Week 3 Recitation Problems

MATH:113, Recitations 304 and 305

The Squeeze and Intermediate Value Theorems

1. $\lim_{t \to 0} t^2 \sin^2 \left(\frac{1}{t} \right)$
2. Show that the function $f(x) = x^4 + x - 3$ has a solution on the interval $[0, 2]$.
3. $\lim_{t \to \infty} \frac{\sin t}{t}$
4. Does the function $f(x) = x^3 + 3x^2 + x - 2$ have a root on $[0, 1]$? If so, approximate it.

Solutions

1. $\lim_{t\to 0} \sin\left(\frac{1}{t}\right)$ doesn't exist, but we know that
$$-1 \le \sin\left(\tfrac{1}{t}\right) \le 1$$
so we can claim that
$$-t^2 \le t^2 \sin\left(\tfrac{1}{t}\right) \le t^2.$$
But then we know that $\lim_{t\to 0} t^2 = 0 = \lim_{t\to 0} -t^2$, so by the squeeze thm, our limit is 0.

2. This function isn't really factorable, so we note that $f(0) = 0+0-3 = -3$ and $f(2) = 16+2-3 = 15$; thus, we have
$$f(0) \le 0 \le f(2),$$
so there must be some $c \in [0,2]$ where $f(c) = 0$.

3. Two ways to solve this one, really:
   (i) as $t \to \infty$, $\sin(t)$ doesn't get bigger (resp. smaller) than 1 (resp. -1), so everything goes to 0;
   (ii) we know that $-1 \le \sin(t) \le 1$, so $-\frac{1}{t} \le \frac{\sin(t)}{t} \le \frac{1}{t}$ and as $\lim_{t\to\infty} -\frac{1}{t} = 0 = \lim_{t\to\infty} \frac{1}{t}$, so $\lim_{t\to\infty} \frac{\sin(t)}{t} = 0$.

4. Yes: by the IVT, it does. We can "binary search" the zero here:
$$f(0.6) < 0,\ f(0.7) > 0$$
$$\Rightarrow f(0.61) < 0,\ f(0.62) > 0$$
$\Rightarrow$ f has a root between 0.61 and 0.62.