Week 10 Recitation Problems

MATH:113, Recitations 304 and 305

Solutions

Suppose, for a certain magical rectangle $\mathscr{R}$, that its length is always exactly three times its height. Describe $\mathscr{R}$'s length by y and its height by x. If the height of $\mathscr{R}$ is decreasing at a rate of 2 inches per minute, how quickly is the length of $\mathscr{R}$ shrinking? If the height of $\mathscr{R}$ is exactly 6 inches and is decreasing at a rate of 2 inches per minute, how quickly is the area of $\mathscr{R}$ shrinking?

1. **Concepts.**

   (i) We treat x as a *function of time* — because the height is changing at a rate of (some number of inches) *per minute*, we know that the height of $\mathscr{R}$ depends on time. Notationally, we have a variable t be a nonnegative real number, and the height of $\mathscr{R}$ is equal to $x(t)$. So, here, *x is a function!*

   (ii) Because the length of $\mathscr{R}$ depends on its height at a given time t, we treat y as a *function of t* as well. For the same reasons as before, we treat y as a function!

   (iii) Based on the previous question and the prompt, we know that $y = 3x$. But as y is a function of t and x is a function of t, it's more precisely written as $y(t) = 3x(t)$. Note also that y depends on t: if the value of t changes, the height of $\mathscr{R}$ (i.e. the value of $x(t)$) changes; depending on the value of $x(t)$, the length (i.e. the value of $y(t)$) changes.

2. **Setup.**

   (i) The area of $\mathscr{R}$ is just the plain old area of a rectangle: $A(t) = y(x) \cdot x(t)$. By substitution, we can simplify this down to $A(t) = 3x(t) \cdot x(t) = 3x(t)^2$.

   (ii) The general tool we have for describing "rates of change" is the derivative: for example, if the height of $\mathscr{R}$ is changing at a rate of -2 inches per minute, and $x(t)$ is the function representing the height, we have $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -2$. If that's the case, what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

   (iii) If we want to figure out how quickly the length $y(x)$ is changing based on how quickly $x(t)$ is changing, we want to figure out the answer to $\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}(3x(t))$: notice that the equals sign relates the derivatives — *that is, relates the rates of change* — giving rise to the name "related rates."

   (iv) Because our function y is implicitly defined in terms of t and our function x is in terms of t — and we don't know what x is — we'll have to use *implicit differentiation*.

3. **Computation.**

(i) We can use the relationship $y(t) = 3x(t)$ and take derivatives: we end up with

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3 \cdot \frac{\mathrm{d}}{\mathrm{d}t} x(t) = 3 \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

or alternatively

$$y'(t) = 3x'(t).$$

But remember that *we know the value of* $\frac{dx}{dt}$: it's -2, so we can conclude that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \cdot -2 = -6.$$

(ii) Given our function $A(t) = 3(x(t))^2$, we use the chain rule to see that

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = 3 \cdot \frac{\mathrm{d}}{\mathrm{d}t} (x(t))^2 = 6 \cdot x(t) \cdot \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Again, *we know what* $\frac{dx}{dt}$ *and* $x(t)$ *are*, so we can substitute those values:

$$6 \cdot x(t) \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = 6 \cdot (6) \cdot (-2) = -72.$$

Imagine that you're a clown with a particularly good set of lungs. You're in charge of making balloon animals at your friend's wife's brother's friend's kid's birthday party — unfortunately, your clown school didn't have the accreditation for awarding Balloon Animal Certificates, so you've only been trained to make extremely spherical "animals."

With your impressive wind power generation, you blow up a balloon $\mathfrak{B}$ so that its volume increases at a rate of 100 cubic inches per second (which is crazy fast). How quickly is the radius of $\mathfrak{B}$ increasing when the diameter is exactly 50 inches?

1. **Setup.**
   (i) The volume of $\mathfrak{B}$ is just the volume of a sphere: ${}^{4\pi}/_{3} \cdot r^3$. Given that we only have information about the *diameter* of $\mathfrak{B}$, that the radius is always half the diameter, and that the diameter depends on time, we describe the diameter by the function $D(t)$, and the radius r by $r(t) = {}^{D(t)}/_{2}$.
   (ii) The "rate of increase" in the volume is simply the value of the derivative $\frac{dV}{dt}$, so $\frac{dV}{dt} = 100$.
   (iii) We'll certainly have to use implicit differentiation again!
   (iv) Suppose that our radius r is only a variable, and our volume function V is expressed in terms of r: that is, $V(r) = {}^{4\pi}/_{3} \cdot r^3$. Taking the derivative with respect to r, we get $\frac{dV}{dr} = 4\pi \cdot r^2$, which is the formula for the surface area of a sphere. Thinking about these relationships is *extremely important* in calculus, especially as we move toward the concepts of *anti-derivatives* and *integrals*.

2. **Computation.**
   (i) Using our definition of $V(t)$, we get that

$$\frac{dV}{dt} = 4\pi \cdot r(t)^2 \cdot \frac{dr}{dt},$$

   and we want to find $\frac{dr}{dt}$. We already know what $\frac{dV}{dt}$ is, and we already know what the diameter $D(t)$ is — thus we know what the radius $r(t)$ is — so all that's left to do is plug in values and do some simplification. Doing so, we get

$$100 = 4\pi \cdot (25)^2 \cdot \frac{dr}{dt}$$

   and, simplifying, we find that

$$\frac{1}{25\pi} = \frac{dr}{dt}.$$

   (ii) The volume of the earth is about 45 trillion cubic inches. If we're increasing the volume of $\mathfrak{B}$ at a rate of 100 cubic inches per minute, it'd take us about 450 billion minutes, which is about 856,164 years.