Week 12 Recitation Problems

MATH:113, Recitations 304 and 305

SOLUTIONS

Critical points and curve sketching

1. **Concepts.** Discuss the following with your group.
   (i) Given a differentiable function on an interval $[a, b]$, a critical point x is a point in $[a, b]$ such that $f'(x) = 0$ *or* $f'(x)$ does not exist.
   (ii) An inflection point is a critical point of the first derivative.
   (iii) As the derivative of f at x tells us how quickly f changes its inputs, critical points tell us when f isn't changing its outputs at all. Inflection points tell us the exact spots where the rate at which f changes its inputs slows down or speeds up.
   (iv) The extreme value theorem tells us that, if f is continuous on a closed interval $[a, b]$, f achieves a maximum and a minimum on $[a, b]$. The mean value theorem says that, if f is continuous on $[a, b]$ and differentiable on (a, b), then there is some input c in $[a, b]$ such that $f'(c) = {}^{f(b)-f(a)}/_{b-a}$. Think about this as the "average" value theorem, so the derivative (at some point) has to be exactly the average values at the endpoints.
2. **Setup.**
   (i) Simplifying $f(x)$, we get that $f'(x) = 4x^3 - 4x$ and $f''(x) = 12x^2 - 4$
   (ii) $f'(x)$ is a degree-three polynomial, so it has three zeros at $x = 1$, $x = -1$, and $x = 0$. $f''(x)$ is a degree-two polynomial so it has two zeros at $x = \pm{}^{1}/_{\sqrt{3}}$.
3. **Computation.** Not much to do here!

L'Hôpital's Rule

1. **Concepts.** Discuss the following with your group.
   (i) If the limit of the ratio of two differentiable functions at a point c is $^0/_0$ or $^\infty/_\infty$, L'Hôpital's rule allows us to take the limit of the ratio of *derivatives* at the point c instead.
   (ii) If the limits of the function *aren't equal* and *aren't* or ∞ at c; if f and g aren't differentiable; if the derivative of g is 0 everywhere *but* c; the limit of the ratio of derivatives has to exist *or itself be an indeterminate form*.
   (iii) We're finding the limit of the ratio of the derivatives at the point c.
2. **Computation.**
   (i) First, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{16n^2+2n}{n^2} &\overset{H}{=} \lim_{n\to\infty}\frac{32n+2}{2n}\\ &\overset{H}{=} \lim_{n\to\infty}\frac{32}{2}\\ &= 16.\end{aligned}$$

Thus $0 \leq c = 16 < \infty$, so $f \in \mathcal{O}(g)$. Next, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{16n^2+2n}{n\cdot\ln(n)} &\overset{H}{=} \lim_{n\to\infty}\frac{32n+2}{1\cdot\ln(n)+n\cdot{}^1/_n}\\ &= \lim_{n\to\infty}\frac{32n+2}{\ln(n)+1}\\ &\overset{H}{=} \lim_{n\to\infty}\frac{32}{^1/_n}\\ &= \lim_{n\to\infty} 32n\\ &= \infty,\end{aligned}$$

so $c = \infty$, and $f \in \Omega(g)$.

3. **As a class, let's prove that L'Hôpital's Rule works in the $^0\!/_0$ case.**

Proof. Suppose that, for some number c, $f(c) = 0 = g(c)$, and $g'(c) \neq 0$. Then, we have

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{f(x)-0}{g(x)-0} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} \cdot \frac{1/x-c}{1/x-c} \\
&= \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}.
\end{aligned}$$

But we know that the limit of the denominator is nonzero and exists; we also know that the limits in the numerator and denominator are *precisely the limits which define the derivative*, so we get

$$\begin{aligned}
\lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} &= \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} \\
&= \frac{f'(c)}{g'(c)} \\
&= \lim_{x\to c} \frac{f'(x)}{g'(x)},
\end{aligned}$$

and we're done. ∎