Week 4 Recitation Problems

MATH:113, Recitations 304 and 305

Determine whether these functions are continuous at the indicated points.

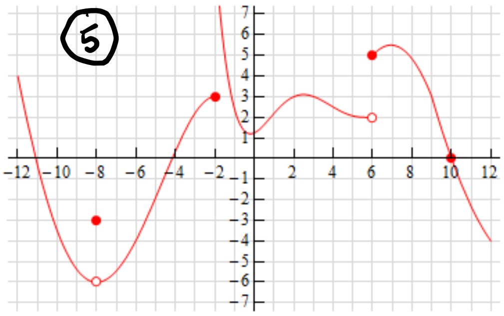
1. $$g(x) = \begin{cases} 2x & x < 6 \\ x - 1 & x \geq 6 \end{cases}$$
$$x = 4, \; x = 6$$

2. $$f(x) = \frac{6}{x^2 - 3x - 10}$$
$$x = -2, \; x = 0, \; x = 3$$

3. $$f(t) = \begin{cases} t^2 & t < -2 \\ t + 6 & t \geq -2 \end{cases}$$
$$t = -2, \; t = 10$$

4. $$h(x) = \frac{5x + 5}{9 - 3x}$$
$$x = -1, \; x = 0, \; x = 3$$

5.



6. $$f(x) = \begin{cases} x^2 & x < 0 \\ e^x & x \geq 0 \end{cases}$$
$$x = 0$$

(How can we make this function continuous?)

. . . .

What are the domains and ranges of these functions? Where are they discontinuous?

1. $$f(x) = \frac{x^2 - 9}{3x^2 + 2x + 8}$$

2. $$H(t) = \frac{8t}{t^2 - 9t - 1}$$

3. $$y(t) = \frac{x}{7 - e^{2t+3}}$$

4. $$r(\theta) = \tan(2\theta)$$

5. $$L(t) = \sin\left(\frac{1}{t}\right)$$

6. $$f(x) = \frac{\sin x}{x - 2}$$

Solve two of these problems using the intermediate value theorem.

(a) Show that the function $f(x) = x^4 + x - 3$ has a root on the interval $[0, 2]$.

(b) Does the function $g(x) = x^3 + 3x^2 + x - 2$ have a root in $[0, 1]$? If so, approximate it.

(c) Show that there exists a positive number c such that $c^2 = 2$.

. . . .

Derivatives.

Definition 1: the limit definition of a derivative.

We shrink the distance from x to our base point a:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or add a bit of distance from x, and shrink that distance:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition 2: the derivative of a function, using only words.

Derivatives measure the "sensitivity" of a function at a point: if we give the function an input, how different will its output be?

Find derivatives for these functions. Find the *value* of the derivative at $x = 0$.

1 $g(x) = x^2$

2 $V(x) = 3 - 14x$

3 $h(x) = \dfrac{5}{x}$

4 $f(x) = |x|$

5 $\cancel{L(x) = \dfrac{x^2 - 4}{x - 2}}$

6 $y(x) = x^{1/3}$

Functions continuous at points

① As $4<6$, we have $g(x)=2x$. Thus, we have

$$\lim_{x\to 4} 2x = 2\lim_{x\to 4} x = 2\cdot 4 = 8,$$

and $g(4)=8$, so $\lim_{x\to 4} g(x) = g(4)$,

so $g(x)$ is cont's @ $x=4$. Now for $x=6$, we have

$$\lim_{x\to 6^-} g(x) = \lim_{x\to 6^-} 2x = 12 \quad \text{and} \quad \lim_{x\to 6^+} g(x) = \lim_{x\to 6^+} x-1 = 5$$

so the left- and right-hand limits are unequal, so g does not have a limit @ $x=6$, and thus cannot be continuous.

② First, note that $z^2-3z-10=(z-5)(z+2)$, so

$$g(z)=\frac{6}{(z-5)(z+2)}.$$

Thus, taking the limits at $z=5$ and $z=-2$ give us 0 in the denominator, so these limits are infinity. (In other words, g has discontinuities at $z=5$ and $z=-2$.) Now, for $z=0$, we get

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{6}{(z-5)(z+2)} = \frac{6}{\lim\limits_{z\to 0}(z-5)\cdot\lim\limits_{z\to 0}(z+2)} = \frac{6}{(-5)\cdot(2)} = \frac{-3}{5},$$

so this limit exists. Further,

$$g(0)=\frac{6}{(z-5)(z+2)} = -\frac{3}{5}, \implies \lim_{x\to 0} g(x) = g(0)$$

so $g(x)$ is continuous at $x=0$.

