Week 11 Recitation Problems

MATH:113, Recitations 304 and 305

SOLUTIONS

Linearization

1. **Concepts.** Discuss the following with your group.
   (i) It's in the name: we want a simple way to guess at values for a particular function!
   (ii) Since *linear* approximations have us use lines, we'll need the value of the function *and the "slope"* of the function at the input $x = a$.
   (iii) This is a more delicate question than might seem: generally, these approximations are really bad, but there's actually a way to *measure* the error between our approximated and actual functions. How would you do it?
2. **Setup.**
   (i) $L(x) = f(a) - f'(a)(x - a)$.
   (ii) $f(a)$ gives you the value of the function at the point $x = a$; $f'(a)$ gives you the slope of the line tangent to f at a.
   (iii) Noting that $f(x) - f(a) \approx f'(a)(x - a)$, we can see that $\Delta y \approx f'(a)\Delta x$. As we shrink these Δs down to be infinitesimally small, we're able to effectively describe a function in terms of its rate of change: that is, $\mathrm{d}y = f'(a)\mathrm{d}x$ admits $\frac{\mathrm{d}x}{\mathrm{d}y} \approx f'(a)$.
3. **Computation.**
   (i) We end up with

$$L(x) = 2^{1/4} + \frac{1}{4}(2^{-3/4})(x - 2).$$

   (ii) We should end up with $L(3) \approx 1.33786$ and $L(10) \approx 2.37841$, which carry $\approx 1.655\%$ and $\approx 33.7481\%$ error respectively.

Critical points and curve sketching

1. **Concepts.** Discuss the following with your group.

   (i) A critical point is a point in the domain of a function at which the derivative of the function is 0 or does not exist.

   (ii) An inflection point is a critical point of the second derivative.

   (iii) Critical points tell us about when the function is increasing or decreasing; inflection points tell us about how quickly the function is increasing/decreasing on those intervals.

   (iv) For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ on a closed interval, the Extreme Value Theorem guarantees that f achieves a maximum and a minimum value on that interval. The Mean Value Theorem tells us that there is some point c on our interval $[a, b]$ such that $f'(c) = {}^{f(b)-f(a)}/_{b-a}$.

2. **Computation.**

   (i) The points at which $f'(x) = 0$ are $x = -1$, $x = 0$, and $x = 1$, and the points at which $f''(x) = 0$ are $x = -1/2$, $x = 1/2$. We can also do a simple sanity check here: f is a degree-4 polynomial, so the first derivative of f is necessarily a degree-3 polynomial, and thus *must* have three roots (though they don't have to be distinct); further, as f' is a degree-3 polynomial, f'' must be a degree-2 polynomial, so it *must* have two roots (though, again, they don't have to be distinct). Notice that, in our calculations, we got three roots for the first derivative and two roots for the second derivative, so we're on the right track!

   (ii) Try doing this by hand instead of using Desmos. You can tell pretty much exactly how a function looks *just* based on information from its derivatives!