval $[a, b]$ are required for the theorems to hold?
2. Setup.
(i) Consider the function $f(x)=(x-1)^{2}(x+1)^{2}$ on the interval $[-3,3]$. Find its first and second derivatives $f^{\prime}(x)$ and $f^{\prime \prime}(x)$.
(ii) Without technological intervention, find the roots (zeros) of the first and second derivatives.
(iii) Using some kind of computer (e.g. Desmos, a graphing calculator), plot the first and second derivatives.

## 3. Computation.

(i) Consider the function $f(x)=(x-1)^{2}(x+1)^{2}$ on the interval $[-3,3]$. What are its critical and inflection points?
(ii) Now, without using the internet, plot critical and inflection points. Note the sign of the first and second derivatives to the left and right of each point. Can you tell (approximately) how the function looks based just on these points? Based on this information, sketch $f(x)$ on the nearest whiteboard.

## L'Hôpital's Rule

1. Concepts. Discuss the following with your group.
(i) What problem does L'Hôpital's Rule help us solve?
(ii) In what situations can we not use L'Hôpital's Rule?
(iii) Suppose we're computing something like

$$
\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}
$$

In words, what are we being asked to compute?

## 2. Computation.

(i) Imagine $n$ represents the size of an input to computer program $P$, and $f(n)$ represents the number of steps it takes to execute $P$. Given a a function $g(n)$, we can characterize the asymptotic running time of the program $P$ by the limit

$$
\lim _{n \rightarrow \infty} \frac{f(n)}{g(n)}=c
$$

If $c=0$, we say that $f$ is $\operatorname{Big}-O$ of $g$ : in other words, for some number $M$, $f(x)$ is upper-bounded by $M \cdot g(x)$ as $x \rightarrow \infty$. Symbolically, $f \in \mathcal{O}(g)$, which means " $f$ is in the class of functions $\mathcal{O}(g)$."
If $0<c<\infty$, we say that $f$ is Big-Theta of $g$, or $f \in \Theta(g)$. This means that $g$ can be upper-bounded by $f$, but $f$ can also be upper-bounded by $g, f$ and $g$ grow at the same asymptotic rate. If $f \in \Theta(g)$, then we've found a tight bound on the running time of $f$.
If $c=\infty$, then $f$ is Big-Omega of $g$, or $f \in \Omega(g)$. In other words, $f$ is an upper bound of $g$.
Suppose we've implemented the QuickSort algorithm and, given a list $L$ of size $n$, the function call QuickSort $(L)$ executes in $f(n)=16 n^{2}+2 n$ steps. If $g(n)=n^{2}$, show that $f \in \mathcal{O}(g)$. If $h(n)=n \cdot \ln (n)$, show that $f \in \Omega(h)$.
3. As a class, let's prove that L'Hôpital's Rule works in the $\%$ case.