③ As f "splits" at 2 and we can't tell anything about f there, we have to take left and right limits:

$$\lim_{t\to -2^-} t^2 = (-2)^2 = 4 \qquad \text{and} \qquad \lim_{t\to -2^+} t+6 = \left(\lim_{t\to -2^+} t\right) + 6 = 4$$

same!

and we have $f(2)=4$, so f is continuous at $t=-2$. For $t=10$, we only care about the $t+6$ piece of $f(t)$; as $t+6$ is a linear function (a degree-1 polynomial), it's continuous on its domain, so f is continuous at $t=10$.

④ Notice that the numerator is a linear function and is thus continuous on its domain; any discontinuities here will come from the denominator. By observation, we can see that $x=-1$ and $x=0$ don't send the denominator to 0, but $x=3$ does, so $f(3)$ doesn't exist in the first place. EXTRA: try to compute the left- and right-sided limits at $x=3$. Are they the same?

⑤ discontinuous at $x=-8$; discontinuous at $x=-2$; discontinuous at $x=6$; continuous at $x=10$.

⑥ Try the left- and right-hand limits:

$$\lim_{x\to 0^-} x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0} e^x = 1,$$

not equal!

So this function isn't continuous at 0. There are infinitely many ways to make this function continuous, but two are

$$f(x) = \begin{cases} x^2+1 & x<0 \\ e^x & x\geq 0 \end{cases} \qquad \text{or} \qquad f(x) = \begin{cases} x^2 & x<-1 \\ e^{x+1} & x\geq -1. \end{cases}$$

Domains, ranges, discontinuities

① First, we can re-write this as

$$f(x) = \frac{(x-3)(x+3)}{(3x-4)(x+2)}$$

so $x=-2$ and $x = 4/3$ evaluate to something over zero; these points are thus our discontinuities

real numbers

D: $\mathbb{R}$ except -2, $4/3$

R: asymptote at $1/3$; $(-\infty, \frac{1}{3})$, and min. value of function is at $\frac{\sqrt{13}+7}{10}$, so $(-\infty, \frac{1}{3}) \cup (\frac{\sqrt{13}+7}{3}, \infty)$

the calculation of the range here is difficult; likely won't see this on an exam.

② Same trick as before: where does division by zero happen? Well, we can't nicely factor the denominator, so we use the quadratic formula:

$$t = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{9 \pm \sqrt{85}}{2},$$

which gives us our points of discontinuity.

D: $\mathbb{R}$ except

R: finding limits as t goes to $-\infty$ and ∞, we find asymptotes at 0, but between the roots we hit all values $(-\infty, \infty)$

③ When is the denominator 0?

$$0 = 7 - e^{2t+3} \Rightarrow 7 = e^{2t+3}$$

$$\Rightarrow \ln(7) = 2t+3$$

$$\Rightarrow t = \frac{\ln(7)-3}{2}$$

D: $\mathbb{R}$ except

R: asymptotes at 0, $\frac{1}{7}x$, but limits as x approaches are $-\infty$ and ∞, so $(-\infty, \infty)$

④ We have a trig function! Note that $\tan(2\theta) = \sin(2\theta)/\cos(2\theta)$, so this function is discontinuous whenever $\cos(2\theta) = 0$. We know that $\cos(t) = 0$ when $t = \frac{\pi}{2} + n \cdot \pi$, where n is an integer; thus, we have

$$2\theta = \frac{\pi}{2} + n\cdot\pi \Rightarrow \theta = \frac{\pi}{4} + \frac{n \cdot \pi}{2}$$

(or another equivalent expression).

D: all $\mathbb{R}$ except odd multiples of $\frac{\pi}{4}$

R: all of $\mathbb{R}$

⑤ This function doesn't have a limit at 0, so it can't be cont's there.

D: $\mathbb{R}$ except 0

R: $[-1, 1]$

D: $\mathbb{R}$ except 2

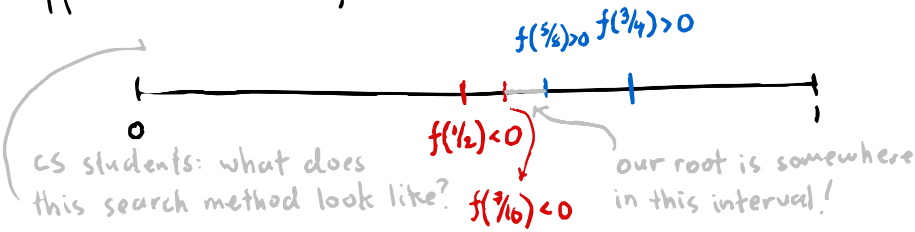
R: all of $\mathbb{R}$

⑥ Again, where's the denominator 0? At $x=2$, the limit of this function goes to $-\infty$ on the left and $+\infty$ on the right, so it doesn't have a limit there. (It also doesn't take a value there.)

Intermediate Value Theorem

(a) First, because f is a polynomial, it's continuous. (This is important to check, as the IVT breaks without continuity!) Now, we have $f(0) = -3$ and $f(2) = 15$; thus, $f(0) < f(2)$. Now, because $f(0) < 0 < f(2)$, the IVT says that there must be some c between 0 and 2 such that $f(0) < 0 = f(c) < f(2)$, so f must have a root in $[0,2]$.

(b) Using the same technique as above, we find $f(0) = -2$ and $f(1) = 3$, so by the IVT, there's a c in $[0,1]$ where $f(0) < 0 = f(c) < f(1)$, so f has a root in $[0,1]$. Now, to approximate the root, use a calculator to test values:



By finding where the function is positive or negative on the interval $[0,1]$, we can make our search window $[a,b]$ progressively narrower by increasing the value of a (when $f(a)$ is negative) and reducing the value of b (when $f(b)$ is positive).

(c) Let the function $f(c) = c^2 - 2$. When $f(c) = 0$, we have

$$f(c) = 0 = c^2 - 2 \Rightarrow c^2 = 2.$$

But then $f(1) = -1 < 0 < f(2) = 2$, so by the IVT, there must be some c such that $f(1) < 0 = f(c) < f(2)$ with c between 1 and 2; thus, there is a real number c such that $c^2 = 2$.

BONUS: how could you approximate c?

Derivatives

① $\lim\limits_{h\to 0} \dfrac{g(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh+h^2}{h}$

$= \lim\limits_{h\to 0} 2x+h$

$g'(x) = 2x$

$\Rightarrow g'(0) = 2(0) = 0.$

② $\lim\limits_{h\to 0} \dfrac{V(x+h)-V(x)}{h} = \lim\limits_{h\to 0} \dfrac{(3-14(x+h))-(3-14x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3-14x-14h-3+14x}{h}$

$= \lim\limits_{h\to 0} \dfrac{14h}{h}$

$= \lim\limits_{h\to 0} 14$

$V'(x) = 14$

$\Rightarrow V'(0) = 14$

③ $\lim\limits_{t\to 0} \dfrac{h(x+t)-h(x)}{t} = \lim\limits_{t\to 0} \dfrac{\frac{5}{x+t} - \frac{5}{x}}{t}$

$= \lim\limits_{t\to 0} \dfrac{\frac{5x-5(x+t)}{(x+t)(x)}}{t}$

$= \lim\limits_{t\to 0} \dfrac{\frac{5x-5x-5t}{x^2+tx}}{t}$

$= \lim\limits_{t\to 0} \dfrac{-5t}{(x^2+tx)(t)}$

$= \lim\limits_{t\to 0} \dfrac{-5}{x^2+tx}$

$V'(x) = -\dfrac{5}{x^2}$

$\Rightarrow V'(0) = \text{DNE!}$

④ $|x| = \begin{cases} x & x>0 \\ 0 & x=0 \\ -x & x<0 \end{cases}$

so let $P(x) = x$, $N(x) = -x$. Then, we have

$\lim\limits_{h\to 0} \dfrac{P(x+h)-P(x)}{h} = \lim\limits_{h\to 0} \dfrac{x+h-x}{h}$

$= \lim\limits_{h\to 0} \dfrac{h}{h}$

$= \lim\limits_{h\to 0} 1$

$= 1$

but

$\lim\limits_{h\to 0} \dfrac{N(x+h)-N(x)}{h} = \lim\limits_{h\to 0} \dfrac{-(x+h)-(-x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{-h}{h}$

$= \lim\limits_{h\to 0} -1$

$= -1.$

But notice that

$|x| = \begin{cases} P(x) & x>0 \\ 0 & x=0 \\ N(x) & x<0 \end{cases}$

so the derivative at $x=0$ must be the derivative of $P(x)$ as $x\to 0^+$ and $Q(x)$ as $x\to 0^-$. But $P'(0) = Q'(0)$, so $|x|$ is not differentiable at 0.

⑤ This one's a little much.

⑥ $\lim\limits_{h\to 0} \dfrac{y(x+h)-y(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^{\frac{1}{2}} - x^{\frac{1}{2}}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(\sqrt{x+h}-\sqrt{x})}{h} \cdot \dfrac{(\sqrt{x+h}+\sqrt{x})}{\sqrt{x+h}+\sqrt{x}}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{x+h}\cdot\sqrt{x+h} - \sqrt{x}\sqrt{x}}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{h}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

$y'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{x^{-1/2}}{2}$

$\Rightarrow y'(0)$ DNE